ON THE NATURE OF MATTER WAVE

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Abstract: In this work we discuss the nature of matter wave of quantum particles whose dynamics are described by Dirac equation. Since we have shown that both Dirac equation and Maxwell field equations can be derived from a general system of linear first order partial differential equations, it is reasonable to suggest that matter wave may have similar physical formation to that of the electromagnetic field in the sense that matter wave is also the result of a coupling of two physical fields, such as the electric field and the magnetic field in electromagnetism. In particular, we show that when Dirac equation is reformulated as a system of real equations, like Maxwell field equations, then Dirac equation describes a quantum particle as a string-like object whose cross-section vibrates as a membrane.

In quantum physics, the wave-particle duality, which shows the coexistence of the wave and the corpuscular aspects of a particle, is a profound, or rather mysterious, property of matter. Furthermore, despite this dual property holds true for all types of matter, the mathematical methods that are used to describe their physical dynamics are also profoundly different. For example, the gravitational field is described by a system of non-linear differential equations, the electromagnetic field by a system of linear first order differential equations and the fields of massive particles by different systems of differential equations, such as Schrödinger and Dirac equations. Since the concept of matter wave was introduced into physics through the universal relationship between the momentum p and the wavelength λ of a quantum particle $\lambda = h/p$, where h is the Planck constant, it seems reasonable to suggest that all types of matter waves manifested by quantum fields should behave in the same manner and they should be described by the same mathematical methods. In our previous works we have shown that both Dirac equation and Maxwell field equations can be formulated from a system of linear first order partial differential equations [1,2]. Except for the dimensions that involve with the field equations, the formulations of Dirac and Maxwell field equations are remarkably similar and a prominent feature that arises from the formulations is that the equations are formed so that the components of the wavefunctions satisfy a wave equation. However, there are essential differences between the physical interpretations of Dirac and Maxwell physical fields. On the one hand, Maxwell electromagnetic field is a classical field which is composed of two different fields that have different physical properties even though they can be converted into each other. On the other hand, despite Dirac field was originally formulated to describe the dynamics of a single particle, such as the electron, it turned out that a solution to Dirac equation describes not only the dynamics of the electron with positive energy but it also describes the dynamics of the same electron with negative energy. The difficulty that is related to the negative energy can be resolved if the negative energy solutions can be identified as positive energy solutions that can be used to describe the dynamics of a positron. This result suggests that Dirac field may also be viewed as being composed of two different fields. In this work we show that, at least from the perspective of mathematical formulation, Dirac field may actually be composed of two physical fields, similar to the case of the electromagnetic field which is composed of the electric field and the magnetic field. A general system of linear first order partial differential equation can be written as [3]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r}\frac{\partial \psi_i}{\partial x_j} = k_1 \sum_{l=1}^{n} b_l^r\,\psi_l + k_2 c^r, \qquad r = 1, 2, \dots, n \tag{1}$$

The system of equations given in Equation (1) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\psi = k_1 \sigma\psi + k_2 J \tag{2}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i, \partial\psi_2/\partial x_i, \dots, \partial\psi_n/\partial x_i)^T$, A_i, σ and J are matrices representing the quantities a_{ij}^k, b_l^r and c^r, and k_1 and k_2 are undetermined constants. Now, if we apply the operator $\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}$ on the left on both sides of Equation (2) then we have

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)(k_1\sigma\psi + k_2 J) \tag{3}$$

If we assume further that the coefficients a_{ij}^k and b_l^r are constants and $A_i\sigma = \sigma A_i$, then Equation (3) can be rewritten in the following form

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j + A_jA_i\right)\frac{\partial^2}{\partial x_i \partial x_j}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2 \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{4}$$

In order for the above systems of partial differential equations to be used to describe physical phenomena, the matrices A_i must be determined. We have shown that, as in the case of Dirac and Maxwell field equations, the matrices A_i must take a form so that Equation (4) reduces to the following equation

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2 \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{5}$$

