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Neutrosophic $\mathcal{N}$-Structures Applied to BCK/BCI-Algebras

Young Bae Jun [1], Florentin Smarandache [2] and Hashem Bordbar [3,*]

1 Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea; skywine@gmail.com
2 Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; fsmarandache@gmail.com
3 Department of Mathematics, Shiraz University, Shiraz 7616914111, Iran
* Correspondence: bordbar.amirh@gmail.com

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Abstract: Neutrosophic $\mathcal{N}$-structures with applications in BCK/BCI-algebras is discussed. The notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra are introduced, and several related properties are investigated. Characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are considered, and relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are stated. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal are provided.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-subalgebra; (closed) neutrosophic $\mathcal{N}$-ideal

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1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (MV-algebras). Additionally, Iséki introduced the notion of a BCI-algebra, which is a generalization of a BCK-algebra (see [2]).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far, most of the generalizations of the crisp set have been conducted on the unit interval $[0,1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [3] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [4] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [5] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components:

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood})$$

For more details, refer to the following site:

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In this paper, we discuss a neutrosophic $\mathcal{N}$-structure with an application to BCK/BCI-algebras. We introduce the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra, and investigate related properties. We consider characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We discuss relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal.

2. Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2,0)$. A *BCI-algebra* refers to a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = \theta,$
(II) $(x * (x * y)) * y = \theta,$
(III) $x * x = \theta,$
(IV) $x * y = y * x = \theta \;\Rightarrow\; x = y.$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \;\Rightarrow\; x * y = \theta)$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\;(x * \theta = x) \tag{1}$$
$$(\forall x, y, z \in X)\;((x * y) * z = (x * z) * y) \tag{2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following:

(I1) $0 \in I,$
(I2) $(\forall x, y \in X)(x * y \in I,\; y \in I \;\Rightarrow x \in I).$

We refer the reader to the books [6,7] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

We denote by $\mathcal{F}(X, [-1,0])$ the collection of functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X, [-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, *$\mathcal{N}$-function* on X). An *$\mathcal{N}$-structure* refers to an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X (see [3]). In what follows, we let X denote the nonempty universe of discourse unless otherwise specified.

A *neutrosophic $\mathcal{N}$-structure* over X (see [8]) is defined to be the structure:

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{3}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X, which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

We note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$$

3. Application in BCK/BCI-Algebras

In this section, we take a BCK/BCI-algebra X as the universe of discourse unless otherwise specified.

Definition 1. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-subalgebra of X if the following condition is valid:*

$$(\forall x, y \in X) \begin{pmatrix} T_N(x*y) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) \leq \bigvee\{F_N(x), F_N(y)\} \end{pmatrix} \tag{4}$$

Example 1. *Consider a BCK-algebra $X = \{\theta, a, b, c\}$ with the following Cayley table.*

$*$	θ	a	b	c
θ	θ	θ	θ	θ
a	a	θ	θ	a
b	b	a	θ	b
c	c	c	c	θ

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.7,-0.2,-0.6)}, \frac{a}{(-0.5,-0.3,-0.4)}, \frac{b}{(-0.5,-0.3,-0.4)}, \frac{c}{(-0.3,-0.8,-0.5)} \right\}$$

over X is a neutrosophic $\mathcal{N}$-subalgebra of X.

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$\begin{aligned} T_N^{\alpha} &:= \{x \in X \mid T_N(x) \leq \alpha\} \\ I_N^{\beta} &:= \{x \in X \mid I_N(x) \geq \beta\} \\ F_N^{\gamma} &:= \{x \in X \mid F_N(x) \leq \gamma\} \end{aligned}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ)-*level set* of $X_{\mathbf{N}}$. Note that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$$

Theorem 1. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subalgebra of X.*

Proof. Let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$ and $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$. If $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (4) that

$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha$,
$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$, and
$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma$.

