Analysis of the Matrix $X_{jk} = [x_{jk}] \in C$ where $x_{jk} = x(j,k) = \delta + \omega(\alpha + \beta j)^{\varphi k}$

Pedro Caceres

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1. Abstract

The function $x(j,k) = x_{jk} = \delta + \omega(\alpha + \beta j)^{\varphi k}$ in $C \to C$ is a generalization of the power function $y(\alpha) = \alpha^k$ in $R \to R$ and the exponential function $y(k) = \alpha^k$ in $R \to R$. In this paper we are going to calculate the values of infinite and partial sums and products involving elements of the matrix $X_{jk} = [x_{jk}] \in C$

$$X_{jk} = [\delta + w(\alpha + \beta j)^{\varphi k}] = \begin{bmatrix} [\delta + w(\alpha+\beta)^{\varphi}] & [\delta + w(\alpha+\beta)^{2\varphi}] & \dots & [\delta + w(\alpha+\beta)^{m\varphi}] \\ [\delta + w(\alpha+2\beta)^{\varphi}] & [\delta + w(\alpha+2\beta)^{2\varphi}] & \dots & [\delta + w(\alpha+2\beta)^{m\varphi}] \\ \vdots & \vdots & & \vdots \\ [\delta + w(\alpha+n\beta)^{\varphi}] & [\delta + w(\alpha+n\beta)^{2\varphi}] & \dots & [\delta + w(\alpha+n\beta)^{m\varphi}] \end{bmatrix}$$

As a result, several new representations will be made for some infinite series, including the Riemann Zeta Function in C.

2. Definition of C-Transformation

The C-Transformation (Caceres Transformation) of a function $f(k): R \to R$ is given by:

$$C\{f(x)\} = \lim_{m\to\infty} \left(\sum_{k=1}^{m} f(k) - \int f(m)\, dm \right)$$

We will call C-values of a function f(x) the results of the C-transformation applied to the function f(x).

3. Functions used in the paper

3.1. Riemann Zeta function $\zeta(k)$:

$$\zeta(k) = \sum_{j=1}^{\infty} j^{-k} \quad \text{converges for } k \neq 1$$

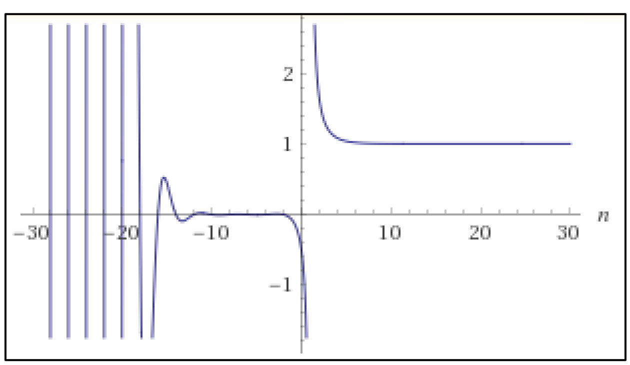

Figure 1. Riemann Zeta function in R

This is one of the best studied functions in mathematics. Some of the most important connections of the Riemann Zeta function and Number Theory are:

a. Euler Product Formula that ties $\zeta(k)$ with the distribution of prime numbers

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \prod_{p\ \mathrm{prime}} \frac{1}{1-p^{-s}}$$

Example for k=2

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{1}{1-2^{-2}}x\frac{1}{1-3^{-2}}x\frac{1}{1-5^{-2}}x\frac{1}{1-7^{-2}}x \ldots$$

b. Integral definition:

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{1}{e^x-1} x^s \frac{dx}{x}$$

Where $\Gamma(s)$ is the Gamma function (described in 4.4)

c. Analytical continuation for:

Re(s)>0: [Dirichlet]

$$\zeta(s) = \frac{1}{s-1}\sum_{k=1}^{\infty} (\frac{n}{(n+1)^s} - \frac{n-s}{n\hat{}s}$$

0<Re(s)<1:

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}$$

-k<Re(s) [Kopp, Konrad. 1945]:

$$\zeta(s) = \frac{1}{s-1}\sum_{k=1}^{\infty}\frac{k(k+1)}{2}(\frac{2k+3+s)}{(k+1)^{s+2}} - \frac{2k-1-s}{k^{s+2}})$$

d. Laurent series at s=1:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=}^{\infty}\frac{(-1)^k\gamma_n}{k!}(s-1)^k)$$

where γ_n are the Stieltjes constants, defined in 4.3.

e. Caceres approximation for ζ(s) [Caceres, Pedro. 2017]:

$$\zeta(s)^{approx} = \frac{1}{1-\pi^{-s}-2^{-s}}$$

	s=3	s=4	s=10	s=14
ζ(s)Actual	1.20206	1.0823	1.000994	1.0000612
ζ(s)Approx	1.18659	1.0784	1.000988	1.0000611

Table 1

3.2. Hurwitz function ζ(k, z):

$$\zeta(k,z) = \sum_{j=0}^{\infty}(j+z)^{-k} = \sum_{j=z}^{\infty} j^{-k} \qquad \text{converges for } k > 1$$

3.3. Generalized Harmonic Function $H_n^{(k)}$:

$$H_n^{(k)} = \sum_{j=1}^{n} j^{-k} = \left(\frac{1}{1^k} + \frac{1}{2^k} + \cdots + \frac{1}{n^k}\right) \quad \text{converges for } k > 1$$

3.4. Gamma Function $\Gamma(n+1)$:

$$\Gamma(n+1) = n!$$

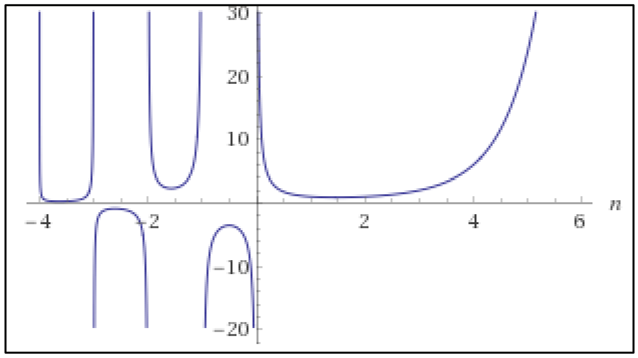

Figure 2. Gamma function in R

$$\Gamma(n) = \frac{n!}{n}$$

$$\Gamma\left(1 + \frac{1}{n}\right) = \frac{1}{n}!\,n$$

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi)}$$

If z is positive even integer, then:

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

Integral representation:

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^z \frac{dx}{x}$$

The derivative of the Gamma function can be given by:

$$\Gamma'(n+1) = n!\,(-\gamma + \sum_{k=1}^{n}\frac{1}{k})$$

Where γ, is the Euler-Mascheroni constant.

3.5. K function $K(n)$:

$$K(n) = \prod_{j=1}^{n-1} j^j = 1^1 \times 2^2 \times \ldots \times 1(n-1)^{(n-1)}$$

3.6. Hyperfactorial function H_n:

$H_n = K(n+1)$

3.7. Digamma function $\psi^{(0)}(z)$:

$$\psi^{(0)}(z) = \frac{d}{dz}\ln\big(\Gamma(z)\big) = \frac{\Gamma'(z)}{\Gamma(z)}$$

For integers n:

$\psi^{(0)}(n) = -\gamma + H_{n-1}$ where γ is the Euler-Mascheroni constant

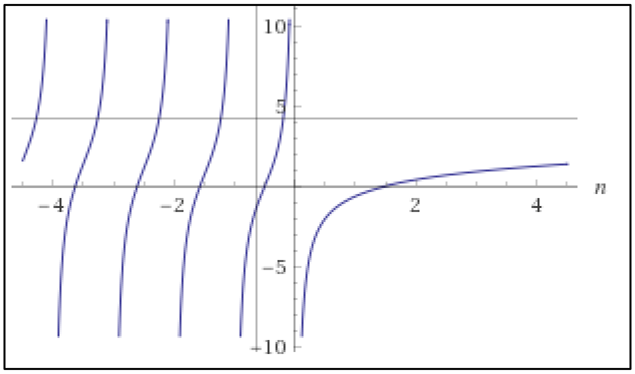

Figure 3. Digamma function in R

3.8. Polygamma function $\psi^{(0)}(z)$:

$$\psi_m(z) = (-1)^{m+1}\, m!\ \zeta(1+m, z)$$

3.9. Pochhammer symbol:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1)$$

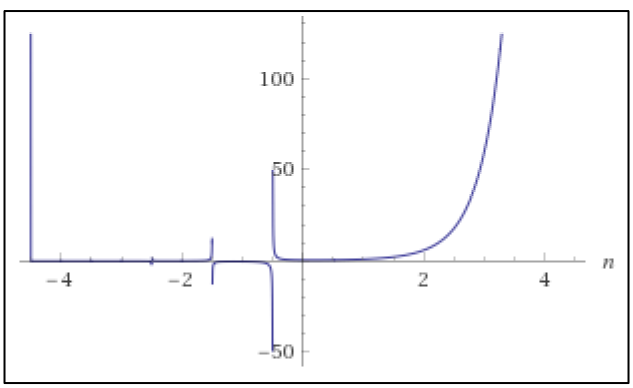

Figure 4. Pochhammer symbols in R

3.10. Hypergeometric function:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\ \frac{z^n}{n!} = 1 + \frac{ab}{1!\,c} z + \frac{a(a+1)b(b+1)}{2!\,c\,(c+1)} z^2 + \cdots$$

Where $(a)_n$ is a Pochhammer symbol. The solution exists if c is not a negative integer (1) for all of $|z|<1$ and (2) on the unit circle $|z|=1$ if $Re(c-a-b)>0$.

This function is related to the Hypergeometric Differential equation:

$z(1-z)y'' + [c-(a+b+1)z]\, y' - aby = 0$

Some particular values:

$$ {}_2F_1(1,1;1;z) = \frac{1}{1-z} $$

$$ {}_2F_1(1,1;2;z) = \frac{-\ln(1-z)}{z} $$

$$ {}_2F_1(1,2;1;z) = \frac{1}{(1-z)^2} $$

$$ {}_2F_1(1,2;2;z) = \frac{1}{1-z} $$

3.11. Polylogarithm $Li_s(z)$ defined by a power series in z, which is also a Dirichlet series in s:

$$ Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \quad \text{converges for } s > 1 \text{ and } |z| < 1 $$

Particular values:

$Li_1(z)$	$Li_0(z)$	$Li_1(1/2)$	$Li_2(1/2)$	$Li_s(\pm i)$
$-\ln(1-z)$	z/(1-z)	$\ln(2)$	$\frac{1}{2}(\zeta(2) - \ln(2)^2)$	$-2^{-s}\eta(s) \pm i\beta(s)$

Table 2

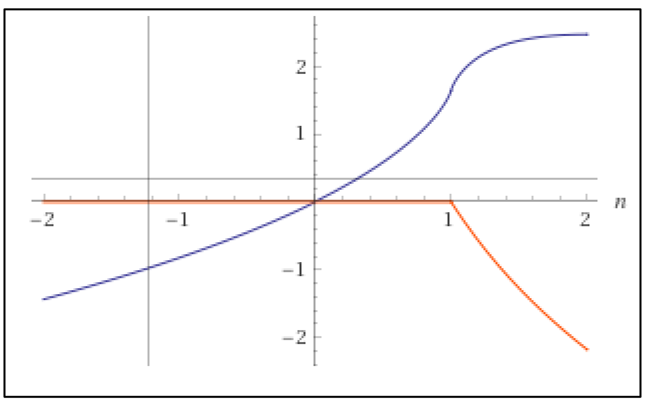

Figure 5. Polylogarithm function in R

Where:

$$ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \text{is the } \eta \text{ (eta)} - \text{Dirichlet function} $$

$$ \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \quad \text{is the } \beta \text{ (beta)} - \text{Dirichlet function} $$

3.12. The Lerch Transcendent function $\Phi(z,s,q)$:

$$ \Phi(z,s,q) = \sum_{k=0}^{\infty} \frac{z^k}{(k+q)^s} $$

4. Constants used in this paper

4.1. Glaisher-Kinkelin constant

$$A = e^{\frac{1}{12}-\zeta'(-1)} = 1.28242712\ldots$$

4.2. Euler-Mascheroni constant:

$$\gamma = \lim_{m\to\infty}(\sum_{k=1}^{m}\frac{1}{k} - \int\frac{dm}{m} = \lim_{m\to\infty}(\sum_{k=1}^{m}\frac{1}{k} - \ln(m) = 0.57721566490\ldots$$

γ is also the C-value of $f(k) = \frac{1}{k}$

4.3. Stieltjes constants:

$$\gamma_n = \lim_{m\to\infty}(\sum_{k=1}^{m}\frac{(\ln k)^n}{k} - \int\frac{(\ln m)^n}{m}\,dk) = \lim_{m\to\infty}(\sum_{k=1}^{m}\frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1})$$

n=0	n=1	n=2	n=3	n=4
0.5772456...	-0.072815...	-0.00969...	0.00205...	0.0007933...

Table 3

γ_n are also the C-values of $f(k) = \frac{(\ln k)^n}{k}$

4.4. Constants involving $\zeta(s)$:

Lemma 1:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

Proof:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = \lim_{n,m\to\infty}\sum_{j=2}^{n}\sum_{k=2}^{m} j^{-k} =$$

$$= \lim_{n,m\to\infty}\sum_{j=2}^{n}\frac{j^{-m-1}(j^m - j)}{j-1} = \lim_{n\to\infty}\sum_{j=2}^{n}\frac{j^{-1}}{j-1} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$$

Lemma 2:

$$\sum_{j=2}^{\infty}(\zeta(j)-1) = 1$$

Proof:

$$\sum_{j=2}^{\infty}(\zeta(j)-1) = \sum_{j=2}^{\infty}(\sum_{k=1}^{\infty} j^{-k} - 1) = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

Per lemma 1

Lemma 3:

$$\lim_{k\to\infty} \zeta(k) = 1$$

Proof:

$$\zeta(k) = \sum_{\mathrm{j}=1}^{\infty} j^{-k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \cdots$$

$$\lim_{k\to\infty} \zeta(k) = \lim_{k\to\infty} 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \cdots = 1$$

Theorem 1.0: The infinite sums $\sum_{\mathrm{j}=1}^{\infty} [\zeta(\mathrm{u} * \mathrm{k} \pm \mathrm{n}) - \zeta(\mathrm{v} * \mathrm{k} \pm \mathrm{m})]$ converge to a value in the interval (-1,1) for all $\mathrm{u} \geq 1, \mathrm{v} \geq 1, \mathrm{n}, \mathrm{m} \in$ N such that $(\mathrm{u} * \mathrm{k} \pm \mathrm{n}) > 1$ and $(\mathrm{v} * \mathrm{k} \pm \mathrm{m}) > 1$ for all $\mathrm{j} \in \mathrm{N}$

Proof:

$$\sum_{\mathrm{j}=1}^{\infty} [\zeta(\mathrm{uj} \pm \mathrm{n}) - \zeta(\mathrm{vj} \pm \mathrm{m})] =$$

$$= \zeta(\mathrm{u} \pm \mathrm{n}) - \zeta(\mathrm{v} \pm \mathrm{m}) + \zeta(2\mathrm{u} \pm \mathrm{n}) - \zeta(2\mathrm{v} \pm \mathrm{m}) + \cdots$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{1}{j^{uk\pm n}} - \frac{1}{j^{vk\pm m}}\right)$$

The largest differences in value between the parameters $\mathrm{uk} \pm \mathrm{n}$ and $\mathrm{vk} \pm \mathrm{m}$ occurs when $\mathrm{u} = 1, \mathrm{n} = 1$, and $\mathrm{v} =$ infinity.

$$\sum_{\mathrm{j}=1}^{\infty} \sum_{\mathrm{k}=1}^{\infty} \left(\frac{1}{\mathrm{j}^{\mathrm{uk}\pm\mathrm{n}}} - \frac{1}{\mathrm{j}^{\mathrm{vk}\pm\mathrm{m}}}\right) \leq \sum_{\mathrm{j}=1}^{\infty} \left(\sum_{\mathrm{k}=1}^{\infty} \frac{1}{\mathrm{j}^{\mathrm{k}+1}}\right) - 1 =$$

$$= \sum_{\mathrm{j}=2}^{\infty} \sum_{\mathrm{k}=1}^{\infty} \frac{1}{\mathrm{j}^{\mathrm{k}+1}} = \sum_{\mathrm{j}=2}^{\infty} \sum_{\mathrm{k}=2}^{\infty} \frac{1}{\mathrm{j}^{\mathrm{k}}} = 1$$

As per lemmas 1,2,3.

Following the same logic, we can also say that:

$$\sum_{\mathrm{j}=1}^{\infty} \sum_{\mathrm{k}=1}^{\infty} \left(\frac{1}{\mathrm{j}^{\mathrm{uk}\pm\mathrm{n}}} - \frac{1}{\mathrm{j}^{\mathrm{vk}\pm\mathrm{m}}}\right) \geq -1$$

Therefore, the infinite sums $\sum_{\mathrm{j}=1}^{\infty} [\zeta(\mathrm{uk} \pm \mathrm{n}) - \zeta(\mathrm{vk} \pm \mathrm{m})]$ converge to a value in the interval (-1, 1) for all $\mathrm{u}, \mathrm{v}, \mathrm{n}, \mathrm{m} \in \mathrm{N}$ such that $(\mathrm{u} * \mathrm{k} \pm \mathrm{n}) > 1$ and $(\mathrm{v} * \mathrm{k} \pm \mathrm{m}) > 1$ for all $\mathrm{j} \in \mathrm{N}$

Similarly, we can propose and formulate the following Theorem.