For the classical electromagnetic field, with the notation $\psi = \left(E_x, E_y, E_z, B_x, B_y, B_z\right)^T = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6)^T$, and $\epsilon\mu = 1$, Maxwell field equations of the electromagnetic field that are derived from Faraday's law and Ampere's law can be written as

$$\frac{\partial\psi_1}{\partial t}+\mu j_1=\frac{\partial\psi_6}{\partial y}-\frac{\partial\psi_5}{\partial z} \tag{6}$$

$$\frac{\partial\psi_2}{\partial t}+\mu j_2=\frac{\partial\psi_4}{\partial z}-\frac{\partial\psi_6}{\partial x} \tag{7}$$

$$\frac{\partial\psi_3}{\partial t}+\mu j_3=\frac{\partial\psi_5}{\partial x}-\frac{\partial\psi_4}{\partial y} \tag{8}$$

$$\frac{\partial\psi_4}{\partial t}=\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y} \tag{9}$$

$$\frac{\partial\psi_5}{\partial t}=\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z} \tag{10}$$

$$\frac{\partial\psi_6}{\partial t}=\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x} \tag{12}$$

The system of equations given in Equations (6-12) can be written the following matrix form

$$\left(A_1\frac{\partial}{\partial t}+A_2\frac{\partial}{\partial x}+A_3\frac{\partial}{\partial y}+A_4\frac{\partial}{\partial z}\right)\psi=A_5J \tag{13}$$

where $J=(j_1,j_2,j_3,0,0,0)^T$ and the matrices A_i are

$$A_1=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&1&0&0&0&0\end{pmatrix}\quad A_3=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\end{pmatrix}$$

$$A_4=\begin{pmatrix}0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\qquad A_5=\begin{pmatrix}\mu&0&0&0&0&0\\0&\mu&0&0&0&0\\0&0&\mu&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix} \tag{14}$$

Furthermore, if an additional condition that imposes on the function ψ that requires that it also satisfies the wave equation given by Equation (5) then Gauss's laws will be recovered. On the other hand, Dirac equation can be derived from Equation (4) by simply imposing the following conditions on the matrices A_i

$$A_i^2=\pm 1 \tag{15}$$

$$A_iA_j+A_jA_i=0 \quad \text{for} \quad i\neq j \tag{16}$$

For the case of $n=4$, the matrices A_i can be shown to take the form

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{17}$$

With $k_1 = m$, $\sigma = 1$ and $k_2 = 0$, the system of linear first order partial differential equations given in Equation (2) reduces to Dirac equation [4]

$$i\frac{\partial\psi_1}{\partial t} + i\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_4}{\partial y} + i\frac{\partial\psi_3}{\partial z} = m\psi_1 \tag{18}$$

$$i\frac{\partial\psi_2}{\partial t} + i\frac{\partial\psi_3}{\partial x} - \frac{\partial\psi_3}{\partial y} - i\frac{\partial\psi_4}{\partial z} = m\psi_2 \tag{19}$$

$$-i\frac{\partial\psi_3}{\partial t} - i\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - i\frac{\partial\psi_1}{\partial z} = m\psi_3 \tag{20}$$

$$-i\frac{\partial\psi_4}{\partial t} - i\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_1}{\partial y} + i\frac{\partial\psi_2}{\partial z} = m\psi_4 \tag{21}$$

For the following discussions, we rewrite Dirac equation given in Equations (18-21) in the following form

$$-\frac{\partial\psi_1}{\partial t} - im\psi_1 = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial\psi_3}{\partial z} \tag{22}$$

$$-\frac{\partial\psi_2}{\partial t} - im\psi_2 = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial\psi_4}{\partial z} \tag{23}$$

$$\frac{\partial\psi_3}{\partial t} - im\psi_3 = \left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_2 - \frac{\partial\psi_1}{\partial z} \tag{24}$$

$$\frac{\partial\psi_4}{\partial t} - im\psi_4 = \left(-\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_1 + \frac{\partial\psi_2}{\partial z} \tag{25}$$