Hence, $x * y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, and therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subalgebra of X. □

Theorem 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are subalgebras of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Proof. Assume that there exist $a, b \in X$ such that $T_N(a * b) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(a * b) > t_\alpha \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$ but $a * b \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\}$$

for all $x, y \in X$. If $I_N(a * b) < \bigwedge\{I_N(a), I_N(b)\}$ for some $a, b \in X$, then

$$I_N(a * b) < t_\beta < \bigwedge\{I_N(a), I_N(b)\}$$

where $t_\beta := \frac{1}{2}\{I_N(a * b) + \bigwedge\{I_N(a), I_N(b)\}\}$. Thus $a, b \in I_N^{t_\beta}$ and $a * b \notin I_N^{t_\beta}$, which is a contradiction. Therefore

$$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(a * b) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}$$

Then $a, b \in F_N^{t_\gamma}$ and $a * b \notin F_N^{t_\gamma}$, which is a contradiction. Hence

$$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

Because $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3. *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subalgebras of X, then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Proposition 1. *If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is a neutrosophic $\mathcal{N}$-subalgebra of X, then $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 4. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_N(a_n) = -1$, $\lim_{n\to\infty} I_N(a_n) = 0$ and $\lim_{n\to\infty} F_N(a_n) = -1$, then $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$.*

Proof. By Proposition 1, we have $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Hence $T_N(\theta) \leq T_N(a_n)$, $I_N(a_n) \leq I_N(\theta)$ and $F_N(\theta) \leq F_N(a_n)$ for every positive integer n. It follows that

$$
\begin{aligned}
-1 \leq T_N(\theta) &\leq \lim_{n\to\infty} T_N(a_n) = -1 \\
0 \geq I_N(\theta) &\geq \lim_{n\to\infty} I_N(a_n) = 0 \\
-1 \leq F_N(\theta) &\leq \lim_{n\to\infty} F_N(a_n) = -1
\end{aligned}
$$

Hence $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$. □

Proposition 2. *If every neutrosophic $\mathcal{N}$-subalgebra $X_{\mathbf{N}}$ of X satisfies:*

$$T_N(x*y) \leq T_N(y), I_N(x*y) \geq I_N(y), F_N(x*y) \leq F_N(y) \tag{5}$$

for all $x, y \in X$, then $X_{\mathbf{N}}$ is constant.

Proof. Using Equations (1) and (5), we have $T_N(x) = T_N(x*\theta) \leq T_N(\theta)$, $I_N(x) = I_N(x*\theta) \geq I_N(\theta)$ and $F_N(x) = F_N(x*\theta) \leq F_N(\theta)$ for all $x \in X$. It follows from Proposition 1 that $T_N(x) = T_N(\theta)$, $I_N(x) = I_N(\theta)$ and $F_N(x) = F_N(\theta)$ for all $x \in X$. Therefore $X_{\mathbf{N}}$ is constant. □

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if the following assertion is valid:*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(\theta) \leq T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(\theta) \geq I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(\theta) \leq F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} \end{array} \right) \tag{6}$$

Example 2. *The neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X in Example 1 is a neutrosophic $\mathcal{N}$-ideal of X.*

Example 3. *Consider a BCI-algebra $X := Y \times \mathbb{Z}$ where $(Y, *, \theta)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see [6]). Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X given by*

$$X_{\mathbf{N}} = \left\{ \tfrac{x}{(\alpha,0,\gamma)} \mid x \in Y \times (\mathbb{N} \cup \{0\}) \right\} \cup \left\{ \tfrac{x}{(0,\beta,0)} \mid x \notin Y \times (\mathbb{N} \cup \{0\}) \right\}$$

where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.

Proposition 3. *Every neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the following assertions:*

$$(x, y \in X)\ (x \preceq y \ \Rightarrow\ T_N(x) \leq T_N(y), I_N(x) \geq I_N(y), F_N(x) \leq F_N(y)) \tag{7}$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. Then $x * y = \theta$, and so

$$
\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} = \bigvee\{T_N(\theta), T_N(y)\} = T_N(y) \\
&I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} = \bigwedge\{I_N(\theta), I_N(y)\} = I_N(y) \\
&F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} = \bigvee\{F_N(\theta), F_N(y)\} = F_N(y)
\end{aligned}
$$

This completes the proof. □

Proposition 4. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then*

(1) $T_N(x*y) \leq T_N((x*y)*y) \ \Leftrightarrow\ T_N((x*z)*(y*z)) \leq T_N((x*y)*z)$
(2) $I_N(x*y) \geq I_N((x*y)*y) \ \Leftrightarrow\ I_N((x*z)*(y*z)) \geq I_N((x*y)*z)$
(3) $F_N(x*y) \leq F_N((x*y)*y) \ \Leftrightarrow\ F_N((x*z)*(y*z)) \leq F_N((x*y)*z)$

for all $x, y, z \in X$.