Theorem 2.0: The infinite sums $\sum_{j=1}^{\infty}[\zeta(u*k \pm n) - \zeta(v*k \pm m)]$ converge to a value in the interval (-∞,∞) for all $u \geq 1, v \geq 1, n, m \in R$ such that $(u*k \pm n) > 1$ and $(v*k \pm m) > 1$ for all $j \in N$

Example:

$$\sum_{j=1}^{\infty} \zeta(1.1*j+2) - \zeta(1.2*j+0.1) = -3.14132\ldots$$

$$\sum_{j=1}^{\infty} \zeta(1.001*j) - \zeta(1.1*j+1) = 999.733\ldots$$

These theorems let us define a set of infinite constants of the type:

$$CZ_{u,v,n,m}^{(q)} = \sum_{j=q}^{\infty}[\zeta(uj \pm n) - \zeta(vj \pm m)] = \text{constant}$$

(By default, q=1 will not be written)

Some of the CZ constants we will use through the paper are:

$$CZ_{2,0,2,1} = \sum_{j=1}^{\infty}[\zeta(2j) - \zeta(2j+1)] = 0.5$$

$$CZ_{4,0,4,-2} = \sum_{j=1}^{\infty}[\zeta(4j) - \zeta(4j-2))] = -0.5766744746\ldots$$

$$CZ_{4,1,4,-1} = \sum_{j=1}^{\infty}[\zeta(4j+1) - \zeta(4j-1))] = -0.171865985524\ldots$$

These constants will be used to calculate the value of values of the Matrix $X_{ij} = [x_{jk}]$ in C complex plane.

The following matrix shows some values of the $CZ_{u,v,1,0}$ constants:

n=1, m=0	v=2	v=3	v=4	v=5	v=200
u=2	-0.500000	0.028310	0.163337	0.212046	0.250000
u=3	-0.658193	-0.129882	0.005144	0.053853	0.091800
u=4	-0.719330	-0.182622	-0.047596	0.001113	0.039067
u=5	-0.732147	-0.203836	-0.068810	-0.020101	0.017853
u=200	-0.750000	-0.221689	-0.086663	-0.037954	0.000000

Table 4

5. **The Matrix $X_{ij} = [x_{jk}] \in R^{j \times k}$ where $x_{jk} = \delta + \omega(\alpha + \beta j)^{\varphi k}$, $\alpha, \beta, \omega, \delta, \varphi \in R$**

$$X_{jk} = [\delta + w(\alpha + \beta j)^{\varphi k}] = \begin{bmatrix} [\delta + w(\alpha+\beta)^{\varphi}] & [\delta + w(\alpha+\beta)^{2\varphi}] & \dots & [\delta + w(\alpha+\beta)^{m\varphi}] \\ [\delta + w(\alpha+2\beta)^{\varphi}] & [\delta + w(\alpha+2\beta)^{2\varphi}] & \dots & [\delta + w(\alpha+2\beta)^{m\varphi}] \\ \vdots & \vdots & & \vdots \\ [\delta + w(\alpha+n\beta)^{\varphi}] & [\delta + w(\alpha+n\beta)^{2\varphi}] & \dots & [\delta + w(\alpha+n\beta)^{m\varphi}] \end{bmatrix}$$

5.1 Elements of the matrix X_{ij} $\quad x_{jk} = \delta + \omega(\alpha + \beta j)^{\varphi k}$

5.2 Rows of the matrix X_{ij} $\quad RX_{jk} = \delta + \omega(\alpha + \beta j)^{\varphi k}$ for $k = 1..m$

5.3 Columns of the matrix X_{ij} $\quad CX_{jk} = \delta + \omega(\alpha + \beta j)^{\varphi k}$ for $j = 1..n$

5.4. Some examples of X_{ij} based on values of $\alpha, \beta, \omega, \delta, \varphi$ in R:

a. If $\alpha = 0, \beta = 1, \omega = 1, \delta = 0, \varphi = -1$ then $x_{jk} = j^{-k}$

$$X_{ij} = \begin{bmatrix} 1^{-1} & \cdots & 1^{-k} \\ \vdots & \ddots & \vdots \\ j^{-1} & \cdots & j^{-k} \end{bmatrix}$$

b. If $\alpha = 0, \beta = 1, \omega = 1, \ \delta = 0$ then $x_{jk} = j^{-\varphi k}$

$$X_{ij} = \begin{bmatrix} 1^{-\varphi} & \cdots & 1^{-\varphi k} \\ \vdots & \ddots & \vdots \\ j^{-\delta} & \cdots & j^{-\delta k} \end{bmatrix}$$

c. If $\alpha = 0, \beta = 1, \omega = 1, \delta = 1, \varphi = -1$ then $x_{jk} = (1 + j^{-k})$

$$X_{ij} = \begin{bmatrix} (1 + 1^{-1}) & \cdots & (1 + 1^{-k}) \\ \vdots & \ddots & \vdots \\ (1 + j^{-1}) & \cdots & (1 + j^{-k}) \end{bmatrix}$$

d. If $\alpha = 1, \beta = -1, \omega = 1, \delta = 1, \varphi = -1$ then $x_{jk} = (1 - j^{-k})$

$$X_{ij} = \begin{bmatrix} (1 - 1^{-1}) & \cdots & (1 - 1^{-k}) \\ \vdots & \ddots & \vdots \\ (1 - j^{-1}) & \cdots & (1 - j^{-k}) \end{bmatrix}$$

e. If $\alpha = 1, \beta = 1, \omega = 1, \delta = 0, \varphi = -1$ then $x_{jk} = (1 + j)^{-k}$

$$X_{ij} = \begin{bmatrix} (1 + 1)^{-1} & \cdots & (1 + 1)^{-k} \\ \vdots & \ddots & \vdots \\ (1 + j)^{-1} & \cdots & (1 + j)^{-k} \end{bmatrix}$$

f. If $\alpha = 0, \beta = 1, \omega = 1, \delta = 0, \varphi = -1$, and $j = re^{i\theta}$ then $x_{jk} = (re^{i\theta})^{-k}$

$$X_{ij} = \begin{bmatrix} (e^{i\theta})^{-1} & \cdots & (e^{i\theta})^{-k} \\ \vdots & \ddots & \vdots \\ (re^{i\theta})^{-1} & \cdots & (re^{i\theta})^{-k} \end{bmatrix}$$

6. **Elements $x_{jk} = \delta + \omega(\alpha + \beta j)^{\varphi k}$ with $\alpha, w, \beta, \delta, \varphi \in R$**

6.1 A general expression for x_{jk} as a power series is given by the binomial expression:

$$x_{jk} = \delta + w(\alpha + \beta j)^{\varphi k} = \delta + w \sum_{p=0}^{\varphi k} \binom{\varphi k}{p} \alpha^{\varphi k - p} (\beta j)^p$$

6.2 For $\delta = 0, \omega = 1, \alpha = 0, \beta = 1, \varphi = 1 \rightarrow x_{jk} = j^k$ can also be expressed as series of the form:

$$x_{jk} = j^k = \left(q + (j-q)\right)^k = \sum_{p=0}^{k} \binom{k}{p} q^{k-p} (j-q)^p$$

For q=1:

$$x_{jk} = j^k = \left(1 + (j-1)\right)^k = \sum_{p=0}^{k} \binom{k}{p} (j-1)^p$$

Example: $3^4 = \binom{4}{0} 2^4 + \binom{4}{1} 2^3 + \binom{4}{2} 2^2 + \binom{4}{3} 2^1 + \binom{4}{4} 2^0 = 81$

$x_{jk} = j^k$ can also be expressed using the exponential function:

$$x_{jk} = \sum_{p=0}^{k} \binom{k}{p} e^{p \ln(j-1)}$$

6.3 For $\delta = 0, \omega = 1, \alpha = 0, \beta = 1, \varphi = -1, k = j \rightarrow y(j,k) = j^{-j}$. Series expansion at j=0:

$$x_{jj} = j^{-j} = 1 - j \log(j) + \frac{1}{2} j^2 \log^2(j) - \frac{1}{6} j^3 \log^3(j) + \cdots = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} j^p \log^p(j)$$

j=1	j=2	j=3	j=4	j=5
1	$\frac{1}{4}$	$\frac{1}{27}$	$\frac{1}{256}$	$\frac{1}{3125}$

Table 5

7. General expressions for the sums of the rows and columns of Matrix $\mathbf{X_{jk}} = [\mathbf{x_{jk}}] \in \mathbf{R^{j\,x\,k}}$ where $\mathbf{x_{jk}} = \boldsymbol{\delta} + \boldsymbol{\omega}(\boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{j})^{\boldsymbol{\varphi}\mathbf{k}}$ with $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\omega}, \boldsymbol{\delta}, \boldsymbol{\varphi} \in \mathbf{R}$

7.1. Sum of Columns, over (j):

A general expression for $\sum x_{jk}$ over j using the Hurwitz function:

$$\sum_{j=1}^{n} \delta + w(\alpha + \beta j)^{\varphi k} = \delta n - w\, \beta^{\varphi k}\, \zeta\left(-\varphi k, \frac{\alpha + \beta}{\beta} + n\right) + w\, \beta^{\varphi k}\, \zeta\left(-\varphi k, \frac{\alpha + \beta}{\beta}\right)$$

7.2. Sum of Rows, over (k):

A general expression for $\sum x_{jk}$ over k using the formula for the sum of geometric progressions when $|(\alpha + \beta j)^{\varphi}| < 1$:

$$\sum_{k=1}^{n} \delta + w(\alpha + \beta j)^{\varphi k} = \delta n + \frac{w\,(\alpha + \beta j)^{\varphi}((\alpha + \beta j)^{\varphi n} - 1}{(\alpha + \beta j)^{\varphi} - 1}$$

8. Case 1: Matrix $X_{ij} = [\mathbf{x_{jk}}] \in \mathbf{R^{j\,x\,k}}$ where $\mathbf{x_{jk}} = \mathbf{j^{-k}}$

8.1. Sum of terms of the rows of $X_{jk} = [j^{-k}]$:

$$\sum_{k=1}^{k=\infty} j^{-k} = \lim_{m \to \infty} \left(\frac{1}{j^1} + \frac{1}{j^2} + \cdots + \frac{1}{j^m}\right) = \frac{1}{j-1} \quad \text{if } j > 1$$

$$\sum_{k=1}^{k=n} j^{-k} = n \qquad \text{if } j = 1 \text{ and } \qquad \sum_{k=1}^{k=\infty} j^{-k} \text{is Divergent}$$

$$\sum_{k=1}^{k=m} j^{-k} = \frac{j^{-k}(j^k - 1)}{j - 1} \qquad \text{if } j > 1$$

Value Table:

J	2	3	4	5
Σ rows$X_{jk} = \Sigma[j^{-k}]$	1	1/2	1/3	1/4

Table 6

$$\sum_{k=p}^{k=m} j^{-k} = \frac{j^{-m} - j^{1-p}}{1 - j}$$

Value Table for row j=3, and p=3

M	4	5	6	7
Σ rows$X_{jk} = \Sigma[j^{-k}]$	4/81	13/243	40/729	121/2187

Table 7

$$\sum_{k=p}^{k=\infty} j^{-k} = \frac{j^{1-p}}{j - 1}$$

Value Table for row j=3

p	2	3	4	5
Σ rows$X_{jk} = \Sigma[j^{-k}]$	1/6	1/18	1/54	1/162

Table 8

8.2. Product of terms of the rows of $X_{jk} = [j^{-k}]$:

$$\prod_{k=1}^{k=\infty} j^{-k} = \lim_{m\to\infty} \left(\frac{1}{j^1} \times \frac{1}{j^2} \times \dots \times \frac{1}{j^m}\right) = 0$$

$$\prod_{k=1}^{k=m} j^{-k} = \left(\frac{1}{j^1} \times \frac{1}{j^2} \times \dots \times \frac{1}{j^m}\right) = j^{-\frac{1}{2m(m+1)}}$$

Value Table:

M	1	2	3	4
Π rows$X_{jk} = \Pi[j^{-k}]$	$\frac{1}{j}$	$\frac{1}{j^3}$	$\frac{1}{j^6}$	$\frac{1}{j^{10}}$

Table 9

Example for j=2

M	1	2	3	4
$\Pi\ \text{rows}X_{jk} = \Pi[j^{-k}]$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{64}$	$\frac{1}{1024}$

Table 10

8.3. Sum of terms of the columns of $X_{jk} = [j^{-k}]$:

$$\sum_{j=1}^{j=\infty} j^{-k} = \lim_{n\to\infty}\left(\frac{1}{1^k}+\frac{1}{2^k}+\cdots+\frac{1}{n^k}\right) = \zeta(k)$$

This is the Riemann-Zeta function.

$$\sum_{j=1}^{j=\infty} j^{-1} = H_n = \zeta(1) \text{ is Divergent}$$

$$\sum_{j=1}^{j=n} j^{-k} = \left(\frac{1}{1^k}+\frac{1}{2^k}+\cdots+\frac{1}{n^k}\right) = H_n^{(k)}$$

Where $H_n^{(k)}$, is the generalized Harmonic function.

K	1	2	3	4
$\Sigma\ \text{Cols}X_{jk} = \Sigma[j^{-k}]$	H_n Divergent	$H_n^{(2)}$	$H_n^{(3)}$	$H_n^{(4)}$

Table 11

Example for k=2

N	1	2	3	4
$\Sigma\ \text{Cols}X_{jk} = \Sigma[j^{-k}]$	$H_1^{(2)} = 1$	$H_2^{(2)} = \frac{5}{4}$	$H_3^{(2)} = \frac{49}{36}$	$H_4^{(2)} = \frac{205}{144}$

Table 12

We can also express this partial sum using the Hurwitz function as:

$$\sum_{j=1}^{j=n} j^{-k} = \left(\frac{1}{1^k}+\frac{1}{2^k}+\cdots+\frac{1}{n^k}\right) = \zeta(k) - \zeta(k, n+1)$$

$$\sum_{j=z}^{j=\infty} j^{-k} = \lim_{n\to\infty}\left(\frac{1}{z^k}+\frac{1}{(z+1)^k}+\cdots+\frac{1}{n^k}\right) = \zeta(k, z)$$

$$\sum_{j=p}^{j=n} j^{-k} = \zeta(k, p) - \zeta(k, n+1)$$

Value Table for column k=3, and n=∞

P	2	3	4	5
$\Sigma\ \mathrm{ColsX}_{jk} = \Sigma\left[j^{-k}\right]$	$\zeta(3)-1$	$\zeta(3)-\frac{9}{8}$	$\zeta(3)-\frac{251}{216}$	$\zeta(3)-\frac{2035}{1728}$

Table 13

Other interesting expressions:

$$\sum_{j=1}^{\infty} j^{-k}\left(1-\frac{1}{\sum_{p=0}^{k}\frac{\binom{k}{p}}{k^p}}\right)=1 \qquad \text{for } \mathrm{Re}(k)>1$$

This expression gives the factor A(k) such that $\sum_{j=1}^{\infty}\frac{A(k)}{j^k}=1$ for all k.

$$\sum_{j=1}^{\infty} j^{-k} = \sum_{j=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-k)^p}{p!}(\log(j))^p \qquad \text{for } \mathrm{Re}(k)>1$$

$$\sum_{j=1}^{n} j^{-k} = \sum_{j=1}^{n}\sum_{p=0}^{\infty}\frac{(-k)^p}{p!}(\log(j))^p = H_n^{(k)} = \zeta(k)+\frac{\psi^{(k-1)}(n+1)}{k-1} \qquad \text{for } \mathrm{Re}(k)>1$$

A similar series:

$$\sum_{j=1}^{n}\sum_{p=1}^{\infty}\frac{k^p}{p!} * j * (-\log(j))^p = H_n^{(k-1)} - \frac{1}{2}n(n+1) \qquad \text{for } \mathrm{Re}(k)>1$$

Using the power series for π^k:

$$\pi^k = \sum_{p=0}^{\infty}\frac{k^p}{p!}(\log(\pi))^p$$

And we can write $\zeta(k)$ as a function of π^k. Example for k=2 for example:

$$\frac{\zeta(2)}{\pi^2} = \frac{\sum_{j=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-2)^p}{p!}(\log(j))^p}{\sum_{p=0}^{\infty}\frac{(-2)^p}{p!}(\log(\pi))^p}$$

We know that these expressions give us rational numbers when k=even:

$$\frac{\zeta(2)}{\pi^2}=\frac{1}{6}$$

$$\frac{\zeta(4)}{\pi^4}=\frac{1}{90}$$

$$\frac{\zeta(6)}{\pi^6}=\frac{1}{945}$$

$$\frac{\zeta(8)}{\pi^8}=\frac{1}{9450}$$

We also know that there is not a defined expression when k is odd. We are proposing the following expression as a very close approximation for $\frac{\zeta(s)}{\pi^s}$ for s=odd positive integer:

$\frac{\zeta(s)}{\pi^s} = \frac{1}{K(s)}$ where $\mathbf{K(s) = Floor}\left(\frac{(2^s-1)\pi^s}{2^s}\right) - \mathbf{1}$ for $\mathbf{s > 1}$

S	K(s) calculated	K(s) actual
2	6	6.0
3	26	25.8
4	90	90.0
5	295	295.1
6	945	945.0
7	2995	2995.3
8	9450	9450.0

Table 14

This approximation leads to the following approximate formulation for ζ(s)

$$\boldsymbol{\zeta(s) = \frac{1}{1-\pi^{-s}-2^{-s}}}$$

Graphically:

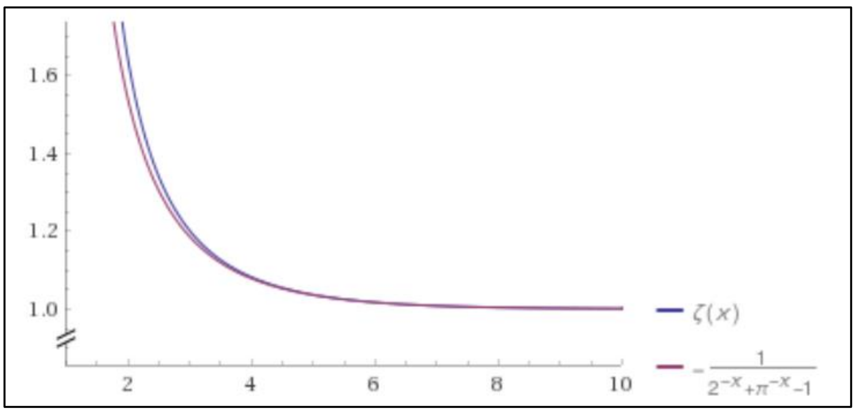

Figure 6. Caceres' approximation for the Riemann Zeta function in R

8.4. Product of terms of the columns of $X_{jk} = [j^{-k}]$:

$$\prod_{j=1}^{j=\infty} j^{-k} = \lim_{n\to\infty}\left(\frac{1}{1^k}\times\frac{1}{2^k}\times\ldots\times\frac{1}{n^k}\right) = 0$$

$$\prod_{j=1}^{j=n} j^{-k} = \left(\frac{1}{1^k}\times\frac{1}{2^k}\times\ldots\times\frac{1}{n^k}\right) = n!^{-k} = \Gamma(n+1)^{-k}$$

Where Γ, is the Gamma function.

