# A Derivation of the Kerr Metric by Ellipsoid Coordinate Transformation

Yu-Ching Chou, M.D.*

*Health 101 clinic, 1F., No.97, Guling St.,*

*Zhongzheng Dist., Taipei City 100, Taiwan*

(Dated: December 19, 2020)

## Abstract

Einstein's general relativistic field equation is a nonlinear partial differential equation that lacks an easy way to obtain exact solutions. The most famous examples are Schwarzschild and Kerr's black hole solutions. The Kerr metric has astrophysical meaning because most of cosmic celestial bodies are rotating. The Kerr metric is even more difficult to derive than the Schwarzschild metric specifically due to off-diagonal term of metric tensor. In this paper, a derivation of Kerr metric was obtained by ellipsoid coordinate transformation, which causes elimination a large amount of tedious derivation. This derivation is not only physics enlightening, but also further deducing some characteristics of the rotating black hole.

Published: International Journal of Physical Science. Vol. 12(11), pp. 130-136, 22 June, 2017. DOI: 10.5897/IJPS2017.4605

(This manuscript is a revised edition)

PACS numbers: 04.20.Jb

Keywords: General relativity, Schwarzschild metric, Kerr metric, ellipsoid coordinate transformation, exact solutions.

## I. INTRODUCTION

The theory of general relativity proposed by Albert Einstein in 1915 was one of the greatest advances in modern physics. It describes the distribution of matter to determine the space-time curvature, and the curvature determines how the matter moves. Einstein's field equation is very simple and elegant, but because Einstein's field equation is a set of nonlinear differential equations, it has proven difficult to find the exact analytic solution. The exact solution has physical meanings, only in some simplified assumptions, the most famous of which include Schwarzschild and Kerr's black hole solution, and Friedman's solution to cosmology. One year after Einstein published his equation, Schwarzschild discovered the spherical symmetry, static vacuum solution with center singularity.[1] Nearly 50 years later, Kerr solved the fixed axis symmetric rotating black hole in 1963.[2] Some of these exact solutions have been used to explain topics related to the gravity in cosmology, such as Mercury's precession of the perihelion, gravitational lens, black hole, expansion of the universe, and gravitational waves.

Today, many solving methods of Einstein field equations have been proposed. For example: Pensose-Newman's method,[3] or Bäcklund transformations.[4] Despite their great success in dealing with the Einstein equation, these methods are technically complex and expert-oriented.

The Kerr solution is important in astrophysics because most cosmic celestial bodies are rotating and rarely completely at rest. Traditionally, the general method of the Kerr solution can be found in *The Mathematical Theory of Black Holes* by the classical works of S.Chandrasekhar.[5] However, the calculation is so lengthy and complicated that college or institute students find it difficult to understand. Recent literature review showed that it is possible to obtain Kerr metric through the oblate spheroidal coordinates transformation.[6] This encourage me to look for a more concise way to solve the vacuum solution of Einstein's field equation through coordinate transformation.

The motivation of this derivation simply came from my desire to use a relatively simple way of Schwarzschild method to derive the Kerr metric, which can enable more students interested in the general relativity to self-deduce the exact solution. In this paper, I will introduce a more enlighted way to find this solution. It is not only a new try, but also the derivation is further linked to some important features of the rotating black hole.

## II. SCHWARZSCHILD AND KERR SOLUTIONS

The exact solution of the Einstein field equation is usually expressed in metric. For example, Minkowski space-time is four-dimension coordinates combining three-dimensional Euclidean space and one-dimension time can be expressed in Cartesian form in Eq. (1):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

and in polar coordinate form in Eq. (2):

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2}$$

Schwarzschild employed a non-rotational sphere-symmetric object with polar coordinate in Eq. (2) with two variables from functions $\nu(r), \lambda(r)$, which was shown in Eq. (3):

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{3}$$

In order to solve the Einstein field equation, Schwarzschild used a vacuum condition, let $R_{\mu\nu} = 0$, calculating Ricci tensor from Eq. (3), and get the first exact solution of the Einstein field equation, Schwarzschild metric, which was shown in Eq. (4).[1]

