Structural Optimization by Using the Stiffness Homogenization.

Ri Yongsop[1], Ri Namhyok[1,3*] Ri Choljin[2], Ri Cholsu[1] and Zhihua Chen[3]

1 Department of Mechanics Engineering, Kim Il Sung University, Pyongyang, DPR of Korea

2 Department of Mining Machine, Chongjin Mining and Metal University, Chongjin, DPR of Korea

3 School of Civil Engineering, Tianjin University, Tianjin, PR of China,300072

*Corresponding Author E-mail: rinamhyok@163.com

Abstract The topology optimization using the homogenization method is to find optimal layout of microstructures which satisfies design demand in the neighborhood of an arbitrary point in a given design domain. The homogenization approach is to compute proper mechanical properties of those microstructures in terms of integral averaging by accepting the periodicity assumption and characteristic function.

In this paper, we described a stiffness homogenization theory that is able to avoid the complication of the homogenization approach and reduce the computation cost by considering the topology optimization process in consistence with the finite element analysis. The method can be applied more flexibly to individual or simultaneous optimization of the topology, shape, size and material layout. We discussed a stiffness homogenization during meshing the space structure with different kinds of elements in detail. An optimal criterion for the minimum weight design problem, as one of typical optimization design problems, was derived and the computational algorithm was presented. Numerical results of an example were compared with previous methods, which show the validation of the method.

Keywords homogenization method, topology optimization, stiffness homogenization, structure design

1 Introduction

In the structural optimization design, the homogenization method was first presented from an idea that topology optimization problems of the structure can be treated in relatively popularized way as well as sizing optimization problems in the late 1980's(Bendsoe and Kikuchi 1988).

The homogenization denotes the process that the heterogeneous medium convert to the homogeneous medium of which property is equivalent to its. In problems of the continuum mechanics including solid mechanics, mathematical consideration is performed under the assumption for homogeneity of medium. The most common tool dealing with these problems is to study behavior of medium using differential and integral calculus, and the fundamental technique for mathematical modeling and analysis and research of behavior for continuum is to solve the initial and boundary value problem of the differential equation.

But in the practices, there are many cases of heterogeneous and discrete medium. Solving such problems using mathematical method is commonly impossible. For this reason, various homogenization methods to solve these problems were presented from attempts to use the parameters assumed as continuous variables for converting heterogeneous-discrete medium to homogeneous-continuous one with properties equivalent to its(Zhikov *et al*. 1979; Marcenko and Khruslov 1974; Makhvalov and Panasenko 1990).

In the homogenization method, it is assumed that material at the neighborhood of an arbitrary point in space consists of a set of periodic microstructures i.e. cells including hole. The resulting medium can be described by effective macroscopic material properties which depend on the geometry of the basic cell, and these properties can be computed by invoking the formulas of homogenization theory based on the multiscale method(Pavliotis and Stuart 2008).

The computations of these effective properties play a key role for the topology optimization. That is, the optimal design problem on heterogenous-discrete domain in topological optimization of the continuum structure by the material layout method is converted to the optimal design problem on the homogeneous-continuum domain in the optimal design by the homogenization method

As a result, the topology optimization problem on the homogenized domain can be solved by determining parameters representing the hole's distribution and size using sizing optimization method. In this sense, this optimization method is called homogenization method or in a broad sense, material layout method.

After this research in the topology optimization field, it attained the highest stage of prosperity and this method was applied to the practical problems.

Most of researches for this method are based on the optimality criteria method(Allaire 2002; Rozvany *et al*. 1994; Bendsoe and Sigmund 2003; Hassani and Hinton 1998), and some of them are based on the mathematical programming methods(Allaire 2002;Yang and Chuang 1994).

Moreover, only the topology optimization problems was investigated at first, but gradually the sizing, shape and material layout optimization problems was investigated(Allaire 2002; Bendsoe and Sigmund 2003; Park 1995). Especially, the material layout optimization based on homogenization approach have also been expanded to several fields including a work for determining the structure with the required conductivity by optimally distributing materials with different heat conductivities(Rodriguese and Fernandes 1995), piezocomposite actuator's optimal design(Jonsmann *et al*. 1999; Emilio and Nelli 1998) and so on.

Besides, there were many researches in which the homogenization method was applied on the optimization of dynamic problems(Bendsoe and Sigmund 2003; Diaz and Kikuchi 1992) and several researches for improving the convergence of the optimization process was presented. But the homogenization method has drawbacks requiring a lot of design variables and great computational cost yet.

Bendsoe(1989) had proposed SIMP (Solid Isotropic Material with Penalization), which introduced material distribution density on the design domain of structure in the research of the topology optimization.

The SIMP interpolation scheme addresses the integer format of the original setting for the topology design problem. It converts this integer problem to a sizing problem that finally results in practical 0-1 designs. Another serious problem associated with the 0-1 problem is that SIMP can lead to nonexistence of solution of the problem. This not only is a serious theoretical drawback but also affects on the estimation of sensitivity of computational results to the fineness of the finite element mesh. As above mentioned, the SIMP interpolation scheme does not directly resolve this problem, and further considerations are in place to assure a well-posed distributed design problem that also is benign in terms of mesh fineness of the finite element model(Hsu and Hsu 2005; Borrvall and Petersson 2001).

Colligating the research results so far, we can find that the homogenization method was generalized and popularized as a major method converting the topology optimization problem to a material layout optimization or a sizing optimization problem. Also according to increase of the computation ability of computers, it is applied to the many optimal design problems for practical structures and its application field is wide more and more.

Its application object and field is optimal design of static structure(Allaire 2002; Bendsoe and Sigmund 2003; Hassani and Hinton 1998; Hsu and Hsu 2005), topology optimization of a eigenvalue and frequency response problem(Bendsoe and Sigmund 2003; Diaz and Kikuchi 1992), topology design of structures with a buckling load constraint(Neves *et al*. 1995), topology optimization of thermo-elastic structures(Rodriguese and Fernandes 1995), optimal design of controlled structures(Ou and Kikuchi 1996), Systematic synthesis of micro compliant mechanisms(Ananthasuresh *et al*. 1993; Ananthasuresh *et al*. 1994), optimal design of piezocomposite materials, piezoelectric transducers and electro-thermal micro actuators(Jonsmann *et al*. 1999; Emilio and Nelli 1998) and so on.