N	1	2	3	4
Π cols $X_{jk} = \Pi[j^{-k}]$	$\Gamma(2)^{-k}$	$\Gamma(3)^{-k}$	$\Gamma(4)^{-k}$	$\Gamma(5)^{-k}$

Table 15

Example for k=2

N	1	2	3	4
Π cols $X_{jk} = \Pi[j^{-k}]$	$\Gamma(2)^{-2} = 1$	$\Gamma(3)^{-2}$=1/4	$\Gamma(4)^{-2}$ =1/36	$\Gamma(5)^{-2}$=1/576

Table 16

8.5. Sum of terms of the Main Diagonal of $X_{jk} = [j^{-j}]$:

$$\sum_{j=1}^{\infty} j^{-j} = \sum_{j=1}^{\infty}\sum_{p=0}^{j}\binom{j}{p}(j-1)^p = \frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}\ldots$$

This infinite sum converges rapidly to:

$$\sum_{j=1}^{\infty} j^{-j} = \frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}\ldots = 1.29129\ldots$$

Other relevant expressions for $y = j^{-j}$

$$\frac{d}{dj} j^{-j} = j^{-j}(\log(j)+1)$$

$$\int j^{-j}\,dj = \int \sum_{p=0}^{\infty}(-1)^p j^p \log^p(j)\ dj = \sum_{p=1}^{\infty}\frac{1}{A_p} j^p \sum_{q=1}^{p}(p-1)(-1)^{q+1}\frac{p^{q-1}}{(q-1)!}\log^{(q-1)}(j))$$

The last expression is accurate for p=1,2,3,4,6,8 and all even, and approximate for p=5,7 and greater odd integers. The terms A_p coincide with the terms of OEIS A055774 of positive numbers that are the least common multiple of n! and n^n.

And it can be calculated that:

$$\int_0^{\infty} j^{-j}\,dj = 1.99545595750014\ldots$$

$$\int_1^{\infty} j^{-j}\,dj = 0.70416996043747\ldots$$

$$\int_2^{\infty} j^{-j}\,dj = 0.131821\ldots\ldots$$

$$\int_0^{\infty} j^{-j}\,dj - \sum_{j=1}^{\infty} j^{-j} = 0.587116\ldots$$

8.6. Product of terms of the Main Diagonal of $X_{jk} = [j^{-j}]$:

$$\prod_{j=1}^{j=\infty} j^{-j} = 0$$

$$\prod_{j=1}^{j=n} j^{-j} = \frac{1}{H(n)}$$

Where H(n) is the Hyperfactorial function

Value Table:

N	1	2	3	4
Π Main Diagonal of X_{jk}	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{108}$	$\frac{1}{27648}$

Table 17

8.7. Sum of terms of the diagonals of X_{jk} with $j \neq k$:

8.7.1. For the diagonal starting in element (j=n, k=1):

$$\sum_{j=n}^{j=\infty} j^{-(j-n+1)} = \lim_{m\to\infty} \left(\frac{1}{n^1} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+m)^{(m+1)}}\right) \text{ converges}$$

N	2	3	4	5
Σ Lower Diagonal of $X_{jk}^{(0,1,-1)}$	0.628474	0.404668	0.297059	0.230955

Table 18

$$\lim_{n\to\infty} \sum_{j=n}^{j=\infty} j^{-(j-n+1)} = 0$$

8.7.2. For the diagonal starting in element (j=1, k=m):

$$\sum_{j=1}^{j=\infty} j^{-(j+m-1)} = \lim_{m\to\infty} \left(\frac{1}{1^{(m)}} + \frac{1}{2^{(m+1)}} + \cdots + \frac{1}{n^{(m+n-1)}}\right) \text{ converges}$$

m	2	3	4	5
Σ Upper Diagonal of X_{jk}	1.13839	1.06687	1.03269	1.0161

Table 19

$$\lim_{m\to\infty} \sum_{j=1}^{j=\infty} j^{-(j+m-1)} = 1$$

8.8. Products of terms of the diagonals of X_{jk} with $j \neq k$::

8.8.1. For the diagonal starting in element (j=n, k=1):

$$\prod_{j=n}^{j=\infty} j^{-(j-n+1)} = \lim_{m\to\infty} \left(\frac{1}{n^1} \times \frac{1}{(n+1)^2} \times \ldots \times \frac{1}{(n+m)^{(m+1)}}\right) = 0$$

$$\prod_{j=n}^{j=n+z} j^{-(j-n+1)} = \frac{\Gamma(n+z+1)^{(n-1)}}{A} \times e^{\frac{1}{12}-\zeta^{(1,0)}(-1,n+z+1)}$$

Where A is the Glaisher-Kinkelin constant

Some values for z=2

N	2	3	4	5
Π Lower Diagonal of X_{jk}	$\frac{1}{1152}$	$\frac{1}{6000}$	$\frac{1}{21600}$	$\frac{1}{61740}$

Table 20

8.8.2. For the diagonal starting in element (j=1, k=m):

$$\prod_{j=1}^{j=\infty} j^{-(j+m-1)} = \lim_{m\to\infty}\left(\frac{1}{1^{(m)}} \times \frac{1}{2^{(m+1)}} \times \dots \times \frac{1}{n^{(m+n-1)}}\right) = 0$$

$$\prod_{j=1}^{j=n} j^{-(j+m-1)} = \frac{\Gamma(n+1)^{(1-m)}}{A} \times e^{\frac{1}{12}-\zeta^{(1,0)}(-1,n+1)}$$

Some values for k=m=2

n	1	2	3	4
Π Upper Diagonal of $X_{jk}^{(0,1,-1)}$	1	$\frac{1}{8}$	$\frac{1}{648}$	$\frac{1}{663552}$

Table 21

8.9. Sum of terms of the Matrix X_{ij}:

Let's call $S_{nm}^{\infty\infty} = \sum_{j=n}^{\infty}\sum_{k=m}^{\infty} j^{-k}$

$S_{nm}^{\infty\infty}$	m=1	m=2	m=3	m=4
n=1	Divergent	Divergent	Divergent	Divergent
n=2	Divergent	1	2-ζ(2)	$3-\zeta(2)-\zeta(3)$
n=3	Divergent	$\frac{1}{2}$	7/4-ζ(2)	$\frac{23}{8}-\zeta(2)-\zeta(3)$
n=4	Divergent	$\frac{1}{3}$	61/36-ζ(2)	$\frac{617}{216}-\zeta(2)-\zeta(3)$
n=5	Divergent	$\frac{1}{4}$	241/144-ζ(2)	$\frac{4927}{1728}-\zeta(2)-\zeta(3)$

Table 22

8.9.1 $S_{2\,2}^{nm}$

$$S_{2\,2}^{nm} = \sum_{j=2}^{n}\sum_{k=2}^{m} j^{-k} = \sum_{j=2}^{n} \frac{1+j^2+j^2+\dots+j^m}{j^m}$$

$$S_{2\,2}^{nm} = (1-m) + \sum_{j=2}^{m} H_n^{(j)}$$

As example:

$$S_{2\,2}^{45} = \sum_{j=2}^{4}\sum_{k=2}^{5} j^{-k} = \sum_{j=2}^{4} \frac{1+j^2+j^3}{j^5} =$$

$$= H_4^{(2)} + H_4^{(3)} + H_4^{(4)} + H_4^{(5)} - 4$$

$$= 0.71636685\ldots$$

Values for several n:

n	1	2	3	4
$H_n^{(2)}+H_n^{(3)}+H_n^{(4)}+H_n^{(5)}-4$	0	0.46875	0.716367	0.766287

Table 23

Let's remember that:

$$H_n^{(m)} = \zeta(m) - \zeta(m, n+1)$$

8.9.2 $S_{2\,2}^{\infty\infty}$

$$S_{2\,2}^{\infty\infty} = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k}$$

$$S_{2\,2}^{\infty\infty} = \lim_{n,m\to\infty} \sum_{j=2}^{n}\sum_{k=2}^{m} j^{-k} = \lim_{n,m\to\infty} \sum_{j=2}^{m} (H_n^{(j)} - 1) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$$

Which is equivalent to the next expression.

8.9.3 $S_{2\,2}^{\infty\infty}$

$$\sum_{j=2}^{\infty} [\zeta(j) - 1] = S_{2\,2}^{\infty\infty} = 1$$

8.9.4 $S_{2\,m}^{\infty\infty}$

$$S_{2\,m}^{\infty\infty} = \sum_{j=2}^{\infty}\sum_{k=m}^{\infty} j^{-k} = (m-1) - \sum_{q=2}^{q-1} \zeta(q)$$

8.9.5 $S_{2\,m}^{\infty\infty}$

$$S_{2\,m}^{\infty\infty} = \sum_{j=2}^{\infty}\sum_{k=m}^{\infty} j^{-k} = S_{22} - \sum_{q=2}^{q-1} (\zeta(q) - 1) = 1 - \sum_{q=2}^{q-1} (\zeta(q) - 1)$$

8.9.6 $S_{2\,2}^{\infty\infty}$

If C_m= Sum of elements of the column k=m, and R_n= Sum of elements of the row j=n, then:

$$S_{n\ m}^{\infty\infty} = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} - \sum_{p=2}^{m-1} C_p - \sum_{q=2}^{n-1} R_q + \sum_{r=2}^{n-1}\sum_{s=2}^{m-1} r^{-s}$$

Where:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

$$\sum_{p=2}^{m-1} C_p = \sum_{p=2}^{m-1} (\zeta(p) - 1)$$

$$\sum_{q=2}^{n-1} R_q = \sum_{q=2}^{n-1} \frac{1}{q(q-1)}$$

and we can rewrite

$$S_{n\ m}^{\infty\infty} = 1 - \sum_{p=2}^{m-1} (\zeta(p) - 1) - \sum_{q=2}^{n-1} \frac{1}{q(q-1)} + \sum_{r=2}^{n-1}\sum_{s=2}^{m-1} r^{-s}$$

Example:

$$S_{3\ 4}^{\infty\infty} = 1 - \sum_{p=2}^{3} (\zeta(p) - 1) - \sum_{q=2}^{2} \frac{1}{q(q-1)} + \sum_{r=2}^{2}\sum_{s=2}^{3} r^{-s}$$

$$S_{3\ 4}^{\infty\infty} = 1 - (\zeta(2) - 1) - (\zeta(3) - 1) - \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$$

$$S_{3\ 4}^{\infty\infty} = \frac{23}{8} - \zeta(2) - \zeta(3)$$

This is the same value shown in the previous table.

8.9.7. Other infinite sums involving $\zeta(s)$ (In parenthesis using CZ_n nomenclature)

$$\sum_{j=2}^{\infty} [\zeta(j) - 1] = 1 \ = CZ_{1,0,\infty,0}^{(2)}$$

$$\sum_{j=2}^{\infty} [2\,\zeta(j) - 1] = \frac{3}{4} \ = CZ_{2,0,\infty,0}^{(2)}$$

$$\sum_{j=2}^{\infty} [\zeta(2j - 1) - 1] = \frac{1}{4} = CZ_{2,-1,\infty,0}^{(2)}$$

$$\sum_{j=2}^{\infty} \frac{\zeta(j-1)}{j} = 1 - \gamma \ = 0.422784 \ldots$$

$$\sum_{j=2}^{\infty} \frac{\zeta(j-1)}{e^j} = -\frac{1}{e}\left(\frac{1}{(e-1)} + \gamma + \psi^{(O)}\left(\frac{e-1}{e}\right)\right) = 0.099151 \ldots$$

8.10. C-values for $X_{ij} = \frac{1}{k^n}$: n>1

$$C_n\{f(n) = \frac{1}{k^n}\} = \lim_{m\to\infty}(\sum_{k=1}^{m}\frac{1}{k^n} - \int\frac{dm}{m^n}) = \lim_{m\to\infty}(\sum_{k=1}^{m}\frac{1}{k^n} - \frac{m^{1-n}}{(n-1)}) = \zeta(n)$$

n=2	n=3	n=4	n=5	$n\to\infty$
$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$	$C_n \to \zeta(n)$

Table 24

9. Case 2: Matrix $X_{ij} = [x_{jk}] \in R^{j\,x\,k}$ where $x_{jk} = j^k$

9.1. Sum of terms of the columns of $X_{jk} = [j^k]$:

$$\sum_{j=1}^{n} j^k = 1^k + 2^k + 3^k + \cdots + n^k = \sum_{j=1}^{n}\sum_{p=0}^{\infty}\frac{k^p}{p!}(\log(j))^p$$

$$\sum_{j=1}^{n} j^k = 1^k + 2^k + 3^k + \cdots + n^k = H_n^{(-k)}$$

$$\sum_{j=1}^{n} j^k = 1^k + 2^k + 3^k + \cdots + n^k = \zeta(-k) - \zeta(-k, n+1)$$

$$\sum_{j=1}^{n} j^k = 1^k + 2^k + 3^k + \cdots = \frac{1}{k+1}\sum_{p=0}^{k}\binom{k+1}{p} B_k^+ n^{k+1-p}$$

Where B_k^+ , are the Bernoulli (k) numbers.

Example for k=2 using Bernoulli's numbers:

$$\sum_{j=1}^{n} j^2 = \frac{1}{3}(B_0 n^3 + 3B_1^+ n^2 + 3B_2 n^1) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n^1}{6}$$

Values for several k:

k=3	k=4
$\sum_{j=1}^{n} j^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$	$\sum_{j=1}^{n} j^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{31n}{30}$

Table 25

A similar series:

$$\sum_{j=2}^{n}\sum_{p=1}^{\infty}\frac{(k-1)^p}{p!} * j * (\log(j))^p = -\binom{n}{2} + (n^k - 1) + \cdots + (2^k - 1) = H_n^{(-k)} - \frac{1}{2}n(n+1)$$

Example for n=4, k=3:

$$\sum_{j=2}^{4}\sum_{p=1}^{\infty}\frac{2^p}{p!} * j * (\log(j))^p = -\binom{4}{2} + (4^3 - 1) + (3^3 - 1) + (2^3 - 1) = 90$$

9.2. Sum of terms of the rows of $X_{jk} = [j^k]$:

$$\sum_{k=1}^{m} j^k = j^1 + j^2 + j^3 + \cdots + j^k = \frac{j(j^m - 1)}{j - 1}$$

9.3. Product of terms of the columns of $X_{jk} = [j^k]$:

$$\prod_{j=1}^{n} j^k = (n!)^k$$

9.4. Product of terms of the columns of $X_{jk} = [j^k]$:

$$\prod_{k=1}^{n} j^k = j^{\left(\frac{1}{2}\right)*n(n+1)}$$

10. Case 3: Matrix $X_{ij} = [x_{jk}] \in R^{j \times k}$ where $x_{jk} = \frac{\ln(j)}{j^k}$

10.1. Sum of terms of the columns of $X_{jk} = \left[\frac{\ln(j)}{j^k}\right]$:

$$\sum_{j=1}^{\infty} \frac{\ln(j)}{j^k} = \zeta^{(1,0)}(k, n+1) - \zeta'(k)$$

$$\sum_{j=1}^{\infty} \frac{\ln(j)}{j^k} = -\zeta'(k)$$

k	$-\zeta'(k)$	$-\zeta'(k)$ Formula n^{th} term
2	0.9375482…	$\zeta^{(1,0)}(2, n+1) - \zeta'(2)$
3	0.1981262…	$\zeta^{(1,0)}(3, n+1) - \zeta'(3)$
4	0.0689112…	$\zeta^{(1,0)}(4, n+1) - \zeta'(4)$

Table 26

10.2. Sum of terms of the rows of $X_{jk} = \left[\frac{\ln(j)}{j^k}\right]$:

$$\sum_{k=1}^{m} \frac{\ln(j)}{j^k} = \frac{j^{-m}(j^m - 1)\ln(j)}{j - 1} \quad \text{when } |j| > 1$$

$$\sum_{k=1}^{\infty} \frac{\ln(j)}{j^k} = \frac{\ln(j)}{j - 1} \quad \text{when } |j| > 1$$

10.3. Sum of terms of the diagonals of $X_{jk} = \left[\frac{\ln(j)}{j^k}\right]$:

$$\sum_{j=1}^{\infty} \frac{\ln(j)}{j^j} = -\text{PolyLog}^{(1,0)}\left(0, \frac{1}{j}\right) = 0.219947 \ldots \quad \text{converges}$$

