A Lower Bound for the Minimal Counter-example to Frankl's Conjecture

Ankush Hore
West Bengal University of Technology
ankushore@gmail.com

Abstract

Frankl's Conjecture, from 1979, states that any finite union-closed family, containing at least one non-empty member set, must have an element which belongs to at least half of the member-sets. In this paper we show that if the minimum value of $\cup\mathcal{A}$ over all counter-examples be q, then any counter-example family must contain at least $4q+1$ sets. As a consequence, we show that the minimal counter-example must contain at least 53 sets.

1 Introduction

A family of sets $\mathcal{A}$ is said to be union-closed if the union of any two member sets is also a member of $\mathcal{A}$. Peter Frankl's conjecture (or the union-closed sets conjecture) states that if $\mathcal{A}$ is finite, then some element must belong to at least half of the sets in $\mathcal{A}$, provided at least one member set is non-empty. A detailed discussion and current standing of the conjecture can be found in [1].

In [2], Ian Roberts and Jamie Simpson showed that if q be the minimum cardinality of $\cup\mathcal{A}$ over all counter-examples, then any counter-example $\mathcal{A}$ must satisfy the inequality $|\mathcal{A}| \geq 4q-1$. In this paper, we expand the ideas presented in [2] to find an improved lower bound $4q+1$. In [3], it was proved that the minimal counter-example must contain at least 13 elements in $\cup\mathcal{A}$. Hence, we show that the minimal counter-example family must contain at least 53 sets.

2 Main results

2.1 Preliminary Lemmas

Throughout this paper, $\mathcal{A}$ will denote the minimal counter-example with $\cup\mathcal{A}$ $= q$, the minimum number of constituent elements across all counter-examples. $|\mathcal{A}|$ must be odd, because if it is even, we can remove a basis set to generate a counter-example with $|\mathcal{A}| - 1$. Let, $|\mathcal{A}|$=$2n+1$.

We denote the family of sets in $\mathcal{A}$ containing an element x as $\mathcal{A}_x$.

The universal set for $\mathcal{A}$ is defined as $S := \cup\mathcal{A}$.

Thus, $|S| = q$.

We define $\mathcal{A}_{\overline{x}} := \{A \in \mathcal{A} : x \notin A\}$.

Let, $C_x := \cup\mathcal{A}_{\overline{x}}$.

We define the family containing all such C_x as $\mathcal{C}$.

$$\mathcal{C} := \{C_x : x \in S\}$$

For any x we define the family $\mathcal{D}_x$ as

$$\mathcal{D}_x := \mathcal{A}_x \setminus \{S\} \setminus \mathcal{C}$$

Next, we define and distinguish the terms *mutually dominant* and *dominant*.

We call 2 elements a and b to be *mutually dominating* if a and b always appear together in the member sets of $\mathcal{A}$.

We say a *dominates* b if $\mathcal{A}_b \subset \mathcal{A}_a$ and $|\mathcal{A}_a| > |\mathcal{A}_b|$.

$\mathcal{A}$ *cannot* contain any *mutually dominating* pair of elements, as they can be replaced by a single element which in turn would violate the minimality of q. Therefore, for any a,$b \in S$, if $a \neq b$, then $C_a \neq C_b$.

However, $\mathcal{A}$ may contain elements which *dominate* other elements.

We define the sets I and J as

$$I := \{a \in S : a \text{ is } \mathit{NOT} \text{ dominated by any other element in } S\}$$

$$J := \{b \in S : b \text{ is dominated by some other element in } S\}$$

If an element is present in n sets of $\mathcal{A}$, then it cannot be dominated by any other element. Hence, they must be present in I. From [4], we know that $\mathcal{A}$ must contain at least 3 elements with abundance n. Thus, $|I| \geq 3$.

Note that every set in $\mathcal{A}$ must contain at least one element from I.

We now prove the following 2 lemmas, a slightly modified form of which is presented in [2].

