Question 431 : A Definite Integral for Pi

Edgar Valdebenito

abstract
This note presents a definite integral for pi

1. INTRODUCTION. The number pi is defined by

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265... \tag{1}$$

2. A DEFINITE INTEGRAL FOR PI.

$$\frac{2\pi}{3\sqrt{3}} = \int_0^{\sqrt[3]{4}/3} \ln\left(\frac{\cos\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}\frac{(3x)^{3/2}}{2}\right)}{\cos\left(\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}\frac{(3x)^{3/2}}{2}\right)}\right)dx \tag{2}$$

Related integrals

$$\frac{2\pi}{3\sqrt{3}} = \int_0^{\sqrt[3]{4}/3} \ln\left(\frac{\sin\left(\frac{\pi}{6}+\frac{1}{3}\cos^{-1}\frac{(3x)^{3/2}}{2}\right)}{\sin\left(\frac{\pi}{6}-\frac{1}{3}\cos^{-1}\frac{(3x)^{3/2}}{2}\right)}\right)dx \tag{3}$$

$$\frac{\pi}{\sqrt{3}\,\sqrt[3]{4}} = \int_0^{\pi/6} \frac{\sin(3x)}{\sqrt[3]{\cos(3x)}} \ln\left(\frac{\sin\left(\frac{\pi}{6}+x\right)}{\sin\left(\frac{\pi}{6}-x\right)}\right)dx \tag{4}$$

$$\frac{\sqrt{3}\,\pi}{\sqrt[3]{4}} = \int_0^1 \frac{1}{\sqrt[3]{x}} \ln\left(\frac{\cos\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}x\right)}{\cos\left(\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}x\right)}\right)dx \tag{5}$$

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