10.4. C-values for $X_{jk} = \left[\frac{\ln k}{k^n}\right]$:

$$C_n\{f(n) = \frac{\ln k}{k^n}\} = \lim_{m \to \infty} (\sum_{k=1}^{m} \frac{\ln k}{k^n} - \int \frac{\ln(m)}{m^n}\, dm)$$

$$= \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{\ln k}{k^n} - \frac{m^{1-n}\left((n-1)\log(m)+1\right)}{(n-1)^2} \right) = \zeta'(n)$$

n=3	0.19813…	$\zeta'(3)$
n=4	-0.06890…	$\zeta'(4)$
n=5	-0.02857…	$\zeta'(5)$

Table 27

11. Case 4: Matrix $X_{ij} = [x_{jk}] \in R^{j \times k}$ where $x_{jk} = \left[\frac{1}{(j-1)j^k}\right]$:

11.1. Sum of terms of the columns of $X_{jk} = \left[\frac{1}{(j-1)j^k}\right]$:

$$\sum_{j=2}^{\infty} \frac{1}{(j-1)j^k} = -k + \sum_{j=2}^{k} \zeta(j)$$

Example: $\frac{1}{1*2^5} + \frac{1}{2*3^5} + \frac{1}{3*4^5} + \cdots = -5 + \zeta(2) + \zeta(3) + \zeta(4) + \zeta(5)$

11.2. Sum of terms of the rows of $X_{jk} = \left[\frac{1}{(j-1)j^k}\right]$:

$$\sum_{k=1}^{m} \frac{1}{(j-1)j^k} = \frac{j^{-m}(j^m-1)}{(j-1)^2} \quad \text{when } |j| > 1$$

$$\sum_{k=1}^{\infty} \frac{1}{(j-1)j^k} = \frac{1}{(j-1)^2} \quad \text{when } |j| > 1$$

J	2	3	4	5
$\frac{1}{(j-1)^2}$ when $\lvert j\rvert > \frac{1}{2}$	1	1/4	1/9	1/16

Table 28

11.3. Sum of terms of the diagonals of $X_{jk} = \left[\frac{1}{(j-1)j^j}\right]$:

$$\sum_{j=2}^{\infty} \frac{1}{(j-1)j^j} = 0.260095 \ldots \text{ converges}$$

11.4. Sum of terms of the Matrix $X_{ij} = \left[\frac{1}{(j-1)j^k}\right]$:

$$\sum_{j=2}^{n} \sum_{k=1}^{\infty} \frac{1}{(j-1)\, j^k} = \zeta(2) - \psi^{(1)}(n)$$

$$\sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j-1)\, j^k} = \zeta(2) = 1.644934 \ldots$$

11.5. C-values for $X_{jk} = \left[\frac{1}{(j-1)j^k}\right]$:

$$C_n\left\{f(n) = \frac{1}{(j-1)j^n}\right\} = \lim_{m\to\infty}\left(\sum_{j=2}^{m}\frac{1}{(j-1)j^n} - \int\frac{dm}{(m-1)j^m}\right)$$

$$= -n + \sum_{j=2}^{n}\zeta(j) - \frac{m^{1-k}\ {}_2F_1(1,1-k;2-k;m)}{k-1} = -n + \sum_{j=2}^{n}\zeta(j)$$

n=2	n=3	n=4	$n\to\infty$
$-2+\zeta(2)$	$-3+\zeta(3)+\zeta(2)$	$-4+\zeta(4)+\zeta(3)+\zeta(2)$	
-0.355065…	-0.15300…	-0.070685	0

Table 29

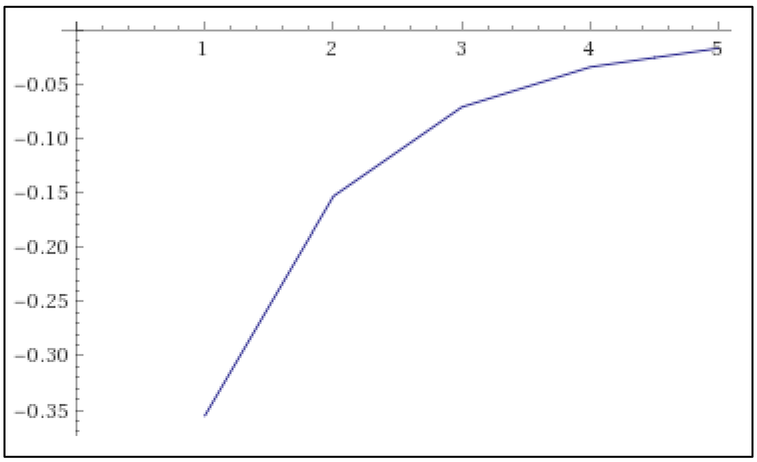

Figure 7. C-values for $X_{jk} = \left[\frac{1}{(j-1)j^k}\right]$

12. Case 5: Matrix $X_{ij} = [x_{jk}] \in R^{j\times k}$ where $x_{jk} = \frac{1}{(2j-1)2^{-k}}$

12.1. Sum of terms of the columns of $CX_{jk} = \left[\frac{1}{(2j-1)2^{-k}}\right]$::

$$\sum_{j=1}^{\infty}\frac{1}{(2j-1)(2j)^k} = \ln(2) - \sum_{j=2}^{k}\frac{\zeta(j)}{2^j}$$

Example: $\frac{1}{1*2^4} + \frac{1}{3*4^4} + \frac{1}{5*6^4} + \cdots = \ln(2) - \frac{\zeta(2)}{2^2} - \frac{\zeta(3)}{2^3} - \frac{\zeta(4)}{2^4}$

12.2. Sum of terms of the rows of $X_{jk} = \left[\frac{1}{(2j-1)(2j)^k}\right]$:

$$\sum_{k=1}^{m}\frac{1}{(2j-1)(2j)^k} = \frac{2^{-m}j^{-m}(2^m j^m - 1)}{(2j-1)^2} \quad \text{when } |j| > \frac{1}{2}$$

$$\sum_{k=1}^{\infty}\frac{1}{(2j-1)(2j)^k} = \frac{1}{(2j-1)^2} \quad \text{when } |j| > \frac{1}{2}$$

J	1	2	3	4
$\frac{1}{(2j-1)^2}$ when $\lvert j\rvert > \frac{1}{2}$	1	1/9	1/25	1/49

Table 30

12.3. Sum of terms of the diagonals of $X_{jk} = \left[\frac{1}{(2j-1)(2j)^k}\right]$:

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)(2j)^j} = 0.521795 \ldots \text{converges}$$

12.4. Sum of terms of the diagonals of $X_{jk} = \left[\frac{1}{(2j-1)(2j)^k}\right]$:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(2j-1)(2j)^k} = \frac{6}{8}\zeta(2) = \frac{\pi^2}{8} \quad \textit{converges}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1}{(2j-1)(2j)^k} = \frac{1}{8}(6\zeta(2) - 2\psi^{(1)}(n + \frac{1}{2}))$$

12.4. C-values for $X_{jk} = \left[\frac{1}{(2j-1)(2j)^k}\right]$:

$$C_n\left\{f(n) = \frac{1}{(2j-1)(2j)^n}\right\} = \lim_{m\to\infty}\left(\sum_{j=2}^{m} \frac{1}{(2j-1)(2j)^n} - \int \frac{dm}{(2m-1)(2m)^n}\right)$$

$$= -\ln(2) - \sum_{j=2}^{n} \frac{\zeta(j)}{2^j} - \frac{2^{-k} m^{1-k} \; {}_2F_1(1, 1-k; 2-k; 2m)}{k-1} = -\ln(2) - \sum_{j=2}^{n} \frac{\zeta(j)}{2^j}$$

n=2	n=3	n=4	$n \to \infty$
$\ln(2) - \frac{\zeta(2)}{2^2}$	$\ln(2) - \frac{\zeta(2)}{2^2} - \frac{\zeta(3)}{2^3}$	$\ln(2) - \frac{\zeta(2)}{2^2} - \frac{\zeta(3)}{2^3} - \frac{\zeta(4)}{2^4}$	
0.281913…	0.131656…	0.064011…	0

Table 31

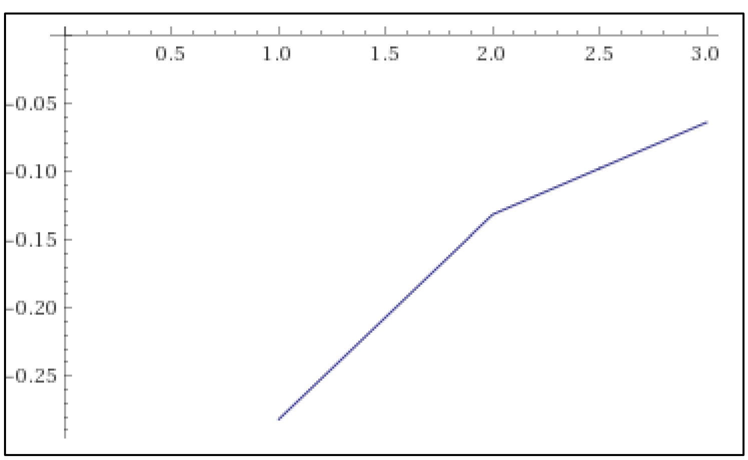

Figure 8. C-values for $X_{jk} = \left[\frac{1}{(2j-1)(2j)^k}\right]$

13. Case 6: Matrix $X_{ij} = [\mathbf{x_{jk}}] \in \mathbf{R}^{j \times k}$ where $\mathbf{x_{jk}} = \mathbf{j^{-ßk}}$ with $|\mathbf{j^{-Re(ß)}}| < 1$

13.1. Sum of terms of the rows of X_{jk}:

$$\sum_{k=1}^{\infty} j^{-ßk} = \frac{1}{j^{ß} - 1}$$

$$\sum_{k=1}^{m} j^{-ßk} = \frac{j^{-ßm}(j^{ßm} - 1)}{j^{ß} - 1}$$

$$\sum_{k=1}^{m} j^{-ßk} = \frac{j^{ß-ßm}}{j^{ß} - 1}$$

13.2. Sum of terms of the columns of X_{jk}:

$$\sum_{j=1}^{\infty} j^{-ßk} = \zeta(\beta k)$$

$$\sum_{j=1}^{n} j^{-ßk} = \zeta(\beta k) - \zeta(\beta k, n+1) = H_n^{(ßk)}$$

$$\sum_{j=n}^{\infty} j^{-ßk} = \zeta(\beta k, n)$$

13.3. Sum of terms of the matrix X_{ij}:

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty} j^{-ßk} = -n + \frac{1}{ß}\psi^{(0)}\left(1 - \frac{1}{ß}\right) - \frac{1}{ß}\psi^{(0)}\left(n - \frac{1}{ß} + 1\right) \quad \textit{diverges}$$

$$\sum_{j=n}^{\infty}\sum_{k=1}^{\infty} j^{-ßk} \text{ diverges for } |n| \le 1 \text{ or } |j^{\beta}| \le 1$$

$$\sum_{j=n}^{\infty}\sum_{k=2}^{\infty} j^{-ßk}$$

ß	n = 2
1	1
2	7/4-ζ(2) = 0.1050659
3	0.01963250
0.8	1.94463
0.7	2.90507
0.6	4.61421
$0.5 \le \frac{1}{n}$	Diverges

Table 32

14. Case 6: Matrix $X_{ij} = [\mathbf{x_{jk}}] \in \mathbf{R}^{j \times k}$ where $\mathbf{x_{jk}} = (\boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{j})^{-\mathbf{k}}$

14.1. Sum of terms of the rows of X_{jk}:

$$\sum_{k=1}^{\infty} (\alpha + \beta j)^{-k} = \frac{1}{(\alpha + \beta j) - 1} \qquad \text{for} \frac{1}{|(\alpha + \beta j)|} < 1$$

$$\sum_{k=1}^{m} (\alpha + \beta j)^{-k} = \frac{(\alpha + \beta j)^{-n} \, [(\alpha + \beta j)^{n} - 1]}{(\alpha + \beta j) - 1} \qquad \text{for} \frac{1}{|(\alpha + \beta j)|} < 1$$

14.1. Sum of terms of the columns of X_{jk}:

$$\sum_{j=1}^{\infty} (\alpha + \beta j)^{-k} = \beta^{-k} \, \zeta(k, \frac{\alpha + \beta}{\beta})$$

$$\sum_{j=1}^{n} (\alpha + \beta j)^{-k} = \beta^{-k} \, (\zeta(k, \frac{\alpha + \beta}{\beta}) - \zeta\left(k, \frac{\alpha + \beta}{\beta} + n\right)$$

14.2. Sum of terms of the diagonals of X_{ij}:

$$\sum_{j=1}^{\infty} (\alpha + \beta j)^{-k} = converges$$

α,ß	1	2	3	4
1	0.628474…	0.376408…	0.271444…	0.212813…
2	0.404668…	0.279835…	0.216403…	0.177041…
3	0.295079…	0.221851…	0.179611…	0.151426…
4	0.230955…	0.183342…	0.153328…	0.132195…

Table 33

14.4. Sum of terms of the matrix of X_{ij}:

$$\sum_{j=1}^{n} \sum_{k=1}^{\infty} (\alpha + \beta j)^{-k} = \frac{\psi^{(0)}\left(\frac{\alpha}{\beta} + n - \frac{1}{\beta} + 1\right) - \psi^{(0)}(\frac{\alpha}{\beta} - \frac{1}{\beta} + 1)}{\beta}$$

$$\sum_{j=1}^{r} \sum_{k=1}^{s} (\alpha + \beta j)^{-k} = \sum_{p=1}^{r} \frac{\sum_{q=0}^{r} (\alpha + p\beta)^{q}}{(\alpha + p\beta)^{s}}$$

Example:

$$\sum_{j=1}^{2} \sum_{k=1}^{3} (\alpha + \beta j)^{-k} = \frac{(\alpha + \beta)^2 + (\alpha + \beta) + 1}{(\alpha + \beta)^3} + \frac{(\alpha + 2\beta)^2 + (\alpha + 2\beta) + 1}{(\alpha + 2\beta)^3}$$

For α=2, $\beta = 4 \rightarrow$

$$\sum_{j=1}^{2} \sum_{k=1}^{3} (2 + 4j)^{-k} = 0.266106$$

15. Case 8: Matrix $X_{ij} = \left[\mathbf{x}_{\mathbf{jk}}\right] \in \mathbf{R}^{\mathbf{j\,x\,k}}$ where $\mathbf{x}_{\mathbf{jk}} = (\boldsymbol{\alpha} + \mathbf{j})^{-\mathbf{k}}$

15.1. Sum of terms of the rows of X_{ij}:

$$\sum_{k=1}^{\infty}(\alpha + j)^{-k} = \frac{1}{(\alpha + j) - 1} \qquad \text{for} \frac{1}{|(\alpha + j)|} < 1$$

$$\sum_{k=1}^{m}(\alpha + j)^{-k} = \frac{(\alpha + j)^{-n}\,[(\alpha + j)^{n} - 1]}{(\alpha + j) - 1} \qquad \text{for} \frac{1}{|(\alpha + j)|} < 1$$

15.2. Sum of terms of the columns of X_{jk}:

$$\sum_{j=0}^{\infty}(\alpha + j)^{-k} = \zeta(k, \alpha)$$

$$\sum_{j=1}^{\infty}(\alpha + j)^{-k} = \zeta(k, \alpha + 1)$$

$$\sum_{j=0}^{n}(\alpha + j)^{-k} = \zeta(k, \alpha) - \zeta(k, \alpha + n + 1)$$

$$\sum_{j=p}^{\infty}(\alpha + j)^{-k} = \zeta(k, \alpha + p)$$

15.3. Sum of terms of the diagonal of X_{ij}:

$$\sum_{j=0}^{\infty}(\alpha + j)^{-j} \quad \text{converges}$$

α=1	α=2	α=3	α=4	α=50
0.628474…	0.404668…	0.295079…	0.230955…	0.0199845…

Table 33

15.4. Sum of terms of the matrix of X_{ij}:

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty}(\alpha + j)^{-k} = \psi^{(0)}(\alpha + n) - \psi^{(0)}(\alpha)$$

The equation simplifies for initial values of j,k=2

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}(\alpha + j)^{-k} = \frac{1}{\alpha + 1} = \sum_{p=0}^{\infty}(-1)^{p}a^{p} = \sum_{p=0}^{\infty}(-1 + \alpha)^{p}(-1)^{p}2^{-(n+1)}$$

$$\sum_{j=2}^{n}\sum_{k=2}^{\infty}(\alpha + j)^{-k} = \frac{1}{\alpha + 1} - \frac{1}{\alpha + n}$$

$$\sum_{j=1}^{r}\sum_{k=1}^{s}(\alpha + j)^{-k} = \sum_{p=1}^{r}\frac{\sum_{q=0}^{r}(a + p)^{q}}{(a + p)^{s}}$$

Example:

$$\sum_{j=1}^{2}\sum_{k=1}^{3}(\alpha+j)^{-k}=\frac{(\alpha+1)^2+(\alpha+1)+1}{(\alpha+1)^3}+\frac{(\alpha+2)^2+(\alpha+2)+1}{(\alpha+2)^3}$$