In this paper, the topology optimization process using stiffness homogenization theory is discussed, in which the periodic assumption is relaxed and the computation cost is reduced without a complicated discussion as integral average by introduction of the characteristic function by simplifying the homogenization approach. The stiffness homogenization method applies individual and simultaneous optimization of sizing, shape, material layout and topology.

This method is more convenient to apply to the computation procedure and structural optimization practices by using the common methods of the finite element analysis in all consideration and formulating of cell and element.

2 Theory of the stiffness homogenization by using area (or volume) rate of element.

In this paper it is assumed that the structural component of the design domain is meshed with elements, each of elements is made of the microstructure, that is, infinitely small cells with holes and the shape of cell and element is triangle or tetrahedron. Of course, the higher order's element can be used and hole's centroid is placed in the centroid of the cell in which it belongs and hole's shape is geometrically similar to that cell.

2.1 Stiffness homogenization by using first order plane triangle element.

First, let's consider the linear triangle plane element.

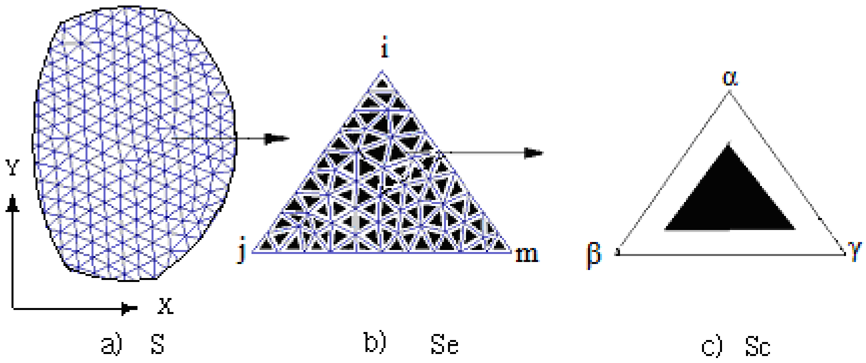


Fig. 1. Geometry model (a) design domain, b) element, c) cell)

In the cells and elements, the hole's size is completely determined with a parameter by using area rate:

$$\varsigma = \frac{S_0}{S} \tag{1}$$

where S is the total area of cell or element and S_0 is area of hole in the cell or the element. The characteristics for the cells is labeled "c" and for the elements is labeled "e".

The stiffness matrix of the cells for the linear element is reduced by the hole rate ζ_c as compared with that without hole and is formulated as following:

$$\overline{\boldsymbol{K}}_c = t_c \int_{S_c(1-\zeta_c)} \boldsymbol{B}_c^{\ T} \boldsymbol{D}_c \boldsymbol{B}_c \ ds = \boldsymbol{K}_c (1-\zeta_c) \tag{2}$$

where

$$\mathbf{K}_c = t_c \int_{S_c} \mathbf{B}_c^{\ T} \mathbf{D}_c \mathbf{B}_c \ ds \ ,$$

$$\mathbf{D}_c = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} ,$$

t_c is thickness of the element(or the cell) and $\mathbf{B}_c$ is strain matrix of the element

The stiffness distribution of the plane triangle element, which is assumed to be made of the infinitely many cells, can be represented in the form interpolated from the stiffnesses per unit area at the nodes.

$$\overline{\overline{\mathbf{K}}}_e = \zeta_{ei}\overline{\mathbf{K}}_{ci} + \zeta_{ej}\overline{\mathbf{K}}_{cj} + \zeta_{em}\overline{\mathbf{K}}_{cm} \tag{3}$$

where i, j and m are node numbers of the element respectively, $\zeta_{e\alpha}(\alpha = \mathrm{i, j, m})$ is the shape function represented by the area coordinates, and $\overline{\mathbf{K}}_{c\alpha}(\alpha = \mathrm{i, j, m})$ is assumed to be continuous in the element.

The stiffness matrix of the plane triangle element is;

$$\overline{\mathbf{K}}_e = \int_{Se} \overline{\overline{\mathbf{K}}}_e ds = \frac{S_e}{3}\left[\mathbf{K}_{ci}(1-\zeta_{ci}) + \mathbf{K}_{cj}(1-\zeta_{cj}) + \mathbf{K}_{cm}(1-\zeta_{cm})\right] \tag{4}$$

That is;

$$\overline{\mathbf{K}}_e = \mathbf{K}_c S_e (1\text{-}\frac{\zeta_{ci} + \zeta_{cj} + \zeta_{cm}}{3})$$

where $\mathbf{K}_c$ is the stiffness of cells without voids, $\mathbf{K}_c S_e$ is the stiffness matrix of the element made of the cells without hole and $(1\text{-}\frac{\zeta_{ci} + \zeta_{cj} + \zeta_{cm}}{3})$ is average area rate. Accordingly, the stiffness of the triangle element is represented as following

$$\overline{\mathbf{K}}_e = \mathbf{K}_e(1-\zeta_e) \tag{5}$$

$$\mathbf{K}_e = \mathbf{K}_c S_e , \quad \zeta_e = \frac{\zeta_{ci} + \zeta_{cj} + \zeta_{cm}}{3}$$

That is, the element's stiffness is reduced by the hole's rate ζ_e as compared with that without hole in cells $\boldsymbol{K}_e$.