Except for the fact that Dirac equation is expressed in complex mathematics and Maxwell field equations are real, the two formulations look remarkably similar. With the form of the field equations given in Equations (22-25), we may interpret that the change of the field (ψ_1, ψ_2) with respect to time generates the field (ψ_3, ψ_4), similar to the case of Maxwell field equations given in Equations (6-8), the change of the electric field generates the magnetic field. With this observation it may be suggested that, like the Maxwell electromagnetic field which is composed of two essentially different physical fields, the Dirac field of massive particles may also be viewed as being composed of two different physical fields, namely the field (ψ_1, ψ_2), which plays the role of the electric field in

Maxwell field equations, and the field (ψ_1, ψ_2), which plays the role of the magnetic field. In the following we show that the similarity between Maxwell field equations and Dirac field equations can be carried forward by showing that it is possible to reformulate Dirac equation as a system of real equations. When we formulate Maxwell field equations from a system of linear first order partial differential equations we rewrite the original Maxwell field equations from a vector form to a system of first order partial differential equations by equating the corresponding terms of the vectorial equations. Now, since, in principle, a complex quantity is equivalent to a vector quantity therefore in order to form a system of real equations from Dirac complex field equations we equate the real parts with the real parts and the imaginary parts with the imaginary parts. In this case Dirac equation given in Equations (22-25) can be rewritten as a system of real equations as follows

$$\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_4}{\partial x}+\frac{\partial\psi_3}{\partial z}=0 \tag{26}$$

$$\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_4}{\partial z}=0 \tag{27}$$

$$\frac{\partial\psi_3}{\partial t}+\frac{\partial\psi_2}{\partial x}+\frac{\partial\psi_1}{\partial z}=0 \tag{28}$$

$$\frac{\partial\psi_4}{\partial t}+\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial z}=0 \tag{29}$$

$$\frac{\partial\psi_4}{\partial y}=m\psi_1 \tag{30}$$

$$-\frac{\partial\psi_3}{\partial y}=m\psi_2 \tag{31}$$

$$-\frac{\partial\psi_2}{\partial y}=m\psi_3 \tag{32}$$

$$\frac{\partial\psi_1}{\partial y}=m\psi_4 \tag{33}$$

The system of Dirac field equations given in Equations (26-33) can be considered as a particular case of a more general system of field equations written in the matrix form

$$\left(A_1\frac{\partial}{\partial t}+A_2\frac{\partial}{\partial x}+A_3\frac{\partial}{\partial y}+A_4\frac{\partial}{\partial z}\right)\psi=m\psi \tag{34}$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T$ and the real matrices A_i are given as

$$A_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad A_2=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{35}$$

The matrices A_i satisfy the following commutation relations

$$A_i^2 = 1 \qquad \text{for} \qquad i = 1, 2, 3, 4 \tag{36}$$

$$A_1 A_i + A_i A_1 = 2A_i \qquad \text{for} \qquad i = 2, 3, 4 \tag{37}$$

$$A_2 A_3 + A_3 A_2 = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{38}$$

$$A_2 A_4 + A_4 A_2 = 0 \tag{39}$$

$$A_3 A_4 + A_4 A_3 = 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{40}$$

By applying $\left(A_1 \frac{\partial}{\partial t} + A_2 \frac{\partial}{\partial x} + A_3 \frac{\partial}{\partial y} + A_4 \frac{\partial}{\partial z}\right)$ to Equation (34) and using the commutation relations given in Equations (36-40), we obtain

$$\begin{aligned}\Bigg(A_1^2 \frac{\partial^2}{\partial t^2} + A_2^2 \frac{\partial^2}{\partial x^2} + A_3^2 \frac{\partial^2}{\partial y^2} + A_4^2 \frac{\partial^2}{\partial z^2} + 2A_1 \frac{\partial^2}{\partial t \partial x} + 2A_2 \frac{\partial^2}{\partial t \partial y} + 2A_3 \frac{\partial^2}{\partial t \partial z} \\ + (A_2 A_3 + A_3 A_2) \frac{\partial^2}{\partial x \partial y} + (A_3 A_4 + A_4 A_3) \frac{\partial^2}{\partial y \partial z}\Bigg)\psi = m^2 \psi\end{aligned} \tag{41}$$