For α=2, β = 4 →

$$\sum_{j=1}^{2}\sum_{k=1}^{3}(2+4j)^{-k}= 0.31007\ldots$$

15.5. C − values for $X_{ij}=\frac{1}{(1-k)^n}$:

$$C_n\left\{f(n)=\frac{1}{(1-k)^n}\right\}=\lim_{m\to\infty}\left(\sum_{k=2}^{m}\frac{1}{(1-k)^n}-\int\frac{dm}{(1-m)^n}\right)$$

$$=\lim_{m\to\infty}\left(\sum_{k=2}^{m}\frac{1}{(1-k)^n}-\frac{(1-m)^{1-n}}{(1-n)}\right)=\sum_{k=2}^{\infty}\frac{1}{(1-k)^n}+\frac{1}{(n-1)}$$

$$=(-1)^n\zeta(n)+\frac{1}{n-1}$$

n=2	n=3	n=4	n=5	$n\to\infty$
$\zeta(2)+1$	$-\zeta(3)+\frac{1}{2}$	$\zeta(4)+\frac{1}{3}$	$-\zeta(5)+\frac{1}{4}$	$(-1)^n\zeta(n)$

Table 34

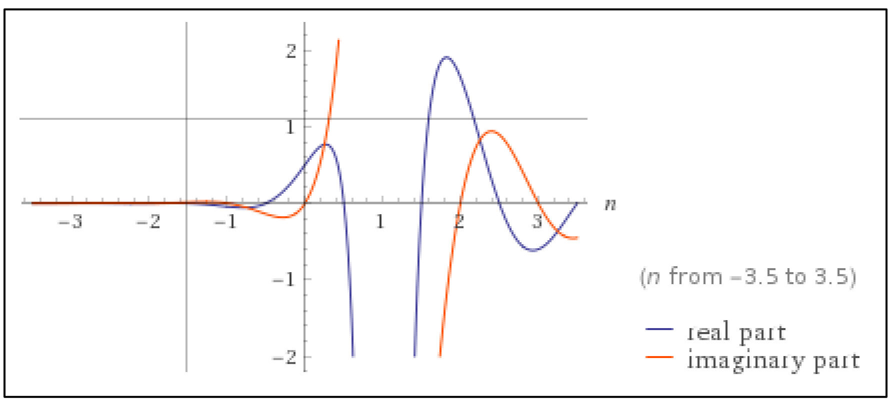

Figure 9. C-values for $X_{jk}=\frac{1}{(1-k)^n}$

16. Case 9: Matrix $X_{ij}=[x_{jk}]\in R^{j\times k}$ where $x_{jk}=(1+\beta j)^{-k}$

16.1. Sum of terms of the rows of X_{ij}:

$$\sum_{k=1}^{\infty}(1+\beta j)^{-k}=\frac{1}{\beta j}$$

Example: $\sum_{k=1}^{\infty}(1+3j)^{-k}=\frac{1}{3j}$

16.2. Sum of terms of the columns of X_{ij}:

$$\sum_{j=1}^{\infty}(1+\beta j)^{-k}=\beta^{-k}\,\zeta\left(k,\frac{\beta+1}{\beta}\right)$$

$$\sum_{j=1}^{n}(1+\beta j)^{-k}=\beta^{-k}(\,\zeta\left(k,\frac{\beta+1}{\beta}\right)-\zeta\left(k,\frac{\beta+1}{\beta}+n\right)$$

Example: $\sum_{j=1}^{\infty}(1+3j)^{-k}=3^{-k}\,\zeta\left(k,\frac{4}{3}\right)$

16.3. Sum of terms of the diagonals of X_{jk}:

$$\sum_{j=1}^{\infty}(1+\beta j)^{-j}=\frac{1}{1+ß}+\frac{1}{(1+2ß)^{2}}+\frac{1}{(1+3ß)^{3}}+\cdots \quad \text{converges}$$

ß=1	ß=2	ß=3	ß=4	ß=50
0.628474…	0.376408…	0.271444…	0.212813…	0.0197062…

Table 35

16.3. Sum of terms of the matrix of X_{ij}:

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty}(1+\beta j)^{-k}=\frac{\psi^{(0)}(n+1)-\psi^{(0)}(1)}{\beta}$$

$$\sum_{j=1}^{r}\sum_{k=1}^{s}(1+\beta j)^{-k}=\sum_{p=1}^{r}\frac{\sum_{q=0}^{r}(1+\beta p)^{q}}{(1+\beta p)^{s}}$$

Example:

$$\sum_{j=1}^{2}\sum_{k=1}^{3}(1+\beta j)^{-k}=\frac{(1+\beta)^{2}+(1+\beta)+1}{(1+\beta)^{3}}+\frac{(1+2\beta)^{2}+(1+2\beta)+1}{(1+2\beta)^{3}}$$

For $\beta=4\rightarrow$

$$\sum_{j=1}^{2}\sum_{k=1}^{3}(1+4j)^{-k}=0.372829$$

16.4. C − values for $X_{ij}=\frac{1}{(1+k)^{n}}$:

$$C_n\left\{f(n)=\frac{1}{(1+k)^{n}}\right\}=\lim_{m\to\infty}(\sum_{k=2}^{m}\frac{1}{(1+k)^{n}}-\int\frac{dm}{(1+m)^{n}})$$

$$=\lim_{m\to\infty}(\sum_{k=2}^{m}\frac{1}{(1+k)^{n}}-\frac{(1+m)^{1-n}}{(1-n)})=\sum_{k=2}^{\infty}\frac{1}{(1-k)^{n}}+\frac{1}{(n-1)})$$

$$=\zeta(n)-1+\frac{1}{n-1}$$

n=2	n=3	n=4	n=5	$n \to \infty$
$\zeta(2)$	$\zeta(3) - \frac{1}{2}$	$\zeta(4) - \frac{2}{3}$	$\zeta(5) - \frac{3}{4}$	0

Table 36

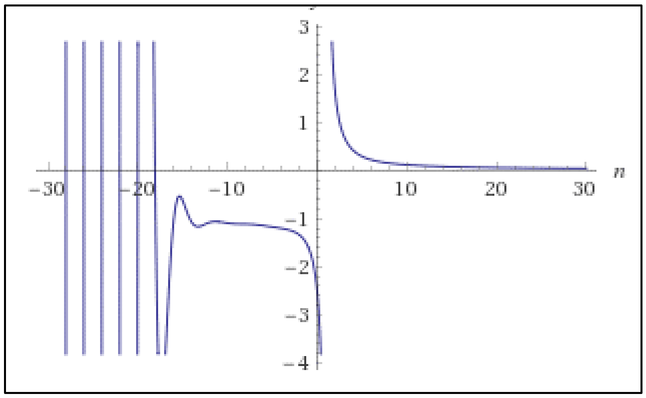

Figure 10. C-values for $X_{jk} = \frac{1}{(1+k)^n}$

17. Case 10: Matrix $X_{ij} = [x_{jk}] \in R^{j\,x\,k}$ where $x_{jk} = 1 + (\alpha + \beta j)^{-k}$

17.1. Sum of terms of the matrix of $X_{jk} = [x_{jk}] \in R^{j\,x\,k}$ where $x_{jk} = 1 + (\alpha + \beta j)^{-k}$:

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty}[1 + (\alpha + \beta j)^{-k}] =$$

$$\sum_{j=1}^{r}\sum_{k=1}^{s}[1 + (\alpha + \beta j)^{-k}] = rs + \sum_{p=1}^{s}\frac{1}{(\alpha + \beta)^{p}} + \sum_{p=2}^{r}\sum_{q=1}^{s}\frac{1}{(\alpha + p\beta)^{s}}$$

Example:

$$\sum_{j=1}^{2}\sum_{k=1}^{2}[1 + (\alpha + \beta j)^{-k}] = \frac{1}{(\alpha + 2\beta)} + \frac{1}{(\alpha + 2\beta)^2} + \frac{1}{(\alpha + \beta)} + \frac{1}{(\alpha + \beta)^2} + 4$$

For α=2, β = 4 →

$$\sum_{j=1}^{2}\sum_{k=1}^{3}[1 + (2 + 4j)^{-k}] = 6.31007$$

17.2. Product of terms of the matrix of $X_{jk} = [\mathrm{x_{jk}}] \in \mathrm{R^{j\,x\,k}}$ where $\mathrm{x_{jk}} = 1 + (\alpha + \beta \mathrm{j})^{-\mathrm{k}}$

A	ß	k	$\prod_{j=1}^{\infty}(1+(\alpha+\beta j)^{-k})$	Value	Partial Product Formula
1	1	2	$\prod_{j=1}^{\infty}(1+(1+j)^{-2})$	1.83804	$= \frac{\sinh(\pi)\,\Gamma(n+(2-i))\Gamma(n+(2+i))}{2\pi\Gamma(n+2)^2}$
1	1	3	$\prod_{j=1}^{\infty}(1+(1+j)^{-3})$	1.21409	$= \frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{2\pi}$
1	1	4	$\prod_{j=1}^{\infty}(1+(1+j)^{-4})$	1.08368	$= -\frac{\sin(\sqrt[4]{-i}\pi)\sin\left((-1)^{\frac{3}{4}}\pi\right)}{2\pi^2}$
2	1	4	$\prod_{j=1}^{\infty}(1+(2+j)^{-4})$	1.01993	$= -\frac{2^3\sin(\sqrt[4]{-i}\pi)\sin\left((-1)^{\frac{3}{4}}\pi\right)}{(2*2^3+1)\pi^2}$

Table 37

17.3. Product of terms of the matrix of $X_{jk} = [\mathrm{x_{jk}}] \in \mathrm{R^{j\,x\,k}}$ where $\mathrm{x_{jk}} = 1 - (\alpha \pm \beta \mathrm{j})^{-\mathrm{k}}$

α	ß	k	Infinite Product	Infinite Product	Partial Product Formula
1	1	2	$\prod_{j=1}^{\infty}(1-(1+j)^{-2})$	$\frac{1}{2}$	$= \frac{n+2}{2n+2}$
1	1	3	$\prod_{j=1}^{\infty}(1-(1+j)^{-3})$	0.809397	$= \frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\Gamma\left(n-\frac{i\sqrt{3}\pi}{2}+\frac{5}{2}\right)\Gamma\left(n+\frac{i\sqrt{3}\pi}{2}+\frac{5}{2}\right)}{3\pi(n+1)^3\Gamma(n+1)^2}$
1	1	4	$\prod_{j=1}^{\infty}(1-(1+j)^{-4})$	0.919019	$= \frac{\sinh(\pi)}{4\pi}$
1	1	2	$\prod_{j=3}^{\infty}(1-(1-j)^{-2})$	$\frac{1}{2}$	$= \frac{n}{2n+1}$
1	1	3	$\prod_{j=1}^{\infty}(1-(1-j)^{-3})$	1.21409	$= \frac{n*\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\Gamma\left(n-\frac{i\sqrt{3}\pi}{2}-\frac{1}{2}\right)\Gamma\left(n+\frac{i\sqrt{3}\pi}{2}-\frac{1}{2}\right)}{3\pi(n+1)^3\Gamma(n+1)^2}$
1	1	4	$\prod_{j=1}^{\infty}(1-(1-j)^{-4})$	0.919019	$= \frac{n*\sinh(\pi)\Gamma(n-i)\Gamma(n+i)}{4\pi(n-1)^3\Gamma(n-1)^2}$

Table 38

18. **Case 11: Matrix $X_{ij} = [x_{jk}] = [x_{jk}] \in R^{j \times k}$ where $x_{jk} = 1 \pm \left(\frac{z}{j}\right)^k$ and $x_{jk} = 1 \pm \left(\frac{z}{2j-1}\right)^k$.**
Euler- Ramanujan Infinite Products.

18.1. If k is EVEN: Given $z \in R$ and $q \in N$ and $k = 2q$, the following expressions are true:

$$\prod_{j=1}^{\infty}\left(1 - \left(\frac{z}{j}\right)^k\right) = i^{q+1}\left(\frac{-1}{\pi z}\right)^q \prod_{j=0}^{q-1} \sin\left(\pi z * e^{\frac{i\pi j}{q}}\right)$$

$$\prod_{j=1}^{\infty}\left(1 + \left(\frac{z}{j}\right)^k\right) = -i^{q}\left(\frac{-1}{\pi z}\right)^q \prod_{j=0}^{q-1} \sin\left(i * \pi z * e^{\frac{i\pi j}{q}}\right)$$

18.2. If k is EVEN: Given $z \in R$ and $q \in N$ and $k = 2q$, the following expressions are true:

$$\prod_{j=1}^{\infty}\left(1 - \left(\frac{z}{2j-1}\right)^k\right) = \prod_{j=0}^{q-1} \cos\left(\frac{\pi z}{2} * e^{\frac{i\pi j}{q}}\right)$$

$$\prod_{j=1}^{\infty}\left(1 + \left(\frac{z}{2j-1}\right)^k\right) = \prod_{j=0}^{q-1} \cos\left(\frac{i * \pi z}{2} * e^{\frac{i\pi j}{q}}\right)$$

18.3. If k is ODD: Given $z \in R$ and $q \in N$, the following expressions are true:

$$\prod_{j=1}^{\infty}\left(1 - \left(\frac{z}{j}\right)^k\right) = 1/\prod_{j=0}^{k-1} \Gamma\left(1 - z * (-1)^j * i^{\frac{2j}{k}}\right)$$

$$\prod_{j=1}^{\infty}\left(1 + \left(\frac{z}{j}\right)^k\right) = 1/\prod_{j=0}^{k-1} \Gamma\left(1 + z * (-1)^j * i^{\frac{2j}{k}}\right)$$

19. **Case 12: Matrix $X_{ij} = [x_{jk}] \in R^{j \times k}$ where $x_{jk} = (1 + j^{-k})$**

19.1. Sum of Rows (diverges)

$$\sum_{k=1}^{m}\left(1 + j^{-k}\right) = \frac{-j^{-m} + jm - m + 1}{1 - j}$$

19.2. Sum of Columns (diverges)

$$\sum_{j=1}^{n}\left(1 + j^{-k}\right) = n + H_n^{(k)} = n + \zeta(k)$$

19.3. Sum of Diagonal (diverges)

$$\sum_{j=1}^{n}\left(1 + j^{-j}\right) = n + \sum_{j=1}^{n}\frac{1}{j^j}$$

19.4. Product of columns (From Euler-Ramanujan products in section 17)

If k=2q even

$$\prod_{j=1}^{\infty}\left(1+j^{-k}\right)=\prod_{j=1}^{\infty}(1+(\frac{1}{j})^{k})= -i^{q}\left(\frac{-1}{\pi}\right)^{q}\prod_{j=0}^{q-1}\sin\left(i*\pi*e^{\frac{i\pi j}{q}}\right)$$

If k is odd

$$\prod_{j=1}^{\infty}\left(1+j^{-k}\right)=\prod_{j=1}^{\infty}(1+(\frac{1}{j})^{k})= 1/\prod_{j=0}^{k-1}\Gamma(1+(-1)^{j}*i^{\frac{2j}{k}})$$

We can also verify that:

$$\frac{\prod_{j=1}^{\infty}\left(1+j^{-k}\right)}{\prod_{j=1}^{\infty}(1+(1+j)^{-k})}=2*\prod_{k=2}^{\infty}\frac{(1+j^{-k})}{(1+j^{-k})}=2 \quad for\ all\ k>3$$

19.5. Product of Rows (converges). Using Pochhammer's notation:

$$\prod_{k=1}^{\infty}\left(1+j^{-k}\right)=\frac{1}{2}\left(-1;\frac{1}{j}\right)_{\infty}$$

We can also express it with:

$$\sum_{j=1}^{n}\sum_{k=1}^{m}(1+j^{-k})=\sum_{j=1}^{n}(\sum_{k=1}^{m}1+\sum_{k=1}^{m}j^{-k})=\sum_{j=1}^{n}(m+\frac{j^{m}(j^{m}-1)}{j-1})=nm+\sum_{j=1}^{n}\frac{j^{m}(j^{m}-1)}{j-1}$$

Example:

$$\sum_{j=1}^{2}\sum_{k=1}^{3}(1+j^{-k})=\frac{79}{8}$$

19.6. Sum (Main Diagonal)

$$\sum_{j=1}^{m}(1+j^{-j})= n+\sum_{j=1}^{n}j^{-j}$$

19.7. Product (Main Diagonal)

$$\lim_{n\to\infty}\prod_{j=2}^{n}(1+j^{-j})=1.30180595229\ldots$$

19.8. C −values for $X_{ij}=1+k^{-n}$:

$$C_n\{f(n)=1+k^{-n}\}=\lim_{m\to\infty}(\sum_{k=1}^{m}(1+k^{-n})-\int(1+m^{-n})\,dm)$$

$$=\lim_{m\to\infty}(\sum_{k=1}^{m}(1+k^{-n})-m+\frac{m^{1-n}}{(1-n)})$$

$$=\lim_{m\to\infty}(\sum_{k=1}^{m}k^{-n}+m-m+\frac{m^{1-n}}{(1-n)})= \zeta(n)$$

$CP_n\{f(n) = 1 + k^{-n}\} = \zeta(\mathrm{n})$

20. Case 13: Matrix $X_{ij} = [\mathbf{x_{jk}}] \in \mathbf{R}^{\mathbf{j\,x\,k}}$ where $\mathbf{x_{jk}} = (\mathbf{1} - \mathbf{j^{-k}})$

20.1. Sum of Rows (diverges)

$$\sum_{k=1}^{m}(1 - j^{-k}) = \frac{j^{-m}(m\,j^{m+1} - mj - j^m + 1)}{j - 1}$$

20.2. Sum of Columns (diverges)

$$\sum_{j=1}^{n}(1 - j^{-k}) = n - H_n^{(k)} = n - \zeta(k)$$

20.3. Sum of Diagonal (diverges)

$$\sum_{j=1}^{n}(1 - j^{-j}) = n - \sum_{j=1}^{n}\frac{1}{j^j}$$

20.4. Product of columns (From Euler-Ramanujan products in section 17)

If k=2q even

$$\prod_{j=1}^{\infty}(1 - j^{-k}) = \prod_{j=1}^{\infty}(1 - (\frac{1}{j})^k) = i^{q+1}\left(\frac{-1}{\pi}\right)^q \prod_{j=0}^{q-1}\sin\left(\pi * e^{\frac{i\pi j}{q}}\right)$$