Considering $\mathbf{K}_e = t_e \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e S_e$, then we can rewrite as following equations:

$$\overline{\boldsymbol{K}}_e = [t_e(1-\varsigma_e)]\boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e S_e \tag{6}$$

$$\overline{\boldsymbol{K}}_e = t_e \boldsymbol{B}_e^T [\boldsymbol{D}_e(1-\varsigma_e)]\boldsymbol{B}_e S_e \tag{7}$$

$$\overline{\boldsymbol{K}}_e = t_e \boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e [S_e(1-\varsigma_e)] \tag{8}$$

$$\overline{\mathbf{K}}_e = \mathbf{K}_e(1-\varsigma_e)$$

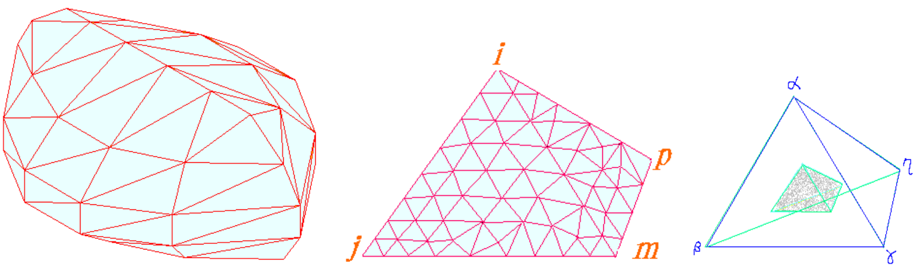
From Equation (6)-(7), it is found that this problem can be considered as the homogenization of the thickness, material constant, area and entire stiffness and in the future, sizing, shape, topology and material layout optimization can be individually or simultaneous performed.

2.2 Stiffness homogenization by using tetrahedron element.

In the space problem, meshing with tetrahedron element,

$$\overline{\mathbf{K}}_c = \int_{V_c(1-\zeta_c)} \mathbf{B}_c{}^T \mathbf{D}_c \mathbf{B}_c \; dv = \mathbf{K}_c(1-\zeta_c) \tag{9}$$

where $\mathbf{B}_c, \mathbf{D}_c$ are strain and elastic matrix of the space problem, respectively.



a) V b) v_e c) V_C

Fig. 2. geometry model(a.design domain, b.element ,c.cell)

By using above similar method

$$\overline{\mathbf{K}}_e = \mathbf{K}_e(1-\zeta_e) \tag{10}$$

where $\mathbf{K}_e = \mathbf{K}_c V_e$ and $\zeta_e = \dfrac{\zeta_{ci} + \zeta_{cj} + \zeta_{cm} + \zeta_{c\rho}}{4}$

From Equation (10) we can also obtain:

$$\overline{\boldsymbol{K}}_e = \boldsymbol{B}_e^T[\boldsymbol{D}_e(1-\varsigma_e)]\boldsymbol{B}_e V_e \tag{11}$$

$$\overline{\boldsymbol{K}}_e = \boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e[V_e(1-\varsigma_e)] \tag{12}$$

$$\overline{\mathbf{K}}_e = \mathbf{K}_e(1-\varsigma_e)$$

In deriving above equations, we used following the integral formulas in area and volume coordinates on triangle and tetrahedron domain.

$$\int_{S_e} \varsigma_i^a \varsigma_j^b \varsigma_m^c ds = 2S_e \frac{a!b!c!}{(a+b+c+2)!},$$

$$\int_{V_e} \varsigma_i^a \varsigma_j^b \varsigma_m^c \varsigma_p^d ds = 6V_e \frac{a!b!c!d!}{(a+b+c+d+3)!}$$

2.3 The stiffness homogenization for the anisotropic material.

If the structure is made of the orthotropic material and we consider the orthotropic property in the stiffness homogenization method, we discuss the triangle of cells and elements at the local coordinates of which axes coincide with the orthotropic axes.

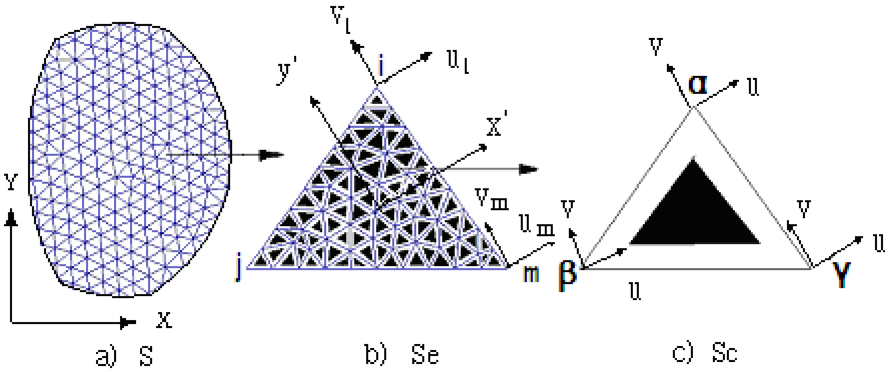


Fig. 3. Geometrical model for orthotropic material
(a.design domain, b.element ,c.cell)

The stiffness matrix of cells is derived at the local coordinate system as following;

$$\hat{\mathbf{K}}_c = t_c \int_{S_c(1-\zeta_c)} \overline{\mathbf{B}}_c{}^T \overline{\mathbf{D}}_c \overline{\mathbf{B}}_c \ ds = \tilde{\mathbf{K}}_c(1-\zeta_c) \tag{13}$$

where $\tilde{\mathbf{K}}_c$ is the cell's stiffness matrix without void in the local coordinate system, $\overline{\mathbf{B}}_c$ is the strain matrix at the local coordinate system and the orthotropic elastic matrix at the local coordinate system $\overline{\boldsymbol{D}}_c$ is:

$$\overline{\boldsymbol{D}}_c = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & 0 \\ \overline{D}_{21} & \overline{D}_{22} & 0 \\ 0 & 0 & \overline{D}_{33} \end{bmatrix} .$$

In the triangle element full of such cells, if the element size is enough small, the local coordinate axes of the inner cells are equal to each other.