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{4}$$

However, Schwarzschild metric cannot be used to describe rotation, axial- symmetry, and charged heavenly bodies. From the examination of the metric tensor $g_{\mu\nu}$ in the Schwarzschild metric, one can obtain the components:

$$\begin{aligned} &g_{00} = 1 - \frac{2M}{r},\ g_{11} = -(1 - \frac{2M}{r})^{-1}, \\ &g_{22} = -r^2,\ g_{33} = -r^2 sin^2\theta \end{aligned}$$

Which can also be presented as:

$$\begin{aligned} &g_{tt} = 1 - \frac{2M}{r},\ g_{rr} = -(1 - \frac{2M}{r})^{-1}, \\ &g_{\theta\theta} = -r^2,\ g_{\phi\phi} = -r^2 sin^2\theta \end{aligned} \tag{5}$$

Differences of metric tensor $g_{\mu\nu}$ between the Schwarzschild metric (4) and Minkowski space-time (2) are only in ($g_{tt}$) and ($g_{rr}$) terms.

Kerr metric is the second exact solution of the Einstein field equation, which can be used to describe space-time geometry in the vacuum area near a rotational, axial-symmetric heavenly body.[2] It is a generalized form of Schwarzschild metric. Kerr metric in Boyer-Lindquist coordinate system can be expressed in Eq. (6):

$$\begin{aligned} ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4Mra\sin^2\theta}{\rho^2} dtd\phi - \frac{\rho^2}{\Delta} dr^2 \\ -\rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \end{aligned} \tag{6}$$

Where define $\rho^2 \equiv r^2 + a^2 cos^2\theta$ and $\Delta \equiv r^2 - 2Mr + a^2$ , $M$ is the mass of the rotational material, $a$ is the spin parameter or specific angular momentum and is related to the angular momentum $J$ by $a = J/M$. In all physical quality, we adopt $c = G = 1$.

By examining the components of metric tensor $g_{\mu\nu}$ in Eq. (6), one can obtain:

$$\begin{aligned} & g_{00} = 1 - \frac{2Mr}{\rho^2}, g_{11} = -\frac{\rho^2}{\Delta}, g_{22} = -\rho^2, \\ & g_{03} = g_{30} = \frac{2Mra\sin^2\theta}{\rho^2} \\ & g_{33} = -\left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right) sin^2\theta \end{aligned} \tag{7}$$

Comparison the components of the Schwarzschild metric (4) with the Kerr metric (6):

1. Both $g_{03}(g_{t\phi})$ and $g_{30}(g_{\phi t})$ off-diagonal terms in Kerr metric are not present in Schwarzschild metric, apparently due to rotation. If the rotation parameter $a = 0$, these two terms vanish.

2. $g_{00}g_{11} = g_{tt}g_{rr} = -1$ in Schwarzschild metric, but not in Kerr metric.

3. When spin parameter $a = 0$, Kerr metric turns into Schwarzschild metric and therefore is a generalized form of Schwarzschild metric.

## III. TRANSFORMATION OF ELLIPSOID SYMMETRIC ORTHOGONAL COORDINATE

To derive Kerr metric, if we start from the initial assumptions, we must introduce $g_{00}, g_{11}, g_{22}, g_{03}, g_{33}$ five variables, all are a function of $(r, \theta)$, and finally we will get monster-like complex equations. Apparently, due to the off-diagonal term, Kerr metric cannot be solved by the spherical symmetry method used in Schwarzschild metric. Besides, Previous study showed that the space-time of Kerr metric is ellipsoidal.[7].