Lemma 1: For any a, $I \subseteq C_a$ if $a \notin I$, or $I \setminus \{a\} \subseteq C_a$ if $a \in I$.

Proof. When $a \notin I$, let $y \in I$. As a cannot dominate y, there must exist a set containing y but not a. So, $y \in C_a$.

When $a \in I$, let $z \in I$. As a cannot dominate z, there must exist a set containing z but not a. So, $z \in C_a$. But, $a \notin C_a$ as $\cup\mathcal{A}_{\overline{a}}$ cannot contain a. □

So, we conclude that if $a \in I$, then it must be present in $q - 1$ sets of $\mathcal{C}$.

Lemma 2: For any a, C_a cannot be a basis set of $\mathcal{A}$.

Proof. Let C_a be a basis. So, we can remove C_a to get a new union-closed $\mathcal{A}'$ with $|\mathcal{A}'| = |\mathcal{A}| - 1$.

From *Lemma 1*, if $a \notin I$, $I \subseteq C_a$. As I must contain all elements with abundance n, removing C_a would generate another counter-example $\mathcal{A}'$ with $|\mathcal{A}'| < |\mathcal{A}|$, a contradiction.

If $a \in I$, $I \setminus \{a\} \subseteq C_a$. Let B_a be a basis set containing a. Removing B_a and C_a we get $\mathcal{A}'$ with $|\mathcal{A}'| = |\mathcal{A}| - 2 = 2n - 1$ and no element contained in more than $n - 1$ sets. Hence, $\mathcal{A}'$ is also a counter-example, a contradiction. $\square$

Definition: For every element a, we define the sets H_a and L_a as

$$H_a := \{b \in S : b \text{ is abundant in } \mathcal{A}_{\overline{a}}\}$$

$$L_a := \{c \in S : c \text{ is abundant in } \mathcal{A}_a\}$$

We now prove a few lemmas which would be repeatedly referenced in the next sections.

Lemma 3: If a,$b \in I$, $b \in H_a$ and $\mathcal{D}_a \cap \mathcal{D}_b \neq \emptyset$, then $|\mathcal{A}| \geq 4q + 3$.

Proof. As $b \in H_a$, it must be present in at least $(n+1)/2$ sets of $\mathcal{A}_{\overline{a}}$. $b \in S$. b must be in $q - 2$ sets of $\mathcal{C} \setminus \{C_a\}$. b must also be present in at least one set of $\mathcal{D}_a$, as $\mathcal{D}_a \cap \mathcal{D}_b \neq \emptyset$. So, we have

$$(n+1)/2 + 1 + (q-2) + 1 \leq n,$$

which yields $|\mathcal{A}| \geq 4q + 3$. $\square$

Lemma 4: If $|\mathcal{A}_x| = |\mathcal{A}_y| = n$, $x \neq y$, then $y \in H_x$ or $y \in L_x$.

Proof. Let $y \notin H_x$ and $y \notin L_x$.

When n is even ($n = 2k$), as $y \notin L_x$, $|\mathcal{A}_x \cap \mathcal{A}_y| \leq k - 1$. As $y \notin H_x$, $|\mathcal{A}_{\overline{x}} \cap \mathcal{A}_y| \leq k$. So, $|\mathcal{A}_y| \leq k - 1 + k = n - 1$, a contradiction.

When n is odd ($n = 2k + 1$), as $y \notin L_x$, $|\mathcal{A}_x \cap \mathcal{A}_y| \leq k$. As $y \notin H_x$, $|\mathcal{A}_{\overline{x}} \cap \mathcal{A}_y| \leq k$. So, $|\mathcal{A}_y| \leq k + k = n - 1$, a contradiction again.

The case, $y \in H_x$ and $y \in L_x$, is not possible as it will render y abundant in $\mathcal{A}$. $\square$

Lemma 5: If $|\mathcal{A}_x| = |\mathcal{A}_y| = n$ and $y \in H_x$, then $x \in H_y$.