Assuming that the stiffness distribution of the element is interpolated by the stiffnesses of cells at nodes as

$$\overline{\hat{\mathbf{K}}}_e = \zeta_{ei}\hat{\mathbf{K}}_{ci} + \zeta_{ej}\hat{\mathbf{K}}_{cj} + \zeta_{em}\hat{\mathbf{K}}_{cm}$$

,the stiffness matrix of the triangular element is;

$$\hat{\mathbf{K}}_e = \int_{Se} \overline{\hat{\mathbf{K}}}_e ds = \frac{S_e}{3}\left[\tilde{\mathbf{K}}_{ci}(1-\zeta_{ci}) + \tilde{\mathbf{K}}_{cj}(1-\zeta_{cj}) + \tilde{\mathbf{K}}_{cm}(1-\zeta_{cm})\right]$$

$$= \tilde{\mathbf{K}}_c S_e (1 - \frac{\zeta_{ci} + \zeta_{cj} + \zeta_{cm}}{3}) \ . \tag{14}$$

Considering $\tilde{\mathbf{K}}_c S_e = \tilde{\mathbf{K}}_e$ and $\frac{\zeta_{ci} + \zeta_{cj} + \zeta_{cm}}{3} = \zeta_e$, then the homogenized stiffness matrix of element at the local coordinate system is represented as following;.

$$\hat{\mathbf{K}}_e = \tilde{\mathbf{K}}_e (1 - \zeta_e)$$

That is, the element's stiffness is reduced by the cell void's rate ζ_e as compared with that without holes.

If the angle between local coordinate's x' axis and global coordinate's x axis is θ_e,

$$\mathbf{K}_e = t_e \mathbf{B}_e^T \mathbf{T}_e \overline{\mathbf{D}}_e \mathbf{T}_e^T \mathbf{B}_e S_e = t_e \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e S_e$$

Therefore the homogenized element stiffness matrix at the global coordinate system is;

$$\overline{\mathbf{K}}_e = \mathbf{K}_e (1 - \zeta_e)$$

where the transformation matrix of coordinate is $\mathbf{T} = \begin{bmatrix} \gamma_e & \beta_e & -2\beta_e\gamma_e \\ \beta_e & \gamma_e & 2\beta_e\gamma_e \\ \beta_e\gamma_e & -\beta_e\gamma_e & \gamma_e^2 - \beta_e^2 \end{bmatrix}$,

$\beta_e = \sin\theta_e$, $\gamma_e = \cos\theta_e$, and $\mathbf{B}_e$ is the strain matrix at the global coordinate system.

That is, replacing the elastic matrix $\overline{\boldsymbol{D}}_e$ and the strain matrix $\overline{\mathbf{B}}_e$ in the homogenized element's stiffness matrixes at the local coordinate system with $\mathbf{D}_e$ and $\mathbf{B}_e$, we obtain the homogenized element's stiffness matrixes at the global coordinate system.

2.4 Stiffness homogenization by using high order element.

In the finite element analysis by using high order element, the stiffness homogenization method can be also used. Using above method, we can find that all homogenized stiffness matrices for high order elements have the form as

$$\overline{\mathbf{K}}_e = \mathbf{K}_e(1-\varsigma_e)$$

At this time, we represented the stiffness distributions of different high order elements by the corresponding shape functions in the area or volume coordinates and used integral formula in the area or volume coordinate system.
For example, shape functions for the second order plane triangle element are presented by the area coordinates $\varsigma_1, \varsigma_2, \varsigma_3$ as

$$N_1 = (2\varsigma_1 - 1)\varsigma_1, \qquad N_2 = (2\varsigma_2 - 1)\varsigma_2$$

$$N_3 = (2\varsigma_3 - 1)\varsigma_3, \qquad N_4 = 4\varsigma_1\varsigma_2$$

$$N_5 = 4\varsigma_2\varsigma_3, \qquad N_6 = 4\varsigma_3\varsigma_1.$$

Thus;

$$\varsigma_e = \frac{1}{6}\left[0(\varsigma_{c1} + \varsigma_{c2} + \varsigma_{c3}) + 2(\varsigma_{c4} + \varsigma_{c5} + \varsigma_{c6})\right] \tag{15}$$

And for the third order plane triangle element the shape function is as below

$$N_1 = \frac{1}{2}\varsigma_1(3\varsigma_1 - 1)(3\varsigma_1 - 2), \quad N_2 = \frac{1}{2}\varsigma_2(3\varsigma_2 - 1)(3\varsigma_2 - 2)$$

$$N_3 = \frac{1}{2}\varsigma_3(3\varsigma_3 - 1)(3\varsigma_3 - 2), \quad N_4 = \frac{9}{2}\varsigma_2\varsigma_3(3\varsigma_2 - 1)$$

$$N_5 = \frac{9}{2}\varsigma_2\varsigma_3(3\varsigma_3 - 1), \qquad N_6 = \frac{9}{2}\varsigma_3\varsigma_1(3\varsigma_3 - 1)$$

$$N_7 = \frac{9}{2}\varsigma_3\varsigma_1(3\varsigma_1 - 1), \qquad N_8 = \frac{9}{2}\varsigma_1\varsigma_2(3\varsigma_1 - 1)$$

$$N_9 = \frac{9}{2}\varsigma_1\varsigma_2(3\varsigma_2 - 1), \qquad N_{10} = 27\varsigma_1\varsigma_2\varsigma_3$$

Therefore

$$\varsigma_e = \frac{1}{10}\left[1(\varsigma_{c1} + \varsigma_{c2} + \varsigma_{c3}) + \frac{3}{4}(\varsigma_{c4} + \varsigma_{c5} + \varsigma_{c6} + \varsigma_{c7} + \varsigma_{c8} + \varsigma_{c9}) + \frac{9}{2}\varsigma_{c10}\right] \tag{16}$$

And for second order tetrahedron element;

$$N_1 = \varsigma_1(2\varsigma_1 - 1), \qquad N_2 = \varsigma_2(2\varsigma_2 - 1),$$

$$N_3 = \varsigma_3(2\varsigma_3 - 1), \qquad N_4 = \varsigma_4(2\varsigma_4 - 1),$$

$$N_5 = 4\varsigma_2\varsigma_3, \quad N_6 = 4\varsigma_3\varsigma_1, \quad N_7 = 4\varsigma_1\varsigma_2$$
$$N_8 = 4\varsigma_1\varsigma_4, \quad N_9 = 4\varsigma_2\varsigma_4, \quad N_{10} = 4\varsigma_3\varsigma_4$$

Therefore

$$\varsigma_e = \frac{1}{10}\left[-\frac{1}{2}(\varsigma_{c1}+\varsigma_{c2}+\varsigma_{c3}+\varsigma_{c4}) + 2(\varsigma_{c5}+\varsigma_{c6}+\varsigma_{c7}+\varsigma_{c8}+\varsigma_{c9}+\varsigma_{c10})\right] \tag{17}$$

From above all, we can find that the average of the void's rate of cell multiplied by some weights for cells at the vertices, nodes on edges and inner nodes of element is corresponding to the stiffness reduction of element's ς_e.