Proof. Note that

$$((x * (y * z)) * z) * z \preceq (x * y) * z \tag{8}$$

for all $x, y, z \in X$. Assume that $T_N(x * y) \leq T_N((x * y) * y)$, $I_N(x * y) \geq I_N((x * y) * y)$ and $F_N(x * y) \leq F_N((x * y) * y)$ for all $x, y \in X$. It follows from Equation (2) and Proposition 3 that

$$\begin{aligned} T_N((x * z) * (y * z)) &= T_N((x * (y * z)) * z) \\ &\leq T_N(((x * (y * z)) * z) * z) \\ &\leq T_N((x * y) * z) \end{aligned}$$

$$\begin{aligned} I_N((x * z) * (y * z)) &= I_N((x * (y * z)) * z) \\ &\geq I_N(((x * (y * z)) * z) * z) \\ &\geq I_N((x * y) * z) \end{aligned}$$

and

$$\begin{aligned} F_N((x * z) * (y * z)) &= F_N((x * (y * z)) * z) \\ &\leq F_N(((x * (y * z)) * z) * z) \\ &\leq F_N((x * y) * z) \end{aligned}$$

for all $x, y \in X$.

Conversely, suppose

$$\begin{aligned} T_N((x * z) * (y * z)) &\leq T_N((x * y) * z) \\ I_N((x * z) * (y * z)) &\geq I_N((x * y) * z) \\ F_N((x * z) * (y * z)) &\leq F_N((x * y) * z) \end{aligned} \tag{9}$$

for all $x, y, z \in X$. If we substitute z for y in Equation (9), then

$$\begin{aligned} T_N(x * z) &= T_N((x * z) * \theta) = T_N((x * z) * (z * z)) \leq T_N((x * z) * z) \\ I_N(x * z) &= I_N((x * z) * \theta) = I_N((x * z) * (z * z)) \geq I_N((x * z) * z) \\ F_N(x * z) &= F_N((x * z) * \theta) = F_N((x * z) * (z * z)) \leq F_N((x * z) * z) \end{aligned}$$

for all $x, z \in X$ by using (III) and Equation (1). □

Theorem 5. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is an ideal of X.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Clearly, $\theta \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$ and $y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x * y) \leq \alpha$, $I_N(x * y) \geq \beta$, $F_N(x * y) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (6) that

$$\begin{aligned} T_N(x) &\leq \bigvee\{T_N(x * y), T_N(y)\} \leq \alpha \\ I_N(x) &\geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \beta \\ F_N(x) &\leq \bigvee\{F_N(x * y), F_N(y)\} \leq \gamma \end{aligned}$$

so that $x \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is an ideal of X. □

Theorem 6. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Proof. If there exist $a, b, c \in X$ such that $T_N(\theta) > T_N(a)$, $I_N(\theta) < I_N(b)$ and $F_N(\theta) > F_N(c)$, respectively, then $T_N(\theta) > a_t \geq T_N(a)$, $I_N(\theta) < b_i \leq I_N(b)$ and $F_N(\theta) > c_f \geq F_N(c)$ for some $a_t, c_f \in [-1, 0)$ and $b_i \in (-1, 0]$. Then $\theta \notin T_N^{a_t}$, $\theta \notin I_N^{b_i}$ and $\theta \notin F_N^{c_f}$. This is a contradiction. Hence, $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_N(a_t) > \bigvee\{T_N(a_t * b_t), T_N(b_t)\}$, $I_N(a_i) < \bigwedge\{I_N(a_i * b_i), I_N(b_i)\}$ and $F_N(a_f) > \bigvee\{F_N(a_f * b_f), F_N(b_f)\}$. Then there exist $s_t, s_f \in [-1, 0)$ and $s_i \in (-1, 0]$ such that

$$\begin{aligned}
&T_N(a_t) > s_t \geq \bigvee\{T_N(a_t * b_t), T_N(b_t)\} \\
&I_N(a_i) < s_i \leq \bigwedge\{I_N(a_i * b_i), I_N(b_i)\} \\
&F_N(a_f) > s_f \geq \bigvee\{F_N(a_f * b_f), F_N(b_f)\}
\end{aligned}$$