If k is odd

$$\prod_{j=1}^{\infty}(1 - j^{-k}) = \prod_{j=1}^{\infty}(1 - (\frac{1}{j})^k) = 1/\prod_{j=0}^{k-1}\Gamma(1 - (-1)^j * i^{\frac{2j}{k}})$$

20.5. Product of Rows (converges). Using Pochhammer's notation:

$$\prod_{k=1}^{\infty}(1 - j^{-k}) = \left(\frac{1}{j};\frac{1}{j}\right)_{\infty}$$

We also can express it with:

$$\sum_{j=1}^{n}\sum_{k=1}^{m}(1 - j^{-k}) = \sum_{j=1}^{n}(\sum_{k=1}^{m}1 - \sum_{k=1}^{m}j^{-k}) = \sum_{j=1}^{n}(m - \frac{j^{-m}(m\,j^{m+1} - mj - j^m + 1)}{j - 1})$$

$$= nm - \sum_{j=1}^{n}\frac{j^{-m}(m\,j^{m+1} - mj - j^m + 1)}{j - 1}$$

Example:

$$\sum_{j=1}^{2}\sum_{k=1}^{3}(1 - j^{-k}) = -11$$

20.6. Sum (Main Diagonal)

$$\sum_{j=1}^{m}(1-j^{-j}) = n - \sum_{j=1}^{n} j^{-j}$$

20.7. Product (Main Diagonal) converges

$$\lim_{n\to\infty}\prod_{j=2}^{n}(1-j^{-j}) = 0.71915450096\,\ldots$$

20.8. The following relationship linking is quite interesting:

$$2 * \sum_{n=p}^{\infty} j^{-j} = \prod_{j=p}^{\infty}\left(1+j^{-j}\right) - \prod_{j=p}^{\infty}\left(1-j^{-j}\right) + O(p^4)$$

With $\lim_{p\to\infty} O(p^4) = 0$

Examples:

P	$2S - P^{+} + P^{-}$
2	-0.0000794572
3	-9.97611 $x10^{-8}$
4	-5.7 x 10^{-11}

Table 39

20.9. C −values for $X_{ij} = 1 - k^{-n}$:

$$C_n\{f(n) = 1 - k^{-n}\} = \lim_{m\to\infty}\left(\sum_{k=1}^{m}(1-k^{-n}) - \int (1-m^{-n})\, dm\right)$$

$$= \lim_{m\to\infty}\left(\sum_{k=1}^{m}(1-k^{-n}) - \frac{m(m^{-n}+(n-1))}{n-1}\right)$$

$$= \lim_{m\to\infty}\left(-\sum_{k=1}^{m} k^{-n} + m - m - \frac{m^{1-n}}{(1-n)}\right) = -\zeta(n)$$

$$C_n\{f(n) = 1 + k^{-n}\} = -\zeta(n)$$

21. Case 14: Matrix $X_{ij} = [x_{jk}] \in R^{j\,x\,k}$ where $x_{jk} = e^{j^{-k}}$

N	$\sum_{j=2}^{n}\sum_{k=2}^{n} e^{j^{-k}}$	Approximation $(n-1)^2+1$
2	$e^{-\left(\frac{1}{4}\right)}=0.7788$	2
3	4.57242	5
4	9.7335	10
5	16.8211	17
6	25.8723	26

Table 40

A good approximation is:

$$\sum_{j=2}^{n}\sum_{k=2}^{n} e^{j^{-k}} = (n-1)^2 + 1 + O(n^2)$$

22. Case 15: Matrix $X_{ij} = [x_{jk}] \in \mathbf{R}^{j \times k}$ **where** $x_{jk} = (j^{-k})!$

N	$\sum_{j=2}^{n}\sum_{k=2}^{n}(j^{-k})!$	Approximation $(n-1)^2 - 0.48$
2	0.9064	0.52
5	15.645	15.52
10	80.5703	80.52
20	360.541	360.52
30	840.532	840.52

Table 41

A good approximation is:

$$\sum_{j=2}^{n}\sum_{k=2}^{n}(j^{-k})! = (n-1)^2 - 0.48 + O(n^4)$$

23. Case 16: Matrix $X_{ij} = [x_{jk}] \in \mathbf{R}^{j \times k}$ **where** $x_{jk} = \cos(j^{-k})$

N	$\sum_{j=2}^{n}\sum_{k=2}^{n}\cos(j^{-k})$	Approximation $(n-1)^2$
2	0.968912	1
3	3.95426	4
4	8.95015	9
5	15.9488	16
10	80.9478	81
20	360.948	361

Table 42

A good approximation is:

$$\sum_{j=2}^{n}\sum_{k=2}^{n}\cos(j^{-k}) = (n-1)^2 + O(n^4)$$

24. Case 17: Matrix $X_{ij} = [x_{jk}] \in R^{j\,x\,k}$ where $x_{jk} = \binom{k}{j}$

24.1. Sum of elements of the Matrix X_{ij}

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\binom{j}{k} = \sum_{j=1}^{n}\sum_{k=1}^{n}\binom{k}{j} = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j!}{k!\,(j-k)!}$$

N	$\sum_{j=1}^{n}\sum_{k=1}^{n}\binom{j}{k}$	Approximation $2^{n+1} - n - 2$
1	1	1
2	4	4
3	11	11
4	26	26
5	57	57

Table 43

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\binom{j}{k} = \sum_{j=1}^{n}\sum_{k=1}^{n}\binom{k}{j} = 2^{n+1} - n - 2$$

This sum is related to Eulerian Numbers (A000295), k-step Lucas numbers(A125127), and many other positive integer relationships and sequences in combinatory theory (permutations, partitions, prism graphs, equivalence classes, pascal triangle, Hamming distances, binary sequences, Hamming perfect error codes, binomial transformations, binary trees transformation, …)

24.2. Sum of Rows

$$\sum_{k=1}^{n}\binom{k}{j} = \frac{(-j+n+1)\binom{n+1}{j} + (j-1)\binom{1}{j}}{j+1} = \frac{n(n+1)}{(j+1)!}\prod_{p=1}^{j-1}(n-p)$$

j	$\sum_{k=1}^{n}\binom{k}{j}$
1	$\frac{1}{2}n(n+1)$
2	$\frac{1}{6}(n-1)n\,(n+1)$
3	$\frac{1}{24}(n-2)(n-1)n(n+1$

Table 44

24.3. Sum of Columns

$$\sum_{j=1}^{n}\binom{k}{j} = -\binom{k}{n+1}\ {}_2F_1(1,-k+n+1;n+2;-1)+2^k-1$$

Where ${}_2F_1(a,b;c,x)$, is the Hypergeometric function.

k	$\sum_{j=1}^{n}\binom{k}{j}$
0	$\frac{1}{2}n(n+1)\binom{0}{n+1}\left(\psi^{(0)}\left(\frac{n}{2}+\frac{1}{2}\right)-\psi^{(0)}\left(\frac{n}{2}+1\right)\right)$
1	$1-\binom{1}{n+1}\ {}_2F_1(1,n;n+2;-1)$
2	$3-\binom{2}{n+1}\ {}_2F_1(1,n-1;n+2;-1)$
3	$7-\binom{3}{n+1}\ {}_2F_1(1,n-2;n+2;-1)$

Table 45

24.4. Sum of Main Diagonal (diverges)

$$\sum_{j=1}^{n}\binom{j}{j} = n$$

25. Case 18: Matrix $X_{ij} = [x_{jk}] \in \mathbf{R}^{j\times k}$ where $x_{jk} = \frac{1}{k}(j^{-k})$

25.1. Sum of members of the Matrix X_{ij}:

$$\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{1}{k}(j^{-k}) = -H_n - \ln(\Gamma(n)) + \ln(\Gamma(n+1)) + 1$$

N	$\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{1}{k}(j^{-k})$
2	0.1250000
3	0.2345679
4	0.2907142
∞	0.420286

Table 46

25.2. Sum of rows:

$$\sum_{k=1}^{n}\frac{1}{k}(j^{-k}) = \left(\frac{1}{j}\right)^{n+1}\left(-\Phi\left(\frac{1}{j},1,n+1\right)\right) - \ln\left(\frac{(j-1)}{j}\right)$$

Where $\Phi(z,s,q)$, is the Lerch Transcendent function:

Example: $\sum_{k=1}^{n}\frac{1}{k}(2^{-k}) = \left(\frac{1}{2}\right)^{n+1}\left(-\Phi\left(\frac{1}{2},1,n+1\right)\right) + \ln(2)$

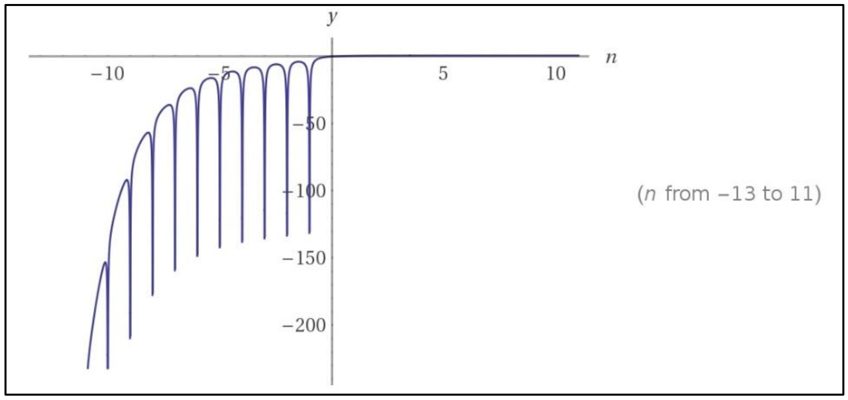

Figure 11. C-values for $X_{jk} = \frac{1}{k} j^{-k}$

25.3. Sum of rows:

$$\sum_{j=1}^{\infty} \frac{1}{k}\left(j^{-k}\right) = \frac{\zeta(k)}{k}$$

25.4. Sum of diagonal converges:

$$\sum_{j=1}^{\infty} \frac{1}{j}\left(j^{-j}\right) = \sum_{j=1}^{\infty} j^{-j-1} = 1.13839 \ldots$$

26. Case 19: Matrix $X_{ij} = [x_{jk}] \in C$ where $x_{jk} = j^{-k}$ with $j = re^{i\theta} = r(\cos\theta + i\sin\theta)$

26.1. Sum of terms of the rows of X_{jk} :

$$R_j = \sum_{k=1}^{\infty} j^{-k} = \sum_{k=1}^{\infty} [r(\cos\theta + i\sin\theta)]^{-k} = \frac{(re^{i\theta})^{-n}\left(-1 + (re^{i\theta})^{n}\right)}{-1 + re^{\theta t}}$$

Example for j=4:

R	**θ**	R_j
4	0	1/3
4	π/6	$\frac{1}{2\sqrt{3} + (-1 + 2i)}$
4	π/4	$\frac{\sqrt{2}}{-\sqrt{2} + (4 + 4i)}$
4	π/3	$\frac{\sqrt{3} + 7i}{-13\sqrt{3} + 13i}$
4	π/2	$\frac{-1 - 4i}{17}$
4	Π	-1/5
4	-π/2	$\frac{-1 + 4i}{17}$

Table 47

A general expression for some values of **θ**:

R	**θ**	R_j
R	0	$\frac{1}{r-1}$
R	π/2	$\frac{-1-r*i}{(r^2+1)}$
R	Π	$\frac{-1}{r+1}$
R	-π/2	$\frac{-1+r*i}{(r^2+1)}$

Table 48

26.2. Sum of terms of the columns of X_{jk}:

$$C_k = \sum_{r=1}^{\infty} [re^{i\theta}]^{-k}$$

$i^k = (-1) \rightarrow$ Example for k=2, 6, 10, ...,2(2n-1):

θ	C_k	$C_k$generic
0	$\zeta(2)$	$\zeta(k)$
π/6	$-e^{\frac{2i\pi}{3}}\zeta(2)$	
π/4	$-i\zeta(2)$	$-i\zeta(k)$??
π/3	$-e^{\frac{i\pi}{3}}\zeta(2)$	
π/2	$-\zeta(2)$	$-\zeta(k)$
Π	$\zeta(2)$	$\zeta(k)$
-π/2	$-\zeta(2)$	$-\zeta(k)$

Table 49

$i^k = (-i) \rightarrow$ Example for k=3:

θ	CC_k
0	$\zeta(3)$
π/6	$-i\zeta(3)$
π/4	$-e^{\frac{i\pi}{4}}\zeta(3)$
π/3	$-\zeta(3)$
π/2	$i\zeta(3)$
Π	$-\zeta(3)$
-π/2	$-i\zeta(3)$

Table 50

If $i^k = 1 \rightarrow$ Example for k=4:

$\boldsymbol{\theta}$	CC_k
0	$\zeta(4)$
$\pi/6$	$e^{\frac{-i\pi}{3}}\zeta(4)$
$\pi/4$	$-\zeta(4)$
$\pi/3$	$e^{\frac{2i\pi}{3}}\zeta(4)$
$\pi/2$	$\zeta(4)$
Π	$\zeta(4)$
$-\pi/2$	$\zeta(4)$

Table 51

$i^k = i \rightarrow$ Example for k=5:

$\boldsymbol{\theta}$	CC_k
0	$\zeta(5)$
$\pi/6$	$-e^{\frac{i\pi}{6}}\zeta(5)$
$\pi/4$	$e^{\frac{3i\pi}{4}}\zeta(5)$
$\pi/3$	$e^{\frac{i\pi}{3}}\zeta(5)$
$\pi/2$	$-i\zeta(5)$
Π	$-\zeta(5)$
$-\pi/2$	$i\zeta(5)$

Table 52

26.3. Sum of terms of the matrix X_{ij}:

$$\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}[re^{i\theta}]^{-k} = \lim_{n\to\infty}(-e^{-it}\,(\psi^{(0)}(1-e^{-i\boldsymbol{\theta}}) - \psi^{(0)}(n+1-e^{-i\boldsymbol{\theta}}))$$

This sum does not converge.

We will calculate the infinite sums starting on r=2, k=2:

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[re^{i\theta}]^{-k} = \lim_{n\to\infty}\frac{(\psi^{(0)}(n+1-e^{-i\boldsymbol{\theta}}) - \psi^{(0)}(n+1) - \psi^{(0)}(2-e^{-i\boldsymbol{\theta}}) - \gamma + 1}{e^{i\boldsymbol{\theta}}})$$

In 4.4. we defined a set of new constants:

$$CZ_{2,0,2,1} = \sum_{j=1}^{\infty}[\zeta(2j) - \zeta(2j+1)] = 0.5$$

$$CZ_{4,0,4,-2} = \sum_{j=1}^{\infty}\ [-\zeta(4j-2)) + \zeta(4j)] = -0.5766744746\ldots$$

$$CZ_{4,1,4,-1} = \sum_{j=1}^{\infty}[-\zeta(4j-1)) + \zeta(4j+1)] = -0.171865985524\ldots$$

Values for some θ:

Θ	*Sum Matrix* $[re^{i\theta}]^{-k}$	Numeric result
0	1	1
π/6	$\dfrac{-2\left(-1+\gamma+\psi^{(0)}\left(\left(2+\frac{i}{2}\right)-\frac{\sqrt{3}}{2}\right)\right)}{\sqrt{3}+i}$	0.2240 – 0.839681 i
π/4	$\dfrac{-(1-i)\left(-1+\gamma+\psi^{(0)}\left(2-\frac{1-i}{2}\right)\right)}{\sqrt{2}}$	−0.240552 – 0.741055 i
π/3	$(-1)^{\frac{2}{3}}\left(-1+\gamma+\psi^{(0)}\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)\right)$	−0.529755 – 0.464869 i
π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right)$	$CZ_{4,0,4,-2} - CZ_{4,1,4,-1}\ i$
Π	1/2	$CZ_{2,0,2,1}$
-π/2	$-i\left(-1+\gamma+\psi^{(0)}(2-i)\right)$	$CZ_{4,0,4,-2} + CZ_{4,1,4,-1}\ i$

Table 53

27. Case 20: Matrix $X_{ij} = [\mathbf{x_{jk}}] \in \mathbf{C}$ where $\mathbf{x_{jk}} = (\mathbf{1} + \mathbf{re^{i\theta}})^{-\mathbf{k}}$

27.1. Sum of terms of the rows of X_{jk}:

$$\sum_{k=2}^{\infty}[1 + r(\cos\theta + i\sin\theta)]^{-k} = \lim_{n\to\infty}\frac{(re^{i\boldsymbol{\theta}})^{-n}\ (-1 + (re^{i\boldsymbol{\theta}})^{n})}{-1 + re^{i\boldsymbol{\theta}}}$$

Example:

r	Θ	Sum Rows
4	0	$\frac{1}{r(r+1)} = \frac{1}{20}$
4	π/6	$-\frac{(-1)^{\frac{5}{6}}}{r\left(1+r(-1)^{\frac{1}{6}}\right)} = 0.0299 - 0.0414\, i$
4	π/4	$-\frac{(-1)^{\frac{3}{4}}}{r\left(1+r(-1)^{\frac{1}{4}}\right)} = 0.00780 - 0.051939\, i$
4	π/3	$-\frac{(-1)^{\frac{2}{3}}}{r\left(1+r(-1)^{\frac{1}{3}}\right)} = -0.0178571 - 0.051549\, i$
4	π/2	$\frac{1}{r(i-r)} = -\frac{1}{17}\,(1+\, i/4)$
4	Π	$\frac{1}{r(r-1)} = \frac{1}{12}$
4	-π/2	$\frac{-1}{r(r+i)} = -\frac{1}{17} * (1 - i/4)$