If ς_{ci} for every cells is the same as ς_c, then $\varsigma_e = \varsigma_c$.

The weight coefficient is the same as in the following table.

	order	node	at the edge	inner
triangle	first	1		
	second	0	2	
	third	1	3/4	9/2
tetrahedron	first	1		
	second	-1/2	2	

In this way, by using the area or volume rate corresponding to the holes' size, we can obtain the homogenized stiffness matrix by using the interpolation's method of the finite element method and can easily solve the optimal design problem.

3 Problem formulation for optimal design and optimal criterion.

Now, we shall formulate the minimum volume design problem by using the stiffness homogenization method and discuss its solving method.

3.1 Optimization Model

The minimum volume design problem is to find the structure with minimum volume(or weight) under the restriction of the global stiffness. That is, the problem is to find ζ_e satisfying the following equations.

$$\sum_{e=1}^{n} \mathrm{V}_e(1-\zeta_e) = \sum_{e=1}^{n} t_e(1-\zeta_e)S_e \Rightarrow \min \tag{18}$$

$$\mathrm{U} = \tfrac{1}{2}\mathbf{u}^T\overline{\mathbf{K}}\mathbf{u} \le \mathrm{U}_0 \left(\mathrm{U}^{-1} = \left(\tfrac{1}{2}\mathbf{u}^T\overline{\mathbf{K}}\mathbf{u}\right)^{-1} \ge \overline{\mathrm{U}}_0\right) \tag{19}$$

$$\overline{\mathbf{K}}\mathbf{u} = \mathbf{P} \tag{20}$$

where $$\overline{\mathbf{K}} = \sum_{e=1}^{n} \overline{\mathbf{K}}_e = \sum_{e=1}^{n} \mathbf{K}_e (1-\zeta_e)$$

Unlike in the conventional optimal design problems, all the sizing, shape, topology and material layout optimization problems in the stiffness homogenization method is reduced to the problem which is finding optimal design parameter ζ_e which is the area rate(volume rate) characterizing hole's size of cells. In the above problems, V_0, U_0: the given volume and strain energy characterizing global stiffness in the structure respectively. And the equality restriction $\overline{\mathbf{K}}\mathbf{u} = \mathbf{P}$ is the stiffness equation, namely, the restriction of the state equation. Also, for convenience, we assume that $\overline{\mathbf{K}}_e$ is the element's stiffness matrix which is expanded in accordance with the dimension of the structure's stiffness matrix $\overline{\mathbf{K}}$.

3.2 Optimality criterion

Let's derive the problem's optimal criterion;

Lagrange's equation is constructed as

$$\mathrm{L}(\zeta_e, \mathbf{u}) = \sum_{e=1}^{n} \mathrm{V}_e (1-\zeta_e) + \lambda\left[\tfrac{1}{2}\mathbf{u}^T \overline{\mathbf{K}}\mathbf{u} - \mathrm{U}_0\right] + \boldsymbol{\mu}^T\left(\overline{\mathbf{K}}\mathbf{u} - \mathbf{P}\right) \tag{21}$$

where λ, $\boldsymbol{\mu}^T$ are Lagrange's uncertain multipliers.

From a necessary condition for the optimal solution of the nonlinear programming problem, the structure's state at the optimal point, namely, optimization criterion is derived.

Using the condition at the optimal point $\dfrac{\partial \mathrm{L}}{\partial \mathbf{u}} = 0$,then

$$\lambda \mathbf{u}^T \overline{\mathbf{K}} + \boldsymbol{\mu}^T \overline{\mathbf{K}} = 0$$

Therefore

$$\boldsymbol{\mu}^T = -\lambda \mathbf{u}^T \tag{22}$$

And using the condition at the optimal point $\dfrac{\partial \mathrm{L}}{\partial \zeta_e} = 0$ (e=1,2,…,n), then

$$-\mathrm{V}_e + \tfrac{1}{2}\lambda \mathbf{u}^T \frac{\partial \overline{\mathbf{K}}}{\partial \zeta_e}\mathbf{u} + \boldsymbol{\mu}^T \frac{\partial \overline{\mathbf{K}}}{\partial \zeta_e}\mathbf{u} + \lambda \mathbf{u}^T \overline{\mathbf{K}} \frac{\partial \mathbf{u}}{\partial \zeta_e} + \boldsymbol{\mu}^T \overline{\mathbf{K}} \frac{\partial \mathbf{u}}{\partial \zeta_e} = 0 \tag{23}$$

$$-V_e + \lambda \, \mathrm{U}_e = 0 \qquad \text{(e=1,2,…,n)} \tag{24}$$

Here, considering $\overline{\mathbf{K}}_e = \mathbf{K}_e(1-\zeta_e)$, then the equation

$$\mathrm{U}_e = -\tfrac{1}{2}\mathbf{u}^T \frac{\partial \overline{\mathbf{K}}}{\partial \zeta_e}\mathbf{u} = -\tfrac{1}{2}\mathbf{u}^T \frac{\partial \sum_{e=1}^{n} \overline{\mathbf{K}}_e}{\partial \zeta_e}\mathbf{u}$$

becomes to $\mathrm{U}_e = \tfrac{1}{2}\mathbf{u}^T \mathbf{K}_e \mathbf{u}$, which presents the element's strain energy when the element has no hole.