Proof. As $y \in H_x$, $y \notin L_x$ (from *Lemma 4*). So, x, y cannot be mutually abundant. So, $x \notin L_y$. Thus, from *Lemma 4* we have $x \in H_y$. $\square$

Definition: For any x, $y \in S$, we define the family $\mathcal{F}_{xy}$ as

$$\mathcal{F}_{xy} := \mathcal{A}_{\overline{x}} \cap \mathcal{A}_{\overline{y}}$$

Note that $\mathcal{F}_{xy}$ is union-closed.

Also note that $\mathcal{F}_{xy}$ cannot contain any set from $\mathcal{C}$ as $\mathcal{A}_x \cup \mathcal{A}_y$ covers $\mathcal{C}$. And $S \notin \mathcal{F}_{xy}$ as S must contain both x and y.

We define the set E_{xy} as

$$E_{xy} := \cup \mathcal{F}_{xy}$$

Lemma 6: If x, $y \in I$, then $E_{xy} \notin \mathcal{C}$.

Proof. If $E_{xy} \in \mathcal{C}$, then it must be either C_x, C_y or some C_k where $k \in I$. E_{xy} cannot be any C_a where $a \in J$, as x or y cannot be dominated by a. If $E_{xy} = C_x$, $\mathcal{A}_{\overline{x}}$ cannot contain y in any of its sets. So, $\mathcal{A}_y \subset \mathcal{A}_x$. Therefore, x dominates y. But, $y \in I$, a contradiction. Similarly, E_{xy} cannot be equal to C_y or any C_k where $k \in I$. □

Now, we prove our central result, $|\mathcal{A}| \geq 4q+1$. To do so, we divide the proof into the 2 following cases.

2.2 Case when $C_x \neq S \setminus \{x\}$ for some x

Theorem 1: If there exists $x \in I$, such that $|\mathcal{A}_x| < n$, then $|\mathcal{A}| \geq 4q+1$.

Proof. $|\mathcal{A}_{\overline{x}}| \geq n+2$. There must exist $y \in I$ abundant in $\mathcal{A}_{\overline{x}}$ (for if y is dominated by some z, then z would also be abundant in $\mathcal{A}_{\overline{x}}$ and we would then choose z instead of y). Thus, y must be in at least $(n+2)/2$ sets of $\mathcal{A}_{\overline{x}}$. y must be in $q-2$ sets of $\mathcal{C} \setminus \{C_x\}$. $y \in S$. So, we have

$$(n+2)/2 + (q-2) + 1 \leq n,$$

which yields $|\mathcal{A}| \geq 4q+1$. □

Theorem 2: If $|\mathcal{A}_x| = n$ for all $x \in I$, then $|\mathcal{A}| \geq 4q+1$.

Proof. Let $y \in I$ and $y \in H_x$. If $\mathcal{D}_x \cap \mathcal{D}_y \neq \emptyset$, then we immediately have $|\mathcal{A}| \geq 4q+3$ from *Lemma 3*. So, let $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$. So, $|\mathcal{F}_{xy}| = q$ (as $|\{S\}| = 1$, $|\mathcal{C}| = q$, $|\mathcal{D}_x| = |\mathcal{D}_y| = n-q$).

As $\mathcal{F}_{xy}$ is union closed, there must exist a $z \in I$ abundant in $\mathcal{F}_{xy}$. We choose z as the element with *maximum abundance* in $\mathcal{F}_{xy}$. If z be present in all q sets of $\mathcal{F}_{xy}$, we have $|\mathcal{A}_z| = 2q$ (as z must be in q sets of $\mathcal{C} \cup \{S\}$). This yields $|\mathcal{A}| \geq 4q+1$.