Different from the derivation methods used in classical works of Chandrasekhar (1983), the author used the changes in coordinate of Kerr metric into ellipsoid symmetry firstly to get a simplified form, and then used Schwarzschild's method to solve Kerr metric. First of all, the following ellipsoid coordinate changes were apply to Eq. (1)[8]:

$$\begin{aligned} x &\to (r^2+a^2)^{1/2} sin\theta cos\phi \\ y &\to (r^2+a^2)^{1/2} sin\theta sin\phi \\ z &\to r cos\theta \\ t &\to t \end{aligned} \tag{8}$$

Where $a$ is the coordinate transformation parameter. The metric under the new coordinate system becomes

$$ds^2 = dt^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \left(r^2+a^2\right) sin^2\theta d\phi^2 \tag{9}$$

Which represents the coordinate with ellipsoid symmetry in vacuum; it can also be obtained by assigning mass $M = 0$ to the Kerr metric (6). Due to the fact that most of the celestial bodies, stars and galaxy for instance, are ellipsoid symmetric, Bijan started from this vacuum ellipsoid coordinate and derived a Schwarzschild-like solution for ellipsoidal celestial objects as following[9]:

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}\frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \left(r^2+a^2\right) sin^2\theta d\phi^2 \tag{10}$$

Eq. (10) morphs into the Schwarzschild's solution (4) when the coordinate transformation parameter $a = 0$ and therefore Eq. (10) is also a generalization of Schwarzschild's solution.

In order to eliminate the difference between Kerr metric and Schwarzschild metric that is described earlier, we can assume to rewrite the Kerr metric in the following coordinates:

$$ds^2 = G'_{00}dT^2 + G'_{11}dr^2 + G'_{22}d\theta^2 + G'_{33}d\Phi^2 \tag{11}$$

To eliminate the off-diagonal term:

$$dT \equiv dt - pd\phi,\ d\Phi \equiv d\phi - qdt \tag{12}$$

to obtain

$$G'_{00}G'_{11} = -1 \tag{13}$$

By comparing the coefficient, Equations (14) to (18) were obtained.

$$G'_{00}p + G'_{33}q = \frac{-2Mrasin^2\theta}{\rho^2} \tag{14}$$

$$G'_{00} + G'_{33}q^2 = 1 - \frac{2Mr}{\rho^2} \tag{15}$$

$$G'_{00}p^2 + G'_{33} = -\left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right) sin^2\theta \tag{16}$$

$$G'_{22} = -\rho^2 \tag{17}$$

$$G'_{11} = \frac{-\rho^2}{\Delta} \tag{18}$$

By solving six variables $G'_{00}, G'_{11}, G'_{22}, G'_{33}, p, q$ in the six dependent Eqs (13) to (18), then we obtain

$$\begin{aligned}
p &= \pm asin^2\theta, take\ positive\ result \\
q &= \pm\frac{a}{r^2 + a^2}, take\ positive\ result \\
G'_{00} &= \frac{\Delta}{\rho^2} \\
G'_{11} &= -\frac{\rho^2}{\Delta} \\
G'_{22} &= -\rho^2 \\
G'_{33} &= -\frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2}
\end{aligned} \tag{19}$$

Substitute into Eq. (8) and we get

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - asin^2\theta d\phi\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2}\left(d\phi - \frac{a}{r^2 + a^2}dt\right)^2 \tag{20}$$

The result can be found in the literature and also textbook by O'Neil. It is also called the Kerr metric with Boyer-Lindquist in orthonormal frame.[10] There is no off-diagonal terms, and $g_{00}g_{11} = -1$ after the coordinate transformation.

## IV. CALCULATING THE RICCI TENSOR

From previous discussion, Eq. (9) can be recognized as the coordinate under the ellipsoid symmetry in vacuum. Therefore, when the mass $M$ approached 0, Kerr metric (20) will also be transformed into Eq. (21), which equals Eq. (9). The differences of metric tensor components are in time-time and radial-radial terms, just the same as between Schwarzschild metric (4) and Minkowski space-time (2). $dT$ and $d\Phi$ defined in Eq. (22) are ellipsoid coordinate transformation functions.

$$ds^2 = \frac{r^2+a^2}{\rho^2}dT^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \frac{\left(r^2+a^2\right)^2 sin^2\theta}{\rho^2}d\Phi^2 \tag{21}$$

$$\begin{aligned} dT &\equiv dt - asin^2\theta d\phi, \\ d\Phi &\equiv d\phi - \frac{a}{r^2+a^2}dt \end{aligned} \tag{22}$$