Summing up the two sides of Equation (24) for all elements repectively,

$$-\sum_{e=1}^{n} \mathrm{V}_e + \lambda \sum_{e=1}^{n} \mathrm{U}_e = 0 \qquad \text{(e=1, 2,…, n)}$$

Multiplying $(1-\zeta_e)$ to the two sides of the above equation and Equation (24), eliminating uncertain multiplier λ and considering $\lambda \neq 0$ from the condition of the optimal point;

$$\lambda\left[\tfrac{1}{2}\mathbf{u}^T \overline{\mathbf{K}}\mathbf{u} - \mathrm{U}_0\right] = 0$$

,then we obtain the equalization criterion about the density of the strain energy at the optimal point as

$$\frac{\mathrm{V}_e(1-\varsigma_e)}{\mathrm{U}_e(1-\varsigma_e)} = \frac{\sum_{e=1}^{n} \mathrm{V}_e(1-\varsigma_e)}{\mathrm{U}_0} \quad \text{(e=1,2,…,n)} \tag{25}$$

where $\overline{\mathrm{U}}_e$ is element's strain energy obtained by the finite element analysis.

From Equation (25) we can find that the structure of the optimal size, shape, topology and material layout is the structure in which element's area (volume) rate is determined so that density of the element's strain energy is equalized to the density of the structure's global strain energy.

4 Algorithm

The optimality criterion can be rewritten as

$$\mathrm{V}_e(1-\varsigma_e) = \frac{\mathrm{U}_e(1-\varsigma_e)\sum_{e=1}^{n} \mathrm{V}_e(1-\varsigma_e)}{\mathrm{U}_0} \qquad \text{(e=1, 2,…, n)}$$

$$\varsigma_e = 1 - \frac{\mathrm{U}_e(1-\varsigma_e)\sum_{e=1}^{n}\mathrm{V}_e(1-\varsigma_e)}{\mathrm{V}_e\mathrm{U}_0} \quad \text{(e=1, 2,…, n)} \qquad (26)$$

Our purpose is to get solution, ς_e^* of the nonlinear algebraic simultaneous Equation (26).

Equation (26) is calculated by using the simple iteration schema as

$$\varsigma_e^{(k+1)} = 1 - \frac{\mathrm{U}_e^{(k)}(1-\varsigma_e^{(k)})\sum_{e=1}^{n}\mathrm{V}_e(1-\varsigma_e^{(k)})}{\mathrm{V}_e\mathrm{U}_0} \text{ (e=1, 2 … n) (k=0, 1, 2 …)} \quad (27)$$

The algorithm is as following:

1. Initialize the design parameters $\zeta_e^{(k)}$ (e=1~n) for k = 0(in general cases, take ζ_e =0 everywhere in the domain). Compute $\mathbf{u}^{(k)}$, $\mathrm{V}_e(1-\varsigma_e^{(k)})$ and $\sum_{e=1}^{n}\mathrm{U}_e^{(k)}(1-\varsigma_e^{(k)})$ by solving the state equation, $\sum_{e=1}^{n}\mathbf{K}_e(1-\varsigma_e^{(k)})\mathbf{u} = \mathbf{P}$.

2. Compute $\boldsymbol{\zeta}^{(k+1)}$ using the iteration schema (27).

3. Repeat the iteration loop for k = k+1, until the convergence is achived. Here, the convergence condition is

$$\left|\sum_{e=1}^{n}\mathrm{V}_e(1-\zeta^{(k+1)}) - \sum_{e=1}^{n}\mathrm{V}_e(1-\zeta^{(k)})\right| \le \varepsilon_1,$$

$$\text{or} \qquad \left|(1-\zeta_e^{(k+1)}) - (1-\zeta_e^{(k)})\right| \le \varepsilon_2 \quad \text{(e=1~n)}$$

The design parameters $\boldsymbol{\zeta}^{(k+1)} = \boldsymbol{\zeta}^*$ (e=1~n) which satisfy the above condition are optimal solutions.

Figure 4 shows the above algorithm.

Obtaining optimal design parameter ς_e^* (e=1, 2 … n), then the optimal structures about every elements can be obtain by using followings;

Size and topology optimization; $\qquad t_e^* = t_e\left(1-\zeta_e^*\right) \qquad (28)$

Shape optimization; $$S_e^* = S_e\left(1-\zeta_e^*\right) \tag{29}$$

Material layout optimization; $$\boldsymbol{D}_{\boldsymbol{e}}^* = \boldsymbol{D}_{\boldsymbol{e}}\left(1-\zeta_{\boldsymbol{e}}^*\right) \tag{30}$$

If Poisson's ratio’s change is neglected, instead of Equation (30),

$$E_e^* = E_e\left(1-\zeta_e^*\right) \tag{31}$$

Simultaneous optimization of the size, shape and topology;

$$V_e^* = V_e\left(1-\zeta_e^*\right) \tag{32}$$

Simultaneous optimization including material layout;

$$\mathbf{K}_e^* = \mathbf{K}_e\left(1-\zeta_e^*\right) \tag{33}$$

Limits of the individual parameters are considered in the computation process. When the simultaneous optimization is performed, the parameter with stronger limit is first determined or the parameters are determined in accordance with weights specified by designer, considering their importance.