It follows that $a_t * b_t \in T_N^{s_t}$, $b_t \in T_N^{s_t}$, $a_i * b_i \in I_N^{s_i}$, $b_i \in I_N^{s_i}$, $a_f * b_f \in F_N^{s_f}$ and $b_f \in F_N^{s_f}$. However, $a_t \notin T_N^{s_t}$, $a_i \notin I_N^{s_i}$ and $a_f \notin F_N^{s_f}$. This is a contradiction, and so

$$\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \\
&I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \\
&F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\}
\end{aligned}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X. □

Proposition 5. *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$(\forall x, y, z \in X) \left(x * y \preceq z \;\Rightarrow \begin{cases} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{cases} \right) \tag{10}$$

Proof. Let $x, y, z \in X$ be such that $x * y \preceq z$. Then $(x * y) * z = \theta$, and so

$$\begin{aligned}
&T_N(x * y) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} = \bigvee\{T_N(\theta), T_N(z)\} = T_N(z) \\
&I_N(x * y) \geq \bigwedge\{I_N((x * y) * z), I_N(z)\} = \bigwedge\{I_N(\theta), I_N(z)\} = I_N(z) \\
&F_N(x * y) \leq \bigvee\{F_N((x * y) * z), F_N(z)\} = \bigvee\{F_N(\theta), F_N(z)\} = F_N(z)
\end{aligned}$$

It follows that

$$\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\} \\
&I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\} \\
&F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\}
\end{aligned}$$

This completes the proof. □

Theorem 7. *In a BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-subalgebra.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a BCK-algebra X. For any $x, y \in X$, we have

$$\begin{aligned}T_N(x*y) &\leq \bigvee\{T_N((x*y)*x), T_N(x)\} = \bigvee\{T_N((x*x)*y), T_N(x)\}\\ &= \bigvee\{T_N(\theta*y), T_N(x)\} = \bigvee\{T_N(\theta), T_N(x)\}\\ &\leq \bigvee\{T_N(x), T_N(y)\}\end{aligned}$$

$$\begin{aligned}I_N(x*y) &\geq \bigwedge\{I_N((x*y)*x), I_N(x)\} = \bigwedge\{I_N((x*x)*y), I_N(x)\}\\ &= \bigwedge\{I_N(\theta*y), I_N(x)\} = \bigwedge\{I_N(\theta), I_N(x)\}\\ &\geq \bigwedge\{I_N(y), I_N(x)\}\end{aligned}$$

and

$$\begin{aligned}F_N(x*y) &\leq \bigvee\{F_N((x*y)*x), F_N(x)\} = \bigvee\{F_N((x*x)*y), F_N(x)\}\\ &= \bigvee\{F_N(\theta*y), F_N(x)\} = \bigvee\{F_N(\theta), F_N(x)\}\\ &\leq \bigvee\{F_N(x), F_N(y)\}\end{aligned}$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of a BCK-algebra X. □

The converse of Theorem 7 may not be true in general, as seen in the following example.

Example 4. *Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table.*

$*$	$\boldsymbol{\theta}$	**1**	**2**	**3**	**4**
θ	θ	θ	θ	θ	θ
1	1	θ	θ	θ	θ
2	2	1	θ	1	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X, which is given as follows:

$$X_{\mathbf{N}} = \left\{\frac{\theta}{(-0.8,0,-1)}, \frac{1}{(-0.8,-0.2,-0.9)}, \frac{2}{(-0.2,-0.6,-0.5)}, \frac{3}{(-0.7,-0.4,-0.7)}, \frac{4}{(-0.4,-0.8,-0.3)}\right\}$$

Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, but it is not a neutrosophic $\mathcal{N}$-ideal of X as $T_N(2) = -0.2 > -0.7 = \bigvee\{T_N(2*3), T_N(3)\}$, $I_N(4) = -0.8 < -0.4 = \bigwedge\{I_N(4*3), I_N(3)\}$, or $F_N(4) = -0.3 > -0.7 = \bigvee\{F_N(4*3), F_N(3)\}$.

Theorem 7 is not valid in a BCI-algebra; that is, if X is a BCI-algebra, then there is a neutrosophic $\mathcal{N}$-ideal that is not a neutrosophic $\mathcal{N}$-subalgebra, as seen in the following example.