In this paper, Schwarzschild method was used to solve Kerr metric starting from Eqs.(21)-(22) by introducing two new functions $e^{2\nu(r,T)}, e^{2\lambda(r,T)}$:

$$ds^2 = e^{2\nu(r,T)}dT^2 - e^{2\lambda(r,T)}dr^2 - \rho^2 d\theta^2 - \frac{\left(r^2+a^2\right)^2 sin^2\theta}{\rho^2}d\Phi^2 \tag{23}$$

Define the parameters $\rho^2$ and $h$ as follows

$$\begin{aligned} &\rho^2 \equiv r^2 + a^2cos^2\theta \\ &h \equiv r^2 + a^2 \end{aligned} \tag{24}$$

Metric tensor has the matrix form

$$g_{\mu\nu} = \begin{pmatrix} e^{2\nu(r,T)} & 0 & 0 & 0 \\ 0 & -e^{2\lambda(r,T)} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -\frac{h^2 sin^2\theta}{\rho^2} \end{pmatrix} \tag{25}$$

Chrostoffel Symbols and Ricci tensors can be obtained by the following steps in Eqs.(26)-(27):

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu_\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \tag{26}$$

$$R_{\alpha\beta} = R^{\rho}_{\alpha\rho\beta} = \partial_\rho \Gamma^{\rho}_{\beta\alpha} - \partial_\beta \Gamma^{\rho}_{\rho\alpha} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\lambda}_{\rho\alpha} \tag{27}$$

Non-zero Chrostoffel symbols are listed in Eqs (28) to (40):

$$\Gamma^0_{00} = \partial_0 \nu, \tag{28}$$

$$\Gamma^0_{10} = \Gamma^0_{01} = \partial_1 \nu \tag{29}$$

$$\Gamma^0_{11} = e^{2(\lambda-\nu)} \partial_1 \lambda \tag{30}$$

$$\Gamma^1_{00} = e^{2(\nu-\lambda)} \partial_1 \nu \tag{31}$$

$$\Gamma^1_{10} = \Gamma^1_{01} = \partial_0 \lambda \tag{32}$$

$$\Gamma^1_{11} = \partial_1 \lambda \tag{33}$$

$$\Gamma^1_{22} = -re^{-2\lambda} \tag{34}$$

$$\Gamma^1_{33} = -re^{-2\lambda} sin^2\theta \left( \frac{2h}{\rho^2} - \frac{h^2}{\rho^4} \right) \tag{35}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{r}{\rho^2} \tag{36}$$

$$\Gamma^2_{22} = -\frac{a^2 sin\theta cos\theta}{\rho^2} \tag{37}$$

$$\Gamma^2_{33} = -sin\theta cos\theta \left( \frac{h^3}{\rho^6} \right) \tag{38}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{2r}{h} - \frac{r}{\rho^2} \tag{39}$$

$$\Gamma^3_{32} = \Gamma^3_{23} = cot\theta \left( \frac{h}{\rho^2} \right) \tag{40}$$

After such calculations, the Ricci tensor of all non-zero components can be obtained as follows:

$$R_{00} = \left[ \partial_0^2 \lambda + (\partial_0 \lambda)^2 - \partial_0 \lambda \partial_0 \nu \right] + e^{2(\nu-\lambda)} \left[ -\partial_1 \nu \partial_1 \lambda + (\partial_1 \nu)^2 + \partial_1^2 \nu + \frac{2r}{h} \partial_1 \nu \right] \tag{41}$$

$$R_{01} = R_{10} = \frac{2r}{h} \partial_0 \lambda \tag{42}$$

$$R_{11} = e^{2(\lambda-\nu)} \left[ \partial_0^2 \lambda + (\partial_0 \lambda)^2 - \partial_0 \lambda \partial_0 \nu \right] + \partial_1 \nu \partial_1 \lambda - (\partial_1 \nu)^2 - \partial_1^2 \nu + \frac{2r}{h} \partial_1 \lambda - \frac{2a^2 (cos^4\theta + r^2)}{\rho^4 h} \tag{43}$$

$$R_{12} = R_{21} = \frac{4ra^2 sin\theta cos\theta}{\rho^4} \tag{44}$$

$$R_{22} = e^{-2\lambda} \left( r \left( \partial_1 \lambda - \partial_1 \nu \right) - 1 + \frac{2r^2}{\rho^2} - \frac{2r^2}{h} \right) + \frac{h^2}{\rho^4} \left( \frac{4r^2 - 3\rho^2}{h} \right) \tag{45}$$