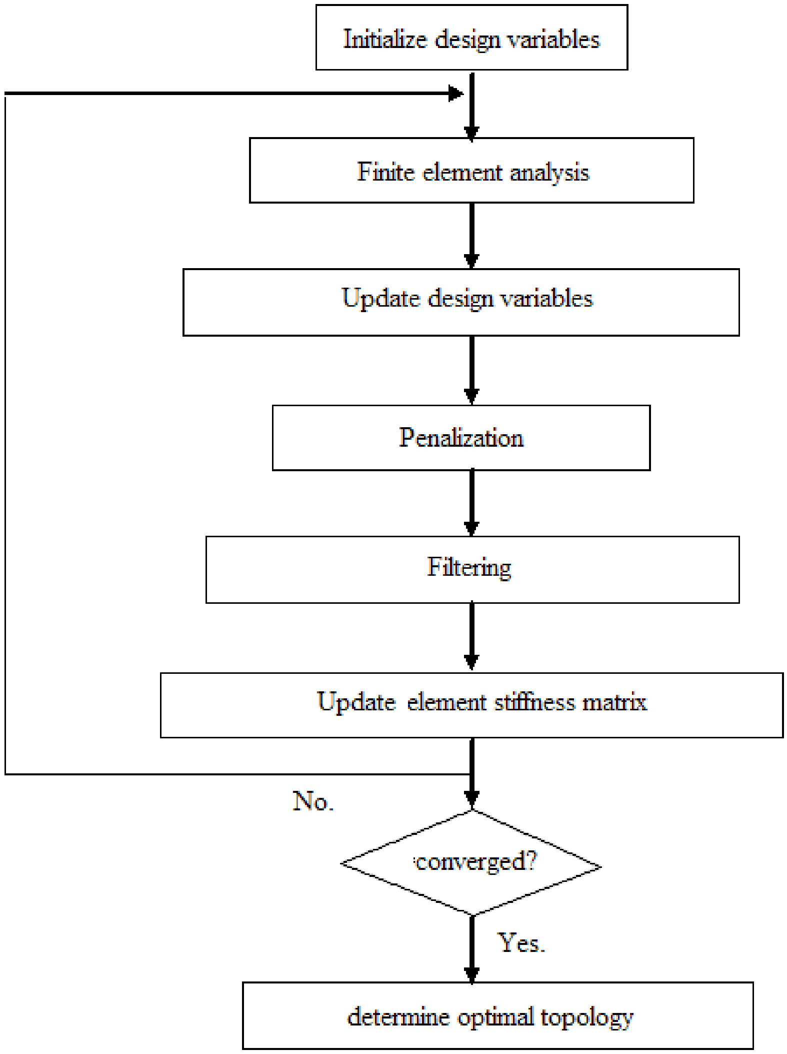
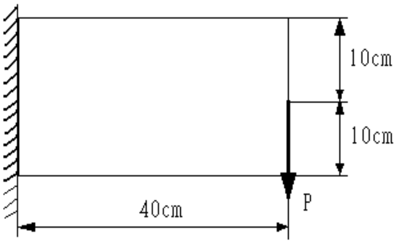


Fig. 4. Optimization algorithm

5 Examples

Consider a thin beam in a plane stress state(the out-of-plane displacement is restrained), which is a standard testing problem(Allaire 2002) for verifying the optimal design method above discussed.



Thickness 2cm, **P**=1 kN

$E_1 = 2.1 \cdot 10^{11} Pa$, $E_2 = 2.5 \cdot 10^{10} Pa$

U_0=0.46E-02 Nm

Fig. 5. The boundary condition about the plate beam problem

The structure is meshed by 1600 linear plain triangle elements.

According to the algorithm as referred above, first ζ_e^* is found and one or several limits is considered when all final parameters is determined.

Figure 6 shows the stress state of the initial structure.

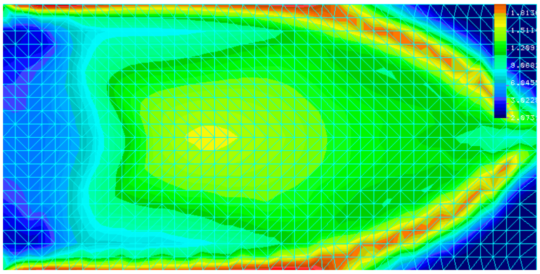


Fig. 6. The stress state of the initial structure.

Figure 7 shows that the checkerboard pattern is generated in the optimization processing. At this time in the repeat process non-converged or impossible structure is obtained. To overcome problems of the generation of the grey domain, checkerboard phenomenon and so on, the previous methods(Allaire 2002; Hsu and Hsu 2005) such as averaging, filtering and penalizing are used.

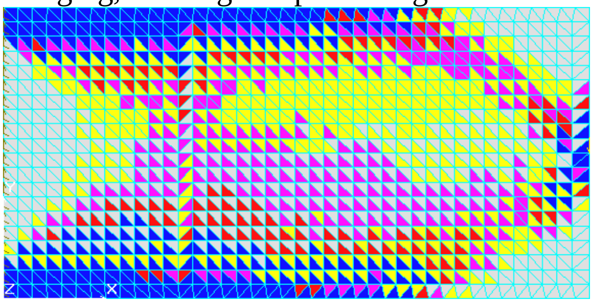


Fig. 7. checkerboard pattern.

The obtained results is shown in Table1 together with pictures. First and second pictures in the table show the shape optimization result at the different limits and third picture shows the topology optimization result.

Forth picture is the calculation result of the simultaneous size and shape optimization under the maximum thickness limit and fifth is the macroscopic material layout optimization using two kind of materials. The sixth shows the simultaneous optimization result of the material layout and topology.

As shown in this example of plane problem, the purposed optimal structure is obtained with 20% of the repeat number and much less computation cost as compared with the previous method. Table 2 shows the three dimensional topology optimization results as compared with the previous homogenization method's results. The compution example is represented as compared with the same objects as in literature(Hsu and Hsu 2005).

The method's validity for the practical problems of the several structures is verified through the results of the three dimensional topology optimization based on the stiffness homogenization method.

Table1. The calculation result of 2-D problem.