Example 5. *Consider the neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X in Example 3. If we take $x := (\theta, 0)$ and $y := (\theta, 1)$ in $Y \times (\mathbb{N} \cup \{0\})$, then $x * y = (\theta, 0) * (\theta, 1) = (\theta, -1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence*

$$\begin{aligned}T_N(x*y) &= 0 > \alpha = \bigvee\{T_N(x), T_N(y)\}\\ I_N(x*y) &= \beta < 0 = \bigwedge\{I_N(x), I_N(y)\} \text{ or}\\ F_N(x*y) &= 0 > \gamma = \bigvee\{F_N(x), F_N(y)\}\end{aligned}$$

Therefore $X_{\mathbf{N}}$ is not a neutrosophic $\mathcal{N}$-subalgebra of X.

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$X_{\mathbf{N}}^{\omega_t} := \{x \in X \mid T_N(x) \leq T_N(\omega_t)\}$$
$$X_{\mathbf{N}}^{\omega_i} := \{x \in X \mid I_N(x) \geq I_N(\omega_i)\}$$
$$X_{\mathbf{N}}^{\omega_f} := \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}$$

Clearly, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

Theorem 8. *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X.*

Proof. Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t}$, $\theta \in X_{\mathbf{N}}^{\omega_i}$ and $\theta \in X_{\mathbf{N}}^{\omega_f}$. Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$T_N(x * y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t)$$
$$I_N(x * y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i)$$
$$F_N(x * y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f)$$

It follows from Equation (6) that

$$T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \leq T_N(\omega_t)$$
$$I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \geq I_N(\omega_i)$$
$$F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \leq F_N(\omega_f)$$

Hence $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X. □

Theorem 9. *Let $\omega_t, \omega_i, \omega_f \in X$ and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X. Then*

(1) *If $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X, then the following assertion is valid:*

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(x) \geq \bigvee\{T_N(y * z), T_N(z)\} \ \Rightarrow\ T_N(x) \geq T_N(y) \\ I_N(x) \leq \bigwedge\{I_N(y * z), I_N(z)\} \ \Rightarrow\ I_N(x) \leq I_N(y) \\ F_N(x) \geq \bigvee\{F_N(y * z), F_N(z)\} \ \Rightarrow\ F_N(x) \geq F_N(y) \end{pmatrix} \tag{11}$$

(2) *If $X_{\mathbf{N}}$ satisfies Equation (11) and*

$$(\forall x \in X)\ (T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x), F_N(\theta) \leq F_N(x)) \tag{12}$$

then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X for all $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$.

Proof. (1) Assume that $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X for $\omega_t, \omega_i, \omega_f \in X$. Let $x, y, z \in X$ be such that $T_N(x) \geq \bigvee\{T_N(y * z), T_N(z)\}$, $I_N(x) \leq \bigwedge\{I_N(y * z), I_N(z)\}$ and $F_N(x) \geq \bigvee\{F_N(y * z), F_N(z)\}$. Then $y * z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, where $\omega_t = \omega_i = \omega_f = x$. It follows from (I2) that $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for $\omega_t = \omega_i = \omega_f = x$. Hence $T_N(y) \leq T_N(\omega_t) = T_N(x)$, $I_N(y) \geq I_N(\omega_i) = I_N(x)$ and $F_N(y) \leq F_N(\omega_f) = F_N(x)$.

(2) Let $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$ and suppose that $X_{\mathbf{N}}$ satisfies Equations (11) and (12). Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ by Equation (12). Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$\begin{aligned} &T_N(x*y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t) \\ &I_N(x*y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i) \\ &F_N(x*y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f) \end{aligned}$$

which implies that $\bigvee\{T_N(x*y), T_N(y)\} \leq T_N(\omega_t)$, $\bigwedge\{I_N(x*y), I_N(y)\} \geq I_N(\omega_i)$, and $\bigvee\{F_N(x*y), F_N(y)\} \leq F_N(\omega_f)$. It follows from Equation (11) that $T_N(\omega_t) \geq T_N(x)$, $I_N(\omega_i) \leq I_N(x)$ and $F_N(\omega_f) \geq F_N(x)$. Thus, $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X. □

Definition 3. *A neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X is said to be closed if it is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Example 6. *Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ with the following Cayley table.*

$*$	θ	1	a	b	c
θ	θ	θ	a	b	c
1	1	θ	a	b	c
a	a	a	θ	c	b
b	b	b	c	θ	a
c	c	c	b	a	θ

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given as follows:

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.9,-0.3,-0.8)}, \frac{1}{(-0.7,-0.4,-0.7)}, \frac{a}{(-0.6,-0.8,-0.3)}, \frac{b}{(-0.2,-0.6,-0.3)}, \frac{c}{(-0.2,-0.8,-0.5)} \right\}$$

Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X.