$R_{33} =$

$$sin^2\theta \left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right) \left[e^{-2\lambda} \left(r\left(\partial_1\lambda - \partial_1\nu\right) - \left(\frac{\rho^2 - r^2}{\rho^2} + \frac{hr^2}{\rho^2(2\rho^2 - h)}\right)\right) + \frac{h^2}{\rho^4}\left(\frac{4r^2 - 3\rho^2}{h}\right)\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)^{-1}\right] \tag{46}$$

## V. FINDING A SOLUTION OF THE VACUUM EINSTEIN FIELD EQUATIONS

To solve this equation, one has to use the limit condition when the angular momentum approaches zero ($a \to 0$), the higher order term may be neglected. The Kerr metric in ellipsoid coordinate will become very close to the Schwarzschild metric. Then we obtain

$$\lim_{a\to 0} h = r^2, \ \lim_{a\to 0} \rho = r \tag{47}$$

$$\lim_{a\to 0} R_{00} = \left[\partial_0^2\lambda + (\partial_0\lambda)^2 - \partial_0\lambda\partial_0\nu\right] + e^{2(\nu-\lambda)}\left[-\partial_1\nu\partial_1\lambda + (\partial_1\nu)^2 + \partial_1^2\nu + \frac{2}{r}\partial_1\nu\right] \tag{48}$$

$$\lim_{a\to 0} R_{01} = R_{10} = \frac{2}{r}\partial_0\lambda \tag{49}$$

$$\lim_{a\to 0} R_{11} = e^{2(\lambda-\nu)}\left[\partial_0^2\lambda + (\partial_0\lambda)^2 - \partial_0\lambda\partial_0\nu\right] + \partial_1\nu\partial_1\lambda - (\partial_1\nu)^2 - \partial_1^2\nu + \frac{2}{r}\partial_1\lambda \tag{50}$$

$$\lim_{a\to 0} R_{12} = R_{21} = 0 \tag{51}$$

$$\lim_{a\to 0} R_{22} = e^{-2\lambda}\left(r\left(\partial_1\lambda - \partial_1\nu\right) - 1\right) + 1 \tag{52}$$

$$\lim_{a\to 0} R_{33} = sin^2\theta\left[e^{-2\lambda}\left(r\left(\partial_1\lambda - \partial_1\nu\right) - 1\right) + 1\right] = sin^2\theta R_{22} \tag{53}$$

From $R_{01} = 0$, we get $\frac{2}{r}\partial_0\lambda = 0$ , $\lambda = \lambda(r), \lambda$ is time dependent. Using the result, and substitute them to Eq. (48) and Eq. (50), we obtain

$$e^{-2\lambda(r)}R_{00} + e^{-2\nu(r,T)}R_{11} = \frac{2}{r}\left(\partial_1\nu(r,T) + \partial_1\lambda(r)\right) = 0 \tag{54}$$

$$\frac{2}{r}\left(\partial_1\nu(r,T) + \partial_1\lambda(r)\right) = 0 \tag{55}$$