In expanded form, we have

$$\begin{aligned}\Bigg(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial x} + 2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial y} \\ + 2\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial z} + 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x \partial y} \\ + 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{\partial^2}{\partial y \partial z}\Bigg)\psi = m^2 \psi\end{aligned} \tag{42}$$

From the expanded form, together with Dirac real equations given in Equations (26-29), it can be shown that all components of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ satisfy the following equation

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}+\frac{\partial^2\psi_i}{\partial y^2}-\frac{\partial^2\psi_i}{\partial z^2}=m^2\psi_i \quad for \qquad i=1,2,3,4 \tag{43}$$

It is observed that due to the positive sign in front of the term related to the differentiation with respect to the y-coordinate, in general the equation given by Equation (43) is not a normal three-dimensional wave equation. However, if the wavefunction ψ satisfies Dirac field equations given in Equations (26-33) then we obtain the following system of equations for all components

$$\frac{\partial^2\psi_i}{\partial^2 y}-m^2\psi_i=0 \tag{44}$$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial z^2}=0 \tag{45}$$

Solutions to Equation (44) are

$$\psi_i=c_1(x,z)e^{my}+c_2(x,z)e^{-my} \tag{46}$$

where c_1 and c_2 are undetermined functions of (x,z), which may be assumed to be constant. The solutions given in Equation (46) give a distribution of a physical quantity, such as the mass of a quantum particle, along the y-axis. On the other hand, Equation (45) can be used to describe the dynamics, for example, of a vibrating membrane in the (x,z)-plane. If the membrane is a circular membrane of radius a, $x^2+z^2<a^2$, then in the polar coordinates $x=r\cos\theta,\ z=r\sin\theta$ the two-dimensional wave equation can be obtained from Equation (45) as

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_i}{\partial r}-\frac{1}{r^2}\frac{\partial^2\psi_i}{\partial^2\theta}=0 \tag{47}$$

The general solution to Equation (47) for the vibrating circular membrane can be found as [5]

$$\begin{aligned}\psi_i(r,\theta,t)=&\sum_{m=1}^{\infty}J_0\left(\sqrt{\lambda_{0m}}r\right)\left(C_{0m}\cos\sqrt{\lambda_{0m}}t+D_{0m}\sin\sqrt{\lambda_{0m}}t\right)\\&+\sum_{m,n=1}^{\infty}J_n\left(\sqrt{\lambda_{nm}}r\right)(A_{nm}\cos n\theta\\&+B_{nm}\sin n\theta)\left(\left(C_{nm}\cos\sqrt{\lambda_{nm}}t+D_{nm}\sin\sqrt{\lambda_{nm}}t\right)\right)\end{aligned} \tag{48}$$

where $J_n\left(\sqrt{\lambda_{nm}}r\right)$ is the Bessel function of order n and the quantities A_{nm}, B_{nm}, C_{nm} and D_{nm} can be specified by the initial and boundary conditions. It is seen that when Dirac equation is reformulated as a system of real equations then the equation describes a quantum particle, such as the electron, as a string-like object whose cross-section vibrates as a membrane.

Now we would like to discuss further whether it is possible the convert the positive sign in front of the term related to the differentiation with respect to the y-coordinate given in Equation (43) to negative. In fact, as shown below, this can be done if the system of equations given in Equations (26-33) is changed to the following system of equations

$$\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_4}{\partial x}+\frac{\partial\psi_3}{\partial z}=0 \tag{49}$$

$$\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_4}{\partial z}=0 \tag{50}$$

$$\frac{\partial\psi_3}{\partial t}+\frac{\partial\psi_2}{\partial x}+\frac{\partial\psi_1}{\partial z}=0 \tag{51}$$

$$\frac{\partial\psi_4}{\partial t}+\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial z}=0 \tag{52}$$

$$\frac{\partial\psi_4}{\partial y}=m\psi_1 \tag{53}$$

$$\frac{\partial\psi_3}{\partial y}=m\psi_2 \tag{54}$$

$$-\frac{\partial\psi_2}{\partial y}=m\psi_3 \tag{55}$$

$$-\frac{\partial\psi_1}{\partial y}=m\psi_4 \tag{56}$$

The system of field equations given in Equations (49-56) can be considered as a special case of a more general system of field equations written in the following matrix form

$$\left(A_1\frac{\partial}{\partial t}+A_2\frac{\partial}{\partial x}+A_3\frac{\partial}{\partial y}+A_4\frac{\partial}{\partial z}\right)\psi=m\psi \tag{57}$$

where the real matrices A_i are given as follows

$$A_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad A_2=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

$$A_3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\qquad A_4=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix} \tag{58}$$