So, let z be present in at most $q-1$ sets of $\mathcal{F}_{xy}$. Hence, there must exist $s \in I$ present in $\mathcal{F}_{xy} \setminus \mathcal{A}_z$. Therefore there exists $G_s \in \mathcal{F}_{xy}$ such that $s \in G_s$ and $z \notin G_s$. As z is maximal in $\mathcal{F}_{xy}$, s must also be present in at most $q-1$ sets of $\mathcal{F}_{xy}$. So, there must exist $G_z \in \mathcal{F}_{xy}$ such that $z \in G_z$ and $s \notin G_z$. Also, as $\mathcal{F}_{xy}$ is union-closed there exists $G_{zs} \in \mathcal{F}_{xy}$ such that $z \in G_{zs}$ and $s \in G_{zs}$. We summarize this as

$$z \in G_z \text{ and } s \notin G_z$$
$$s \in G_s \text{ and } z \notin G_s$$
$$s \in G_{zs} \text{ and } z \in G_{zs}$$

where G_z, G_s, $G_{zs} \in \mathcal{F}_{xy}$.

Our set-up is depicted in Figure 1 below.

Based on *Lemma 4*, we have the following 3 sub-cases:

a) $z \in H_x$*:*

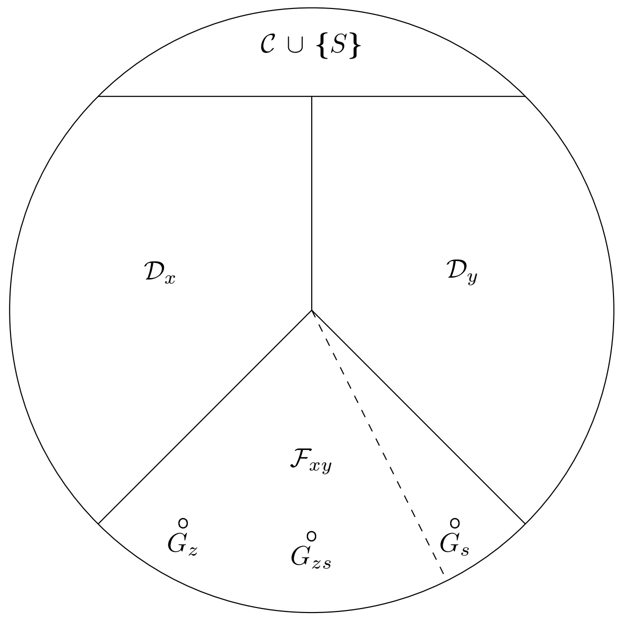

Figure 1: Representation of $\mathcal{A}$

We consider the family $\mathcal{F}_{sy}$. There exists a basis B_x where $x \in B_x$ and $s \notin B_x$, as s cannot dominate x. As $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$, $y \notin B_x$. Hence, $B_x \in \mathcal{F}_{sy}$. As $G_z \in \mathcal{F}_{xy}$, so $y \notin G_z$. Also, $s \notin G_z$. Therefore, $G_z \in \mathcal{F}_{sy}$.

As, B_x, $G_z \in \mathcal{F}_{sy}$, therefore x, $z \in E_{sy}$. From *Lemma 6*, $E_{sy} \notin \mathcal{C}$. Hence, $E_{sy} \in \mathcal{D}_x \cap \mathcal{D}_z$. Thus, as $\mathcal{D}_x \cap \mathcal{D}_z \neq \emptyset$ and $z \in H_x$, from *Lemma 3*, we have $|\mathcal{A}| \geq 4q + 3$.

b) $z \in H_y$:

We consider the family $\mathcal{F}_{sx}$. There exists a basis B_y where $y \in B_y$ and $s \notin B_y$, as s cannot dominate y. As $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$, $x \notin B_y$. Hence, $B_y \in \mathcal{F}_{sx}$. As $G_z \in \mathcal{F}_{xy}$, so $x \notin G_z$. Also, $s \notin G_z$. Therefore, $G_z \in \mathcal{F}_{sx}$.