Solving this equation yields $\nu(r,T) + \lambda(r) =$ const. Next, the time coordinate is redefined in Eq. (21) by replacing $dT \to e^{const.}dT$, so that $\nu(r,T) = \nu(r) = -\lambda(r)$. Therefore

$$e^{2\nu(r)} = e^{-2\lambda(r)}. \tag{56}$$

Substitute into Eq.(53), we obtain

$$\lim_{a\to 0} R_{22} = e^{-2\lambda}\left(r\left(\partial_1\lambda - \partial_1\nu\right) - 1\right) + 1 = 0 \tag{57}$$

$$e^{2\nu} = 1 + \frac{C}{r},\ let\ C = -2M \tag{58}$$

So, under the limit condition when angular momentum approaches zero ($a \to 0$), the equations could be solved as follows

$$\begin{aligned} \lim_{a\to 0} e^{2\nu} &= 1 - \frac{2M}{r} = \frac{r^2 - 2Mr}{r^2} \\ \lim_{a\to 0} e^{2\lambda} &= \left(1 - \frac{2M}{r}\right)^{-1} = \frac{r^2}{r^2 - 2Mr} \end{aligned} \tag{59}$$

One could also demand another limit condition of flat space-time, where the mass approaches zero ($M \to 0$) in the Eq. (18), then we have

$$\begin{aligned} \lim_{M\to 0} e^{2\nu} &= \frac{r^2 + a^2}{\rho^2} = \frac{r^2 + a^2}{r^2 + a^2 cos^2\theta} \\ \lim_{M\to 0} e^{2\lambda} &= \frac{\rho^2}{r^2 + a^2} = \frac{r^2 + a^2 cos^2\theta}{r^2 + a^2} \end{aligned} \tag{60}$$

Deduced from the above conditions in Eqs. (59) to (60), the Ricci tensors could be solved

$$\begin{aligned} e^{2\nu} &= \frac{r^2 - 2Mr + a^2}{r^2 + a^2 cos^2\theta} \\ e^{2\lambda} &= \frac{r^2 + a^2 cos^2\theta}{r^2 - 2Mr + a^2} \end{aligned} \tag{61}$$

Finally, we get the Kerr metric

$$\begin{aligned} ds^2 = &\frac{r^2 - 2Mr + a^2}{\rho^2}\left(dt - asin^2\theta d\phi\right)^2 - \frac{\rho^2}{r^2 - 2Mr + a^2}dr^2 - \rho^2 d\theta^2 \\ &- \frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2}\left(d\phi - \frac{a}{r^2 + a^2}dt\right)^2 \end{aligned} \tag{62}$$

## VI. DISCUSSION

It is proven that the Kerr metric can be obtained by combining the ellipsoid coordinate transformation and the assumptions listed in Eqs. (21) to (23) following these steps: transforming the Euclidian four-dimention space-time in Equation (1) to vacuum Minkowski space-time with ellipsoid symmetry in Equation 9; transforming from $(t, r, \theta, \phi)$ to $(T, r, \theta, \Phi)$ under the new coordinate system to eliminate the major difference in metric tensor components between the Kerr metric and the Schwarzschild metric: there are no off-diagonal terms and the product of $g_{00}g_{11}$ becomes -1; solving vacuum Einstein's equation by using the Schwarzschild method from Eq. (23); applying limit method to calculate Ricci curvature tensor; and finally deducting the Kerr metric.

Table I shows the list of the metric tensor components discussed in previous sections, including the Minkowski space-time, the Schwarzschild solution, empty ellipsoid, a Schwarzschild-like ellipsoid solution, and the Kerr solution. The Minkowski space-time and the Schwarzschild solution have spherical symmetry, and the others have ellipsoid symmetry.

Further, some of the characteristics with deeper physics meaning of ellipsoid symmetry, Kerr metric, and rotating black hole can be obtained from this new coordinate function $dT, d\Phi$. Remember, when $a$ approaches to zero $(a \to 0), dT, d\Phi$ degenerates to $dt, d\phi$.