The only difference is the change of the matrix A_3. The matrices A_i given in Equation (58) satisfy the following commutation relations

$$A_3^2=-1 \tag{59}$$

$$A_i^2 = 1 \qquad \text{for} \qquad i = 1, 2, 4 \tag{60}$$

$$A_1 A_i + A_i A_1 = 2A_i \qquad \text{for} \qquad i = 2, 3, 4 \tag{61}$$

$$A_2 A_3 + A_3 A_2 = 0 \tag{62}$$

$$A_2 A_4 + A_4 A_2 = 0 \tag{63}$$

$$A_3 A_4 + A_4 A_3 = 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{64}$$

By applying $\left(A_1 \frac{\partial}{\partial t} + A_2 \frac{\partial}{\partial x} + A_3 \frac{\partial}{\partial y} + A_4 \frac{\partial}{\partial z}\right)$ to Equation (57) and using the commutation relations given in Equations (59-64), we obtain

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2A_1 \frac{\partial^2}{\partial t \partial x} + 2A_2 \frac{\partial^2}{\partial t \partial y} + 2A_3 \frac{\partial^2}{\partial t \partial z} + (A_3 A_4 + A_4 A_3) \frac{\partial^2}{\partial y \partial z}\right)\psi = m^2 \psi \tag{65}$$

In expanded form, we have

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial x} + 2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial y} + 2\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial z} + 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \frac{\partial^2}{\partial y \partial z}\right)\psi = m^2 \psi \tag{66}$$

It can be shown from the expanded form given in Equation (66), together with the field equations given in Equations (49-52), that the components of the wavefunction ψ satisfy the following equations

$$\frac{\partial^2 \psi_i}{\partial t^2} - \frac{\partial^2 \psi_i}{\partial x^2} - \frac{\partial^2 \psi_i}{\partial y^2} - \frac{\partial^2 \psi_i}{\partial z^2} - 2\frac{\partial^2 \psi_i}{\partial x \partial y} = m^2 \psi_i \qquad \text{for} \qquad i = 1, 2 \tag{67}$$

$$\frac{\partial^2 \psi_i}{\partial t^2} - \frac{\partial^2 \psi_i}{\partial x^2} - \frac{\partial^2 \psi_i}{\partial y^2} - \frac{\partial^2 \psi_i}{\partial z^2} + 2\frac{\partial^2 \psi_i}{\partial x \partial y} = m^2 \psi_i \qquad \text{for} \qquad i = 3, 4 \tag{68}$$

The partial derivatives that involve the (x, y)-coordinates in Equations (67) and (68) form a parabolic partial differential equation therefore these equations can be transformed into a two-dimensional wave equation which can be used to describe a vibrating membrane, as previously discussed for the case of Dirac real equations [6]. In particular, if the wavefunction ψ satisfies the field equations given in Equations (49-56) then we obtain the following system of equations for the components

$$\frac{\partial^2\psi_i}{\partial^2 y}+m^2\psi_i=0 \tag{69}$$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial z^2}-2\frac{\partial^2\psi_i}{\partial x\partial y}=0 \qquad \text{for} \quad i=1,2 \tag{70}$$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial z^2}+2\frac{\partial^2\psi_i}{\partial x\partial y}=0 \qquad \text{for} \quad i=3,4 \tag{71}$$

Solutions to Equation (69) are

$$\psi_i=c_1(x,z)e^{imy}+c_2(x,z)e^{-imy} \tag{72}$$

where c_1 and c_2 are undetermined functions of (x,z), which may be assumed to be constant. The solutions given in Equation (72) give a periodic distribution of a physical quantity, such as the mass of the particle, along the *y*-axis.

References

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