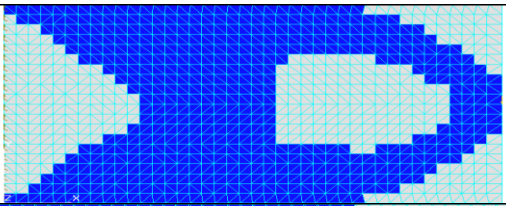
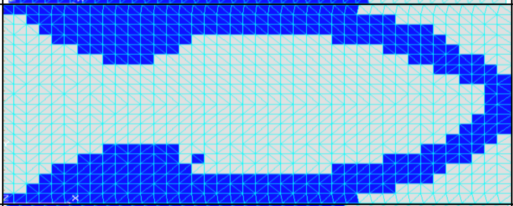
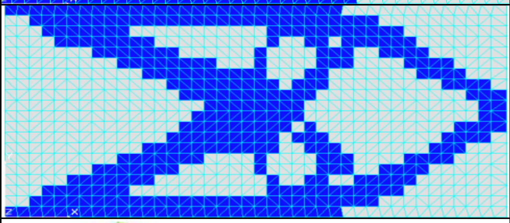
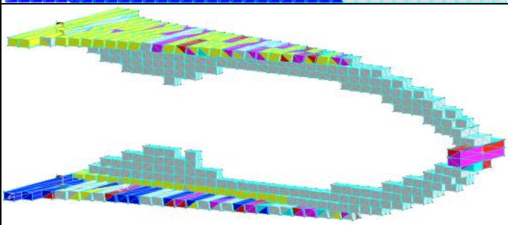
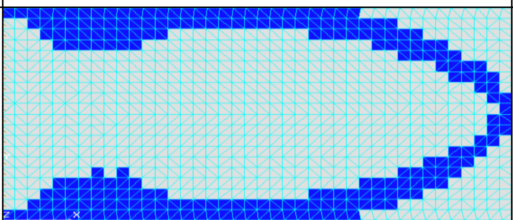
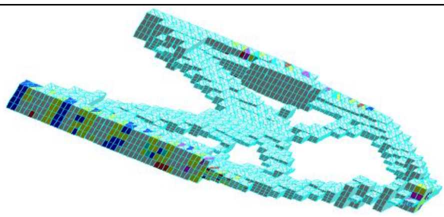
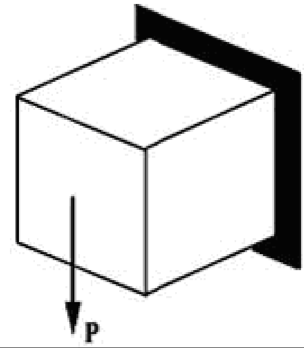
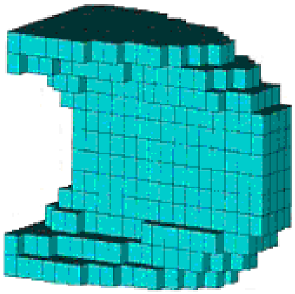
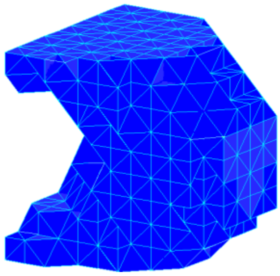
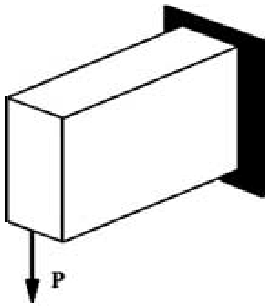
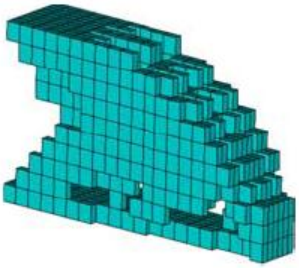
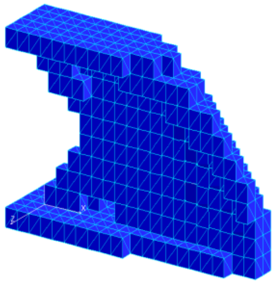
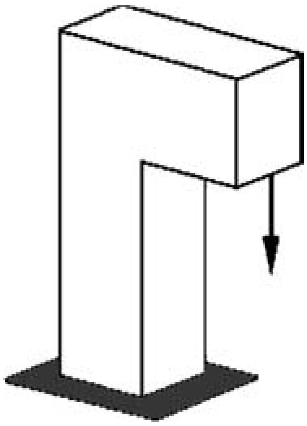
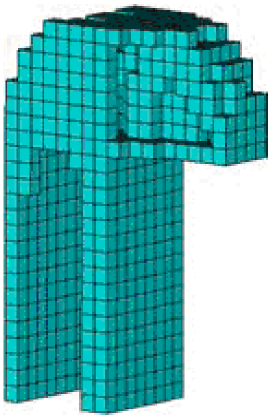
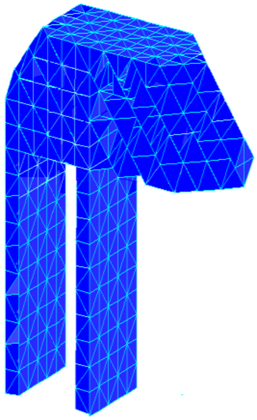
	Optimal structure	optimization initial	Homogenization Method(Allaire 2002)	Etc.
1		0.588	iteration 21 (------)	ζ_e limit 0.9
2		0.443	iteration 21 (------)	ζ_e limit 0.8
3		0.34	iteration 21 (100)	
4		0.413	iteration 25 (------)	$t_{\max}$ = 5.0cm
5		------	iteration 30 (------)	E1, E2
6		------	iteration 22 (------)	out edge: $4 \cdot 10^{11}$ Pa in: $2 \cdot 10^{11}$ Pa

Table2. 3-D topology optimization results

example	homogenization method(Hsu and Hsu 2005)	stiffness homogenization method
		
		
		

6 Conclusions

As above-mentioned, the stiffness homogenization method proposed for structural optimization is a kind of material layout method, in which the components of the space structure are corresponding to each element and the stiffness matrix is estimated according to the microstructure hole's rate.

The structure optimization by the stiffness homogenization method makes it possible to use the theory and method of the finite element method in all the processes of homogenization, structure analysis and optimization in a unified way.

The derivation of the optimality criterion for minimum volume design problem and numerical example of the two and three dimensional problems shows that the theory for the structural optimization by the stiffness homogenization is easily extend to the different kinds of optimization problems and practices.

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