### A. Ellipsoid symmetry and the Kerr metric

While metric with spherical symmetry in vacuum has the following expression:

$$-r^2 d\theta^2 - r^2 sin^2\theta d\theta^2 \tag{63}$$

And metric of ellipsoid symmetric in vacuum has the following expression

$$\begin{aligned} &-\rho^2 d\theta^2 - \left(r^2 + a^2\right) sin^2\theta d\phi^2 \\ &= -\rho^2 d\theta^2 - \left(\frac{r^2 + a^2}{a}\right)\left(asin^2\theta\right) d\phi^2 \end{aligned} \tag{64}$$

Where $\frac{a}{r^2+a^2}$ and a $sin^2\theta$ term can be seen in multiply and divide combination. Terms of $d\theta^2, d\phi^2$ in the Kerr metric is showed as follows

$$\begin{aligned} &-\rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right) sin^2\theta d\phi^2 \\ &= -\rho^2 d\theta^2 - \left(\frac{r^2 + a^2}{a} + \frac{2Mrasin^2\theta}{\rho^2}\right)\left(asin^2\theta\right) d\phi^2 \end{aligned} \tag{65}$$

Where $\frac{a}{r^2+a^2}$ and $asin^2\theta$ term can also be seen in linear combination. The $a$ in Eq. (64) represents a parameter in the ellipsoid symmetric coordinate transformation, however,$a$ in Kerr metric (65) represents a spin parameter, which is proportional to angular momentum. Both $a$ 's are equivalent in mathematical perspective and used to transform the space-time into ellipsoid symmetry with a rotational symmetric z-axis. When $a \to 0$, both Eq. (64) and Eq. (65) degenerate into spherical symmetry in Eq. (63).

### B. Frame-dragging angular momentum

In physics, a spinning heavenly body with a non-zero mass will generate a frame-dragging phenomenon along the equator's direction, which has been proven by Gravity Probe B experiment.[11] Therefore, an extra term $\frac{2Mra^2 sin^2\theta}{\rho^2}$ in the Kerr metric (65) compared to the vacuum ellipsoid symmetry in Eq. (64). When the mass approaches zero $M \to 0$, Eq. (65) degenerates into Eq. (64).

To order to describe frame-dragging, Kerr metric can be re-written as follows

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\ &= \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right)dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}\left(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 \end{aligned} \tag{66}$$

The definition of angular momentum ($\Omega$) in frame-dragging:

$$\begin{aligned} \Omega = -\frac{g_{t\phi}}{g_{\phi}\phi} &= \frac{\frac{2Mrasin^2\theta}{\rho^2}}{\left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right)sin^2\theta} \\ &= \frac{2Mra}{\rho^2\left(r^2+a^2\right) + 2Mra^2sin^2\theta} \\ &= \frac{2Mr}{\rho^2\left(\frac{r^2+a^2}{a}\right) + 2Mr\left(asin^2\theta\right)} \end{aligned} \tag{67}$$

So, we see both the $asin^2\theta$ and $\frac{a}{r^2+a^2}$ term in $\Omega$ , which means $dT, d\Phi$ would have some relation with frame-dragging angular momentum.

### C. Black hole angular velocity

Its close relationship to the black hole angular velocity ($\Omega_H$) can be easily identified by examining $d\Phi$ term as follows

$$\begin{aligned} & d\Phi = d\phi - \frac{a}{r^2+a^2}dt \\ & \Omega_H = \frac{a}{r_+^2 + a^2} \\ & from\ \Delta = 0,\ solve\ r_\pm = M \pm \sqrt{M^2 - a^2} \end{aligned} \tag{68}$$

Based on this derivation, in the future we will further study whether the method mentioned in this paper can be extended to other more general cases. For example, suppose we start with three functions $e^{2\nu(r)}, e^{2\lambda(r)}, e^{2\mu(r,\theta)}$ such as

$$ds^2 = \frac{e^{2\nu(r)}}{\rho^2}dT^2 - \frac{\rho^2}{e^{2\lambda(r)}}dr^2 - \rho^2 d\theta^2 - \frac{e^{2\mu(r,\theta)}}{\rho^2}d\Phi^2 \tag{69}$$

Besides, as $dT, d\Phi$ is shown to be related with ellipsoid symmetry, frame-dragging angular momentum, and black hole angular velocity, which are all rotation parameters, it deserves further study to determine if this method could be extended to solve the other axial-symmetry exact solutions of vacuum Einstein's field equation.