Table 54

27.2. Sum of terms of the columns of X_{jk}:

$$\sum_{r=2}^{\infty}[1+re^{i\theta}]^{-k} = \lim_{n\to\infty}(e^{i\theta})^{-k}(\zeta\left(k, e^{-i\theta}(1+2e^{i\theta})\right) - \zeta(k, e^{-i\theta}(1+2e^{i\theta})+n-1)$$

k	θ	Sum Columns
2	0	$\zeta(k,3) = \zeta(2) - 5/4$
2	π/6	$(-1)^{\frac{k}{6}}\,\zeta\left(k, 2-(-1)^{\frac{5}{6}}\right) = 0.2712754 - 0.305228\, i$
2	π/4	$(-1)^{\frac{k}{4}}\,\zeta\left(k, 2-(-1)^{\frac{3}{4}}\right) = 0.126667 - 0.40693\, i$
2	π/3	$(-1)^{\frac{k}{3}}\,\zeta\left(k, 2-(-1)^{\frac{2}{3}}\right) = -0.057305 - 0.449735\, i$
2	π/2	$i^{-k}\zeta(k, 2-i) = -0.4630 - 0.294233\, i$
2	Π	$(-1)^{k}\,\zeta(k)$
2	-π/2	$i^{-k}\zeta(k, 2+i) = -0.4630 + 0.294233\, i$

Table 55

27.3. Sum of terms of the matrix X_{ij}:

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[1+re^{i\theta}]^{-k} =$$

$$= \lim_{n\to\infty} \frac{(-\psi^{(0)}(n+1-e^{-i\theta}) + \psi^{(0)}(n+1) + \psi^{(0)}(2+e^{-i\theta}) + \gamma - 1}{e^{i\theta}})$$

Θ	*Sum Matrix* $[1+re^{i\theta}]^{-k}$
0	$CZ_{2,0,2,1}$=1/2
π/6	$-(-1)^{\frac{5}{6}}\left(1+\gamma+\psi^{(0)}\left(\frac{1}{2}((4-i)+\sqrt{3})\right)\right)$
π/4	$-(-1)^{\frac{3}{4}}\left(-1+\gamma+\psi^{(0)}\left(2-(-1)^{\frac{3}{4}}\right)\right)$
π/3	$-(-1)^{\frac{2}{3}}\left(-1+\gamma+\psi^{(0)}\left(2-(-1)^{\frac{2}{3}}\right)\right)$
π/2	$-i\left(-1+\gamma+\psi^{(0)}(2-i)\right) = CZ_{4,0,4,-2} + CZ_{4,1,4,-1}\, i$
Π	$2 * CZ_{2,0,2,1} = 1$
-π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right) = CZ_{4,0,4,-2} - CZ_{4,1,4,-1}\, i$

Table 56

28. Case 21: Matrix $X_{ij} = [x_{jk}] \in C$ where $x_{jk} = (1 - re^{i\theta})^{-k}$

28.1. Sum of terms of the rows of X_{jk}:

$$\sum_{k=2}^{\infty}[1-re^{i\theta}]^{-k} = \lim_{n\to\infty} \frac{e^{-i\theta}(1-re^{i\theta})^{-n}((1-re^{i\theta})^{n} + re^{i\theta} - 1)}{r(-1+re^{i\theta})}$$

Example:

r	θ	Sum Rows
4	0	$\frac{1}{r(r-1)} = 1/12$
4	π/6	$-\frac{(-1)^{\frac{5}{6}}}{r\left(-1+r(-1)^{\frac{1}{6}}\right)} = 0.02814 - 0.07357\, i$
4	π/4	$-\frac{(-1)^{\frac{3}{4}}}{r\left(-1+r(-1)^{\frac{1}{4}}\right)} = -0.015584 - 0.0725745\, i$
4	π/3	$-\frac{(-1)^{\frac{2}{3}}}{r\left(-1+r(-1)^{\frac{1}{3}}\right)} = -0.04807 - 0.049963\, i$
4	π/2	$\frac{1}{r(i+r)} = \frac{1}{17}(\frac{i}{4} - 1)$
4	Π	$\frac{1}{r(r+1)} = \frac{1}{20}$
4	-π/2	$\frac{1}{r(i-r)} = \frac{-1}{17}(\frac{i}{4} + 1)$

Table 57

28.2. Sum of terms of the columns of X_{jk}:

$$\sum_{r=2}^{\infty}[1-\mathrm{re}^{\mathrm{i}\theta}]^{-k} = \lim_{n\to\infty}(-e^{i\theta})^{-k}(\zeta\left(k,-e^{-i\theta}(1-2e^{i\theta})\right)-\zeta(k,-e^{-i\theta}(1-2e^{i\theta})+n-1)$$

k	Θ	Sum Columns
2	0	$(-1)^{k}\,\zeta(k)$
2	π/6	$(-1)^{\frac{-k}{6}}\,\zeta\left(k,2+(-1)^{\frac{5}{6}}\right)=-0.06049-1.17639\,i$
2	π/4	$(-1)^{\frac{-k}{4}}\,\zeta\left(k,2+(-1)^{\frac{3}{4}}\right)=-0.56851-0.72593\,i$
2	π/3	$(-1)^{\frac{-k}{3}}\,\zeta\left(k,2+(-1)^{\frac{2}{3}}\right)=-0.694162-0.274495\,i$
2	π/2	$-i^{-k}\zeta(k,2+i)=\ -0.4630+0.294233\,i$
2	Π	$\zeta(k,3)=\zeta(2)-5/4$
2	-π/2	$i^{-k}\zeta(k,2-i)=\ -0.4630-0.294233\,i$

Table 58

28.3. Sum of terms of the matrix X_{ij}:

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[1-\mathrm{re}^{\mathrm{i}\theta}]^{-k}$$

$$=\lim_{n\to\infty}\frac{(\psi^{(0)}(n+1-e^{-i\theta})-\psi^{(0)}(n+1)-\psi^{(0)}(2-e^{-i\theta})-\gamma+1}{e^{i\theta}})$$

$$=\frac{\left(\psi^{(0)}(e^{-i\theta}-2\right)-\gamma+1}{e^{i\theta}} \quad \textit{when}\ e^{i\theta}\neq 0$$

Values:

Θ	*Sum Matrix* $[1-\mathrm{re}^{\mathrm{i}\theta}]^{-k}$
0	$2*CZ_{2,0,2,1}=1$
π/6	$(-1)^{\frac{5}{6}}\,(-1+\gamma+\psi^{(0)}\left(2+(-1)^{\frac{5}{6}}\right)$
π/4	$(-1)^{\frac{3}{4}}\,(-1+\gamma+\psi^{(0)}\left(2+(-1)^{\frac{3}{4}}\right))$
π/3	$(-1)^{\frac{2}{3}}\,(-1+\gamma+\psi^{(0)}\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right))$
π/2	$i\,\left(-1+\gamma+\psi^{(0)}(2+i)\right)=CZ_{4,0,4,-2}-CZ_{4,1,4,-1}\,i$
Π	$CZ_{2,0,2,1}=1/2$
-π/2	-i $\left(-1+\gamma+\psi^{(0)}(2-i)\right)=CZ_{4,0,4,-2}+CZ_{4,1,4,-1}\,i$

Table 59

29. Case 22: Matrix $X_{ij} = [\mathbf{x_{jk}}] \in \mathbf{C}$ where $\mathbf{x_{jk}} = \mathbf{j^{-z}}$ with $z \in C$

29.1 Let's define $z = \alpha + ßi$, then:

$$\mathbf{x_{jk}} = \mathbf{j^{-z}} = \mathbf{j^{\alpha+ßi}} = \mathbf{j^{\alpha}}\,(\mathbf{cos}(ß\,\mathbf{ln}(\mathbf{j})) + \mathbf{i\,sin}(ß\,\mathbf{ln}(\mathbf{j}))$$

The C-Transformation of x_{jk} is given by:

$$C_n\{f(z) = j^{-z}\} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(z) - \int f(n)\,dn\right)$$

and

$$\int f(n)\,dn = \int n^{-z}\,dn = \frac{1}{(1-\alpha) - iß}\,n^{(1-\alpha)-ßi}$$

That we can write:

$$\int f(n)\,dn = [n^{(1-\alpha)}\,[\cos(ß * ln(n) - i\,sin(ß * ln(n))]] * \frac{[(1-\alpha) + iß]}{[(1-\alpha)^2 + ß^2]}$$

We can then write the real and imaginary part of $C_n\{f(z) = j^{-z}\}$:

(a) $Re[C_n\{f(z) = j^{-z}\}] = \sum_{k=1}^{n} k^{-\alpha}\,(\cos(ß * \ln(k))-$

$- n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))]$

(b) $Im[C_n\{f(z) = j^{-z}\}] = \sum_{k=1}^{n} k^{-\alpha}\,(\sin(ß * \ln(k))+$

$+ n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))]$

We can calculate the following table:

$z = \alpha + iß$	$\lim_{n\to\infty} C_n\{f(z) = j^{-z}\}$	ζ(z)
(2,0)	1.644934 + i*0	ζ(2,0)
(3,0)	1.202057 + i*0	ζ(3,0)
(1, 1)	0.582096 + i* 0.9269	ζ(1,1)
(1/2, 14.134725…)	0 + i*0	Zero of the ζ function

Table 60. Values of $C_n\{f(z) = j^{-z}\}$

We can see that if $z = \alpha + i\beta \in C$ with α>0, $then \lim_{m\to\infty} C_n\{f(z) = j^{-z}\}$= ζ(z) when Re(z)= α≥0

29.2. Let's define four functions derived from $C_n\{f(z) = j^{-z}\}$:

f1(z) = $\sum_{k=1}^{n} k^{-\alpha}$ (cos(ß $*$ ln(k))

f2(z) = [$n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß*sin(ß*ln(n))]]

f3(z) = $\sum_{k=1}^{n} k^{-\alpha}$ (sin(ß $*$ ln(k))

f4(z) = - $n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))]

And let's call:

X(z) = f1(z) +i* f3(z)

Y(z) = f2(z) +i* f4(z)

In general, we can now express that any solution in Z of ζ(z) as:

$$\zeta(z) = \lim_{n\to\infty}[f1(z) - f2(z)] + i * [f3(z) - f4(z)]$$

Or

$$\zeta(z) = \lim_{n\to\infty}[X(z) - Y(z)]$$

Where:

X(z) = $\sum_{k=1}^{n} k^{-\alpha}$ (cos(ß $*$ ln(k)) + i* $\sum_{k=1}^{n} k^{-\alpha}$ (sin(ß $*$ ln(k))

Y(z)= [($n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß*sin(ß*ln(n))])

+ i ($n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))])]

Let's observe that if z*= α+iß as a non-trivial zero of ζ(z), then:

X(z)=Y(z)

f1(z) = f2(z) and f3(z) = f4(z)

and

Re(ζ(z*))= 0 and Im(ζ(z*))= 0 implies:

$\sum_{k=1}^{n} k^{-\alpha}$ (cos(ß $*$ ln(k))= $n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß*sin(ß*ln(n))]

$\sum_{k=1}^{n} k^{-\alpha}$ (sin(ß $*$ ln(k))= - $n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))]

Let's represent graphically the following four wave functions f1(z), -f2(z), f1(z), -f2(z) when $n \to \infty$ for different values of z=α+iß :

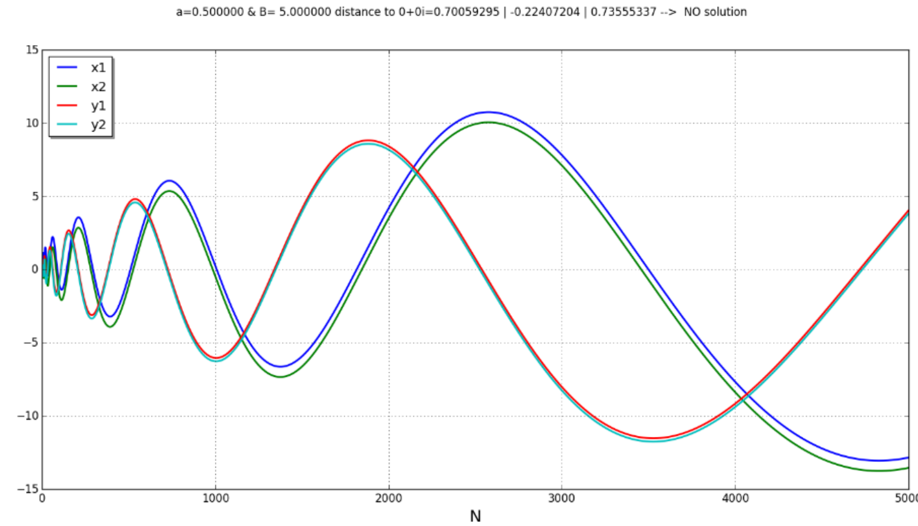

*Figure 12: z= ½ + i*5*

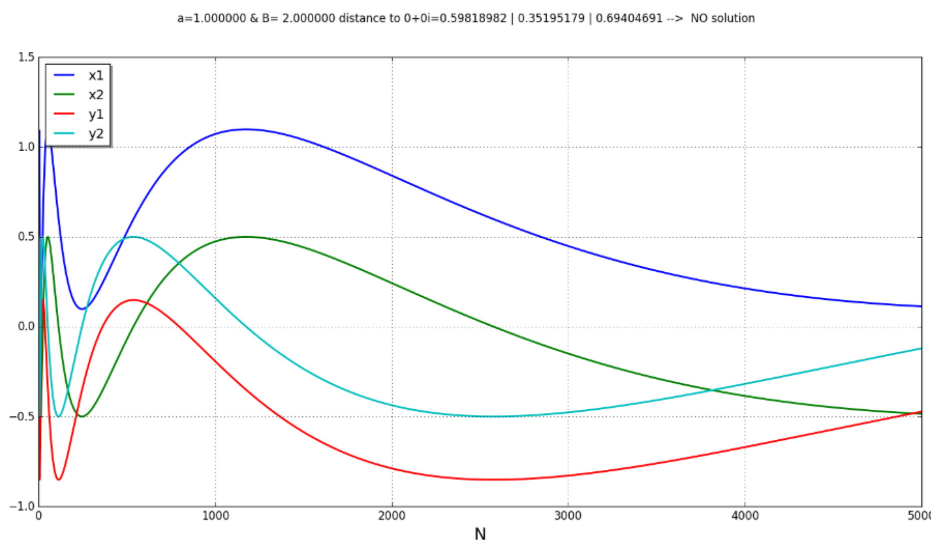

Figure 13: z=1+2i

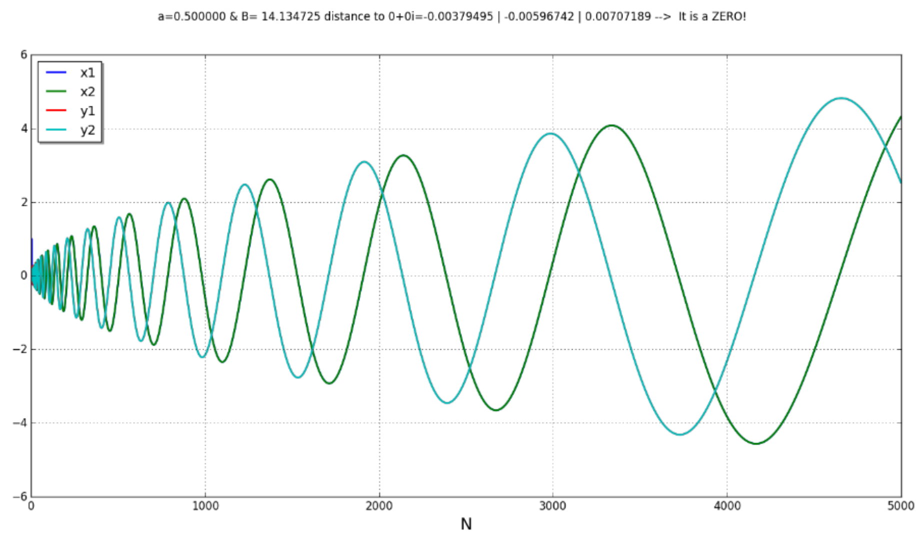

*Figure 14 : z= ½ + 14.134725… *i*

The observations that can be drawn from these charts for multiple iterations over α,ß are :

- The waves evolve around x=0 if Re(z)=α<1
- The graphs, as wave functions, evolve around the x axis if α=1/2 as all four partial functions are of the type f(x)=f($\sqrt{n}$):

  $f(n) = A\sqrt{n}\ (B * \cos(\ln(Cn) \pm D * \sin(\ln(Cn))$ with A, B, C, D =constant

- If z* is a known non-trivial zero of ζ(z), such as R(1)=1/2+i*14.134725… the 4 graphs collapse into 2, as in figure 4. We just checked the obvious evidence that for z* non-trivial zeros of ζ(z) the following obvious equalities happen:

  f1(z) = - f2(z) and
  f3(z) = - f4(z)

29.3. Modulus of X(z) and Y(z)

Let's calculate and plot the distance of ζ(z) to the origin given by its modulus defined for *z=a+ib* as:

$$| z |^2 = x^2 + b^2$$

The modulus is the distance of z to the origin.

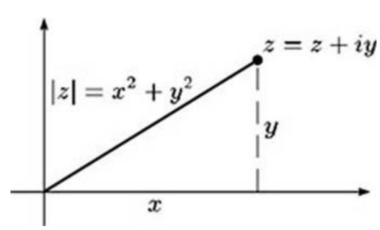

Figure 15. Modulus of a complex number

The modulus of ζ(z) will be given by:

$$|\zeta(z)|^2 = [f1(z)) - f2(z)]^2 + [(f3(z) - f4(z)]^2$$

This modulus must be zero when z=z* a zero of ζ(z).