As, B_y, $G_z \in \mathcal{F}_{sx}$, therefore y, $z \in E_{sx}$. From *Lemma 6*, $E_{sx} \notin \mathcal{C}$. Hence, $E_{sx} \in \mathcal{D}_y \cap \mathcal{D}_z$. Thus, as $\mathcal{D}_y \cap \mathcal{D}_z \neq \emptyset$ and $z \in H_y$, from *Lemma 3*, we have $|\mathcal{A}| \geq 4q + 3$.

c) $z \in L_x$ and $z \in L_y$:

$z \in L_x$ implies $x \in L_z$ as $|\mathcal{A}_x| = |\mathcal{A}_z| = n$. Similarly, as $z \in L_y$, we have $y \in L_z$. Therefore, from *Lemma 4*, we have x, $y \notin H_z$. As, x, $y \notin H_z$, let, $r \in I$ be an element of H_z.

If r be present in any set of $\mathcal{F}_{xy}$, then we have a set $G_{rz} \in \mathcal{F}_{xy}$ containing both r and z, as $\mathcal{F}_{xy}$ is union-closed. Also, as $G_{rz} \notin \mathcal{C}$, $G_{rz} \in \mathcal{D}_r \cap \mathcal{D}_z$ and as $r \in H_z$ we have $|\mathcal{A}| \geq 4q + 3$ from *Lemma 3*.

So, let us assume that r is not in any sets of $\mathcal{F}_{xy}$. So, $D_r \subset \mathcal{D}_x \cup \mathcal{D}_y$. As r

cannot be dominated by s, there must exist a basis B_r such that $r \in B_r$ and $s \notin B_r$.
If $B_r \in \mathcal{D}_x$, then $B_r \in \mathcal{F}_{sy}$ (because $y \notin B_r$, since $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$). Also, $G_z \in \mathcal{F}_{sy}$ (shown in *Case a*). So, $z, r \in E_{sy} \notin \mathcal{C}$.
If $B_r \in \mathcal{D}_y$, then $B_r \in \mathcal{F}_{sx}$ (because $x \notin B_r$, since $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$). Also, $G_z \in \mathcal{F}_{sx}$ (shown in *Case b*). So, $z, r \in E_{sx} \notin \mathcal{C}$.
So, at least one of E_{sx} and E_{sy} must be present in $\mathcal{D}_r \cap \mathcal{D}_z$, and as $r \in H_z$ we have $|\mathcal{A}| \geq 4q + 3$ from *Lemma 3*.

□

2.3 Case when $C_x = S \setminus \{x\}$ for all x

In this case, no element can be dominated by any other element. Thus, all elements must be present in $q - 1$ sets of $\mathcal{C}$.

Theorem 3: If there exists x such that $|\mathcal{A}_x| < n$, then $|\mathcal{A}| \geq 4q + 1$.

Proof. The proof is similar to that of *Theorem 1*. $|\mathcal{A}_{\overline{x}}| \geq n + 2$. Let $y \in H_x$. y must be in at least $(n + 2)/2$ sets of $\mathcal{A}_{\overline{x}}$. y must be in $q - 2$ sets of $\mathcal{C} \setminus \{C_x\}$. $y \in S$. So,
$(n + 2)/2 + (q - 2) + 1 \leq n$, which yields $|\mathcal{A}| \geq 4q + 1$. □

Theorem 4: If for all x, $|\mathcal{A}_x| = n$, then $|\mathcal{A}| \geq 4q + 1$.

Proof. As $|\mathcal{A}_x| = n$ for all x, no element can dominate any other element. Therefore, $I = S$. As, in the proof of *Theorem 2*, we did not consider any element from J, this just becomes a special case of *Theorem 2*. □

Theorem 5: The minimal counter-example to Frankl's conjecture must contain at least 53 sets.

Proof. Combining *Theorems 1, 2, 3 and 4*, we get $|\mathcal{A}| \geq 4q + 1$. As it is shown in [3] that $q \geq 13$, we have $|\mathcal{A}| \geq 53$. □

3 Remarks

With more research using the methods exploited in this paper, we may further improve this lower bound, or even end up proving that the minimal counter-example cannot exist.

References

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