## VII. CONCLUSION

In this paper, we derive the Kerr metric from the coordinate transformation method. First, we obtain the Kerr Metric with Boyer-Lindquist in orthonormal frame, and then we prove that it is possible to derive the Kerr metric from the vacuum ellipsoid symmetry, and this derivation allows us to better understand the physical properties of the rotating black hole, such as the frame-dragging effect, and the angular velocity. This deduction method is different from classical papers written by Kerr and Chandrasekhar, and is expected to encourage future study in this subject.


## ACKNOWLEDGMENTS

I would like to especially thank Ruby Lin, Dr. Simon Lin of Academia Sinica, Prof. Anthony Zee, and a wonderful NTU OpenCourseWare[12] taught by Prof. Yeong-Chuan Kao. I also feel indebted to my sister, Shanrong, for her advice in the refinement of my English writing.


TABLE I. Metric Tensor Components and Symmetry

| Metric Tensor | $dt^2(dT^2)$ | $dr^2$ | $d\theta^2$ | $d\phi^2(d\Phi^2)$ | Symmetry and State |
|---|---|---|---|---|---|
| Minkowski | $1$ | $-1$ | $-r^2$ | $-r^2 sin^2\theta$ | Spherical, Empty |
| Schwarzschild | $\frac{r^2-2Mr}{r^2}$ | $-\frac{r^2}{r^2-2Mr}$ | $-r^2$ | $-r^2 sin^2\theta$ | Spherical, Static, Mass |
| Ellipsoid | $\frac{r^2+a^2}{\rho^2}$ | $-\frac{\rho^2}{r^2+a^2}$ | $-\rho^2$ | $-\frac{\left(r^2+a^2\right)^2 sin^2\theta}{\rho^2}$ | Ellipsoid, Empty |
| Schwarzschild-like | $\frac{r^2-2Mr}{r^2}$ | $-\frac{r^2}{r^2-2Mr}\frac{\rho^2}{r^2+a^2}$ | $-\rho^2$ | $-\left(r^2+a^2\right) sin^2\theta$ | Ellipsoid, Static, Mass |
| Kerr | $\frac{r^2-2Mr+a^2}{\rho^2}$ | $-\frac{\rho^2}{r^2-2Mr+a^2}$ | $-\rho^2$ | $-\frac{\left(r^2+a^2\right)^2 sin^2\theta}{\rho^2}$ | Ellipsoid,Axisymmetric, Mass |

* unclejoe0306@gmail.com; permanent address: Health 101 clinic, 1F., No.97, Guling St., Zhongzheng Dist., Taipei City 100, Taiwan

[1] K. Schwarzschild, "On the Gravitational Field of a Point-Mass, According to Einstein's Theory," Abraham Zelmanov J. **1**, 10–16 (1916).

[2] R. P. Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics," Phys. Rev. Lett. **11**, 237–238 (1963).

[3] R. Penrose and W. Rindler, *Spinors and Space-Time* (Cambridge University Press, Cambridge, 1984).

[4] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Equations* (Cambridge University Press, Cambridge, 1981).

[5] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

[6] J. Enderlein, "A heuristic way of obtaining the Kerr metric," Am. J. Phys. **65**(9), 897–902 (1997).

[7] A. Krasinski, "Ellipsoidal Space-Time, Sources for the Kerr Metric," Ann. Phys. **112**(1), 22–40 (1978).

[8] L. Landau and E. Lifshitz, *The Mathematical Theory of Black Holes*, Volume 2 (Pergamon Press, 1987).

[9] N. Bijan, "Schwarzschild-Like Solution for Ellipsoidal Celestial Objects," Int. J. Phys. Sci. **51**, 477–492 (2011).

[10] B. O'Neil, *The Geometry of Kerr Black Holes* (A. K. Peters, Wellesley, Massachusetts, 1995).

[11] C. W. F. Everitt. et al, "Gravity Probe B: Final results of a space experiment to test general relativity," Phys. Rev. Lett. **106**, 201101 (2011).

[12] *National Taiwan University OpenCourseWare: general relativity"* Web Site, `<http://ocw.aca.ntu.edu.tw/ntu-ocw/index.php/ocw/cou/099S134>`.