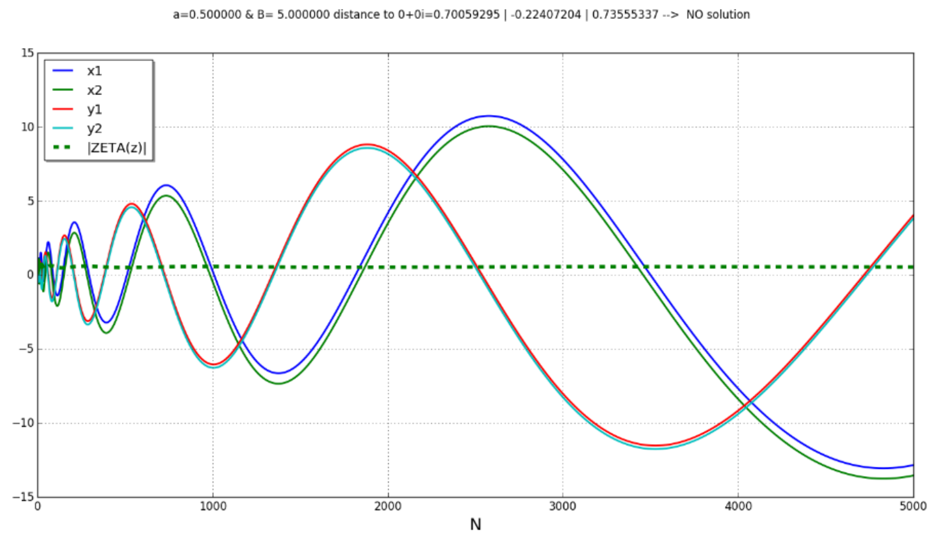

*Figure 16: z= ½ + i*5 |ζ(z)|² > 0.7005… for n>5000 and growing as n → ∞*

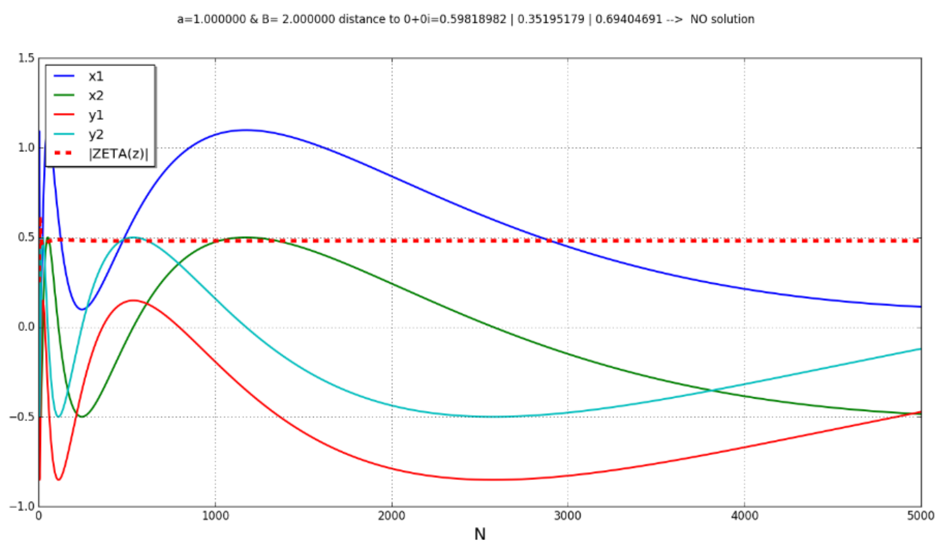

*Figure 17: z= 1 + i*2 |ζ(z)|² > 0.5981… for n>5000 n → ∞*

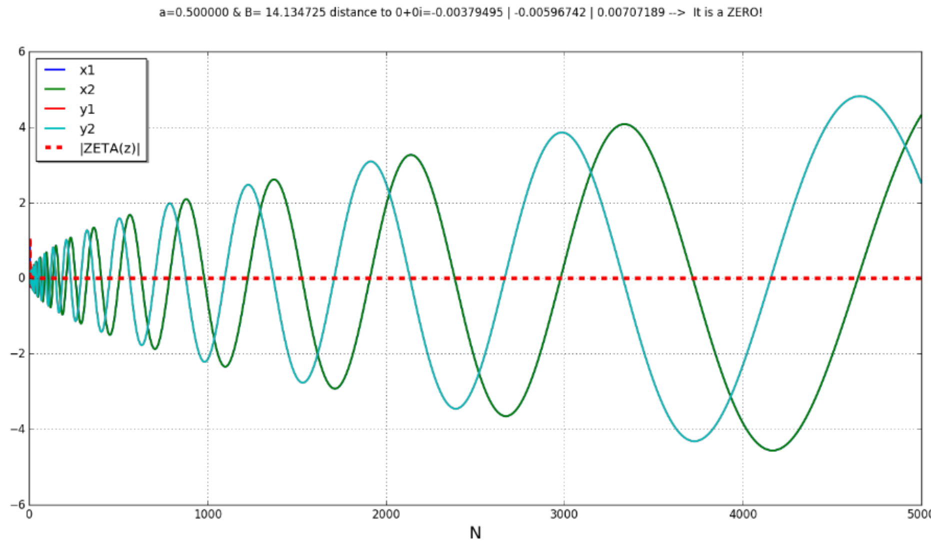

*Figure 18: z= ½ + 14.1347251417346 *i |ζ(z)|² < infinitesimal for n>5000*

29.4 The function Y(z) and its modulus |Y(z)|²

We defined

$$Y(z)= [(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))])$$

$$+ i (n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))])]$$

the value of |Y(z)|² is therefore:

$$|Y(z)|^2= [(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))])^2$$

$$+ (n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))])^2]$$

Simplifying, we obtain a very important formula for |y(z)|:

$|Y(z)|^2 = n^{2(1-\alpha)} * \frac{1}{[ß^2+(1-\alpha)^2]}$ -> |y(z)|² is a polynomial function.

29.5 Lemma xx: |Y(z)|² is a line for α=1/2

The slope for any |Y(z)|² with respect to n is given by:

$$slope(|y(z)|2) = d(|Y(z)|\ 2)/dn$$

Which equals to:

$$slope(|Y(z)|2) = 2(1-\alpha)\ n^{1-2\alpha} * \frac{1}{[ß^2 + (1-\alpha)^2]}$$

|Y(z)| ² is a line when the slope is constant, which can only happen if and only if

$$(1 - 2\alpha) = 0 \;\; => \;\; \alpha = 1/2$$

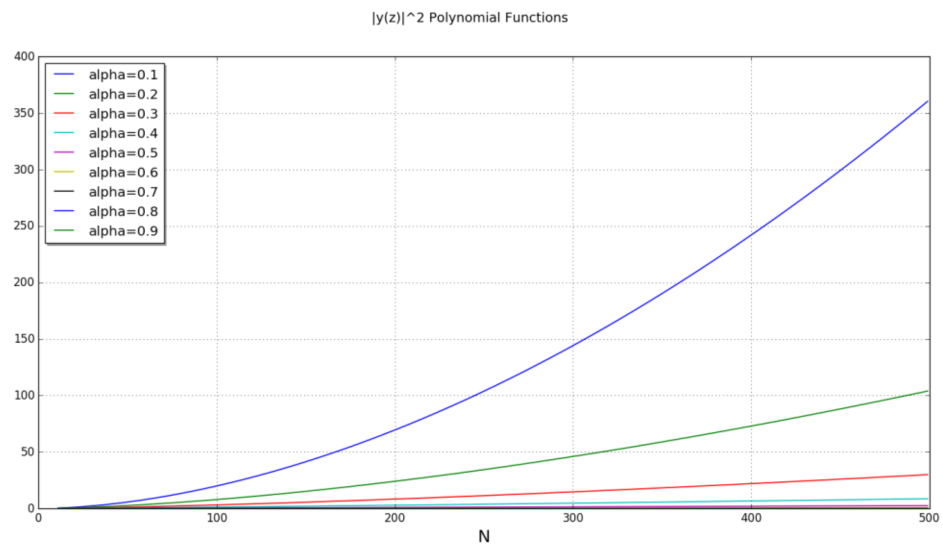

Figure 19. Polynomial representations of |y(z)|²

When α=1/2, the slope of $|y(z)|^2$ is:

$$\text{slope } |y(z)|^2 = \frac{1}{[ß^2+1/4]} \text{ for } \alpha=1/2 \qquad \text{with } z=\alpha+iß$$

29.5. The function X(z) and its modulus |x(z)|²

We have defined:

$$X(z) = \sum_{k=1}^{n} k^{-\alpha} (\cos(ß * \ln(k)) \; + \; i^* \sum_{k=1}^{n} k^{-\alpha} (\sin(ß * \ln(k))$$

The modulus of x(z) will be calculated by:

$$|x(z)|^2 = (\sum k^{-\alpha} \cos(ß \ln(n))^2 + (\sum k^{-\alpha} \sin(ß \ln(n))^2$$

The square of an infinite series will need some algebraic manipulation. We will simplify this calculation using the following expressions:

a) $(\sum_{n=1}^{N} a_n)(\sum_{n=1}^{N} b_n) = \sum_{n=1}^{N} a_n b_n + \sum_{n=1}^{N} \sum_{m \neq n}^{N} a_n * b_n$

b) $\cos(ß\ln(k))*\cos(ß\ln(j)) + \sin(ß\ln(k))*\sin(ß\ln(j)) = \cos(ß\ln(k)-ß\ln(k)) = cos(ß(\ln\left(\frac{k}{j}\right)))$

to obtain a workable expression for |x(z)|:

$$|x(z)|^2 = \sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß(\ln\left(\frac{k}{j}\right)\right)$$

If we change the indices of the sums we have to deduct the main diagonal:

$$|x(z)|^2 = \sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) - \sum_{k=1}^{n} k^{-2\alpha} \cos(ß\ln(1))$$

$$|x(z)|^2 = \sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) - \sum_{k=1}^{n} k^{-2\alpha}$$

And:

$$|x(z)|^2 = \sum_{k=1}^{n}\sum_{j1}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)$$

Examples:

α	ß	$\lim_{n\to\infty} \|x(z)\|^2$	$\|\zeta(\alpha,z)\|^2$
2.0	7	1.074756	1.074756
3.0	2	0.968687	0.968687

Table 61

We can see that:

$$\lim_{n\to\infty} |x(z)|^2 = |\zeta(\alpha,z)|^2 = \zeta(\alpha + ßi) * \zeta(\alpha - ßi) \quad \text{for } \alpha>1$$

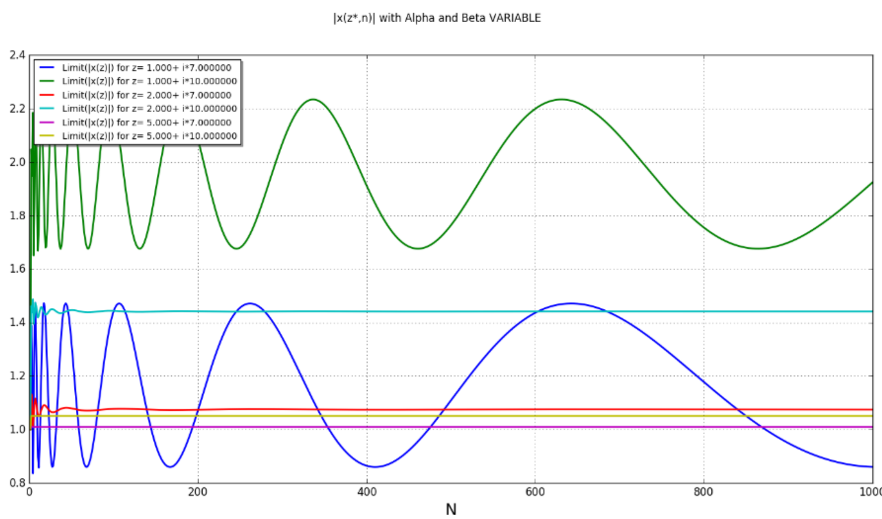

Figure 20. $|x(n)|^2$ converges when $n \to \infty$ and α>1

The graphs for α=1 do not converge while all other graphs for α>1 they all converge to a a constant of value ζ(α,ß) as horizontal straight line with slope equal to zero. This observation can be used to prove that there are no zero values of ζ(z) for z with Re(z)=α>1 based on the following Lemma where we prove that if α+ßi is a zero of ζ(z) then $|x(z)|^2$ tends to a straight line with slope not equal to zero.

29.6 Lemma xx: $|X(z)|^2$ is a line for α=1/2

The proposition says that the following limit exists only for Re(z) = 1/2

$$\lim_{n\to\infty} \frac{|(X(n)|^2}{n} = S$$

We have already formulated the expression:

$$\lim_{n\to\infty} (|\mathrm{x(z)}|^2/n) = \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(\text{ß}\left(\ln\left(\frac{k}{j}\right)\right)\right))$$

29.6.1. For α>1/2 , $\lim_{n\to\infty} \frac{|(X(n)|^2}{n} = 0$

$\lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha}) = 0$ because 2α>1 and the series is convergent

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(\text{ß}\left(\ln\left(\frac{k}{j}\right)\right)\right) < \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} (k^{-\alpha} * j^{-\alpha})$$

$$< \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha})$$

So:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(\text{ß}\left(\ln\left(\frac{k}{j}\right)\right)\right) = 0$$

29.6.2. For α<1/2, $\lim_{n\to\infty} \frac{|(X(n)|^2}{n} = \infty$

$$\lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha}) < \lim_{n\to\infty} \frac{1}{n} \left(n * \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

And:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(\text{ß}\left(\ln\left(\frac{k}{j}\right)\right)\right) > \lim_{n\to\infty} (\frac{1}{n} * n^2 * \frac{1}{n^{2\alpha}}) = \infty$$

Where we replaced the summations by the number of elements in the matrix (n x n) times the smallest value in each row (1/n) and 1>(2-1-2α)>0 when α<1/2

29.6.3. $\lim_{n\to\infty} \frac{|(X(n)|^2}{n}$ exists when α=1/2.

Before calculating this limit, let's see graphically that the limit actually exists for certain z=z* (in the graph ß=R(1)=14.134725…) with α=1/2:

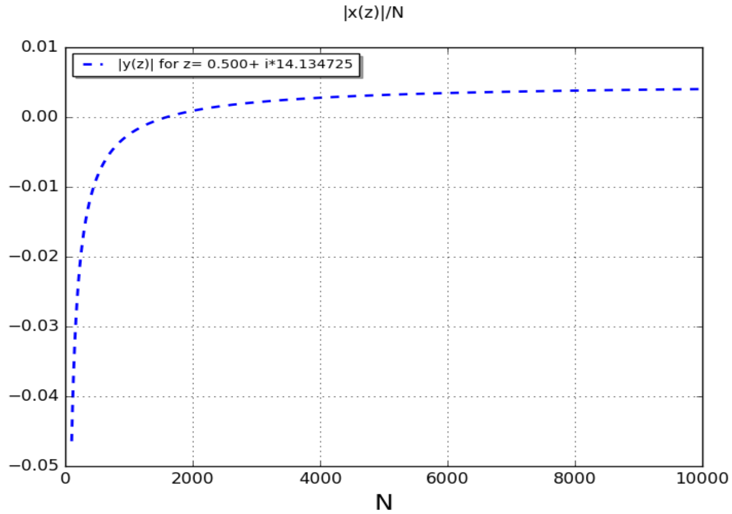

Figure 21: The chart shows that for z=1/2+i 14.134725 the limit |x(z)|²/n =0.0044999 for n → ∞*

When α=1/2, we can express ($|x(z)|^2/n$) as:

$$\lim_{n\to\infty}(|x(z)|^2/n) = \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n}k^{-1} + \sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right))=$$

$$= \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n}k^{-1}) + \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)) =$$

$$= 0 + \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right))=$$

$$= \lim_{n\to\infty}\frac{2n}{n}(\sum_{j=1}^{n-1}n^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{n}{j}\right)\right)\right))=$$

$$= \lim_{n\to\infty} 2(n^{-\frac{1}{2}}\sum_{j=1}^{n-1} * j^{-\frac{1}{2}} * \cos\left(ß\left(\ln\left(\frac{n}{j}\right)\right)\right))=$$

Using the integral approximation of the infinite series

$$= 2^* \lim_{n\to\infty}\frac{2*\sqrt{n}*\cos\left(ß*ln\left(\frac{n}{n}\right)\right)-2*ß*\sin(ß*\ln(\frac{n}{n})}{4*ß^2+1} * n^{-\frac{1}{2}}$$

$$= 2^*\frac{2*\sqrt{n}}{4*ß^2+1}n^{-\frac{1}{2}} = 2^*\frac{2}{4*ß^2+1} = \frac{1}{ß^2+1/4}$$

So, if $\lim_{n\to\infty}(|x(z)|^2/n)$ exists will be equal to:

$$\lim_{n\to\infty}\frac{|(X(n)|^2}{n} = \frac{1}{ß^2+\frac{1}{4}}\ iff\ z = \frac{1}{2} + iß$$

The fact that:

$\zeta(z) = X(z) - Y(z)$

$|(X(z)|^2$ collapses to a line with slope $\frac{1}{ß^2+\frac{1}{4}}$ when α=1/2 and $n \to \infty$

$|(Y(z)|^2$ collapses to a line with slope $\frac{1}{ß^2+\frac{1}{4}}$ when α=1/2 and $n \to \infty$

$|(X(z)|^2$ and $|(Y(z)|^2$have only same form representation as a straight line when α=1/2

are the basic elements of the proposed proof by the author of the Riemann Hypothesis (Caceres, 2017)

30. REFERENCES

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