Theorem 10. *Let X be a BCI-algebra, For any $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [-1, 0)$ and $\beta_1, \beta_2 \in (-1, 0]$ with $\alpha_1 < \alpha_2$, $\gamma_1 < \gamma_2$ and $\beta_1 > \beta_2$, let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X given as follows:*

$$T_N : X \to [-1, 0],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in X_+ \\ \alpha_2 & \text{otherwise} \end{cases}$$

$$I_N : X \to [-1, 0],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in X_+ \\ \beta_2 & \text{otherwise} \end{cases}$$

$$F_N : X \to [-1, 0],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in X_+ \\ \gamma_2 & \text{otherwise} \end{cases}$$

where $X_+ = \{x \in X \mid \theta \preceq x\}$. Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X.

Proof. Because $\theta \in X_+$, we have $T_N(\theta) = \alpha_1 \leq T_N(x)$, $I_N(\theta) = \beta_1 \geq I_N(x)$ and $F_N(\theta) = \gamma_1 \leq F_N(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$\begin{aligned} &T_N(x) = \alpha_1 \leq \bigvee\{T_N(x*y), T_N(y)\} \\ &I_N(x) = \beta_1 \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ &F_N(x) = \gamma_1 \leq \bigvee\{F_N(x*y), F_N(y)\} \end{aligned}$$

Suppose that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$, and if $y \in X_+$ then $x * y \notin X_+$. In either case, we have

$$T_N(x) = \alpha_2 = \bigvee\{T_N(x * y), T_N(y)\}$$
$$I_N(x) = \beta_2 = \bigwedge\{I_N(x * y), I_N(y)\}$$
$$F_N(x) = \gamma_2 = \bigvee\{F_N(x * y), F_N(y)\}$$

For any $x, y \in X$, if any one of x and y does not belong to X_+, then

$$T_N(x * y) \leq \alpha_2 = \bigvee\{T_N(x), T_N(y)\}$$
$$I_N(x * y) \geq \beta_2 = \bigwedge\{I_N(x), I_N(y)\}$$
$$F_N(x * y) \leq \gamma_2 = \bigvee\{F_N(x), F_N(y)\}$$

If $x, y \in X_+$, then $x * y \in X_+$. Hence

$$T_N(x * y) = \alpha_1 = \bigvee\{T_N(x), T_N(y)\}$$
$$I_N(x * y) = \beta_1 = \bigwedge\{I_N(x), I_N(y)\}$$
$$F_N(x * y) = \gamma_1 = \bigvee\{F_N(x), F_N(y)\}$$

Therefore $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X. □

Proposition 6. *Every closed neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of a BCI-algebra X satisfies the following condition:*

$$(\forall x \in X)\ (T_N(\theta * x) \leq T_N(x),\ I_N(\theta * x) \geq I_N(x),\ F_N(\theta * x) \leq F_N(x)) \tag{13}$$

Proof. Straightforward. □

We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be closed.

Theorem 11. *Let X be a BCI-algebra. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X that satisfies the condition of Equation (13), then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and hence is a closed neutrosophic $\mathcal{N}$-ideal of X.*

Proof. Note that $(x * y) * x \preceq \theta * y$ for all $x, y \in X$. Using Equations (10) and (13), we have

$$T_N(x * y) \leq \bigvee\{T_N(x), T_N(\theta * y)\} \leq \bigvee\{T_N(x), T_N(y)\}$$
$$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(\theta * y)\} \geq \bigwedge\{I_N(x), I_N(y)\}$$
$$F_N(x * y) \leq \bigvee\{F_N(x), F_N(\theta * y)\} \leq \bigvee\{F_N(x), F_N(y)\}$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and is therefore a closed neutrosophic $\mathcal{N}$-ideal of X. □

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