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Neutrosophic Soft Matrix and its application to Decision Making

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Abstract: The motivation of this paper is to extend the concept of Neutrosophic soft matrix (NSM) theory. Some basic definitions of classical matrix theory in the parlance of neutrosophic soft set theory have been presented with proper examples. Then, a theoretical studies of some traditional operations of NSM have been developed. Finally, a decision making theory has been proposed by developing an appropriate solution algorithm, namely, score function algorithm and it has been illustrated by suitable examples.

Keywords: Intuitionistic fuzzy soft matrix, Neutrosophic soft set, Neutrosophic soft matrix and different operators, Application in decision making.

1 Introduction

Researchers in economics, sociology, medical science, engineering, environment science and many other several fields deal daily with the vague, imprecise and occasionally insufficient information of modeling uncertain data. Such uncertainties are usually handled with the help of the topics like probability, fuzzy sets [1], intuitionistic fuzzy sets [2], interval mathematics, rough sets etc. But, Molodtsov [3] has shown that each of the above topics suffers from inherent difficulties possibly due to inadequacy of their parametrization tool and there after, he initiated a novel concept 'soft set theory' for modeling vagueness and uncertainties. It is completely free from the parametrization inadequacy syndrome of different theories dealing with uncertainty. This makes the theory very convenient, efficient and easily applicable in practice. Molodtsov [3] successfully applied several directions for the applications of soft set theory, such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration and probability etc. In 2010, Cagman and Enginoglu [4] introduced a new soft set based decision making method which selects a set of optimum elements from the alternatives. Maji et al. [5, 6] have done further research on soft set theory.

Presence of vagueness demanded 'fuzzy soft set' [7] to come into picture. But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information situation. For that, Maji et al. [8, 9] have introduced the notion 'intuitionistic fuzzy soft set' in 2001. Matrices play an important role in the broad area of science and engineering. Classical matrix theory sometimes fails to solve the problems involving uncertainties. Hence, several authors proposed the matrix representation of soft set, fuzzy soft set, intuitionistic fuzzy soft set and applied these in certain decision making problems, for instance Cagman and Enginoglu [10], Yong and Chenli [11], Borah et al. [12], Neog and Sut [13], Broumi et al. [14], Mondal and Roy [15], Chetia and Das [16], Basu et al. [17], Rajarajeswari and Dhanalakshmi [18].

Evaluation of non-membership values is also not always possible for the same reason as in case of membership values and so, there exist an indeterministic part upon which hesitation survives. As a result, Smarandache [19, 20] has introduced the concept of **Neutrosophic Set (NS)** which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji [21] has introduced a combined concept **Neutrosophic soft set (NSS)**. Using this concept, several mathematicians have produced their research works in different mathematical structures, for instance Deli [22, 24], Broumi and Smarandache [25]. Later, this concept has been modified by Deli and Broumi [26]. Accordingly, Bera and Mahapatra [23, 27-31] introduce some view on algebraic structure on neutrosophic soft set. The development of decision making algorithms over neutrosophic soft set theory are seen in the literatures [32-37].

The present study aims to extend the NSM theory by developing the basic definitions of classical matrix theory and by establishing some results in NSS theory context. The organisation of the paper is as following :

Section 2 deals some preliminary necessary definitions which will be used in rest of this paper. In Section 3, the concept of NSM has been discussed broadly with suitable examples. Then, some traditional operators of NSM are proposed along with some properties in Section 4. In Section 5, a decision making algorithm has been developed and applied in two different situations. Firstly, it has been adopted in a class room to select the best student in an academic year and then in national security system to emphasize the security management in five mega cities. This algorithm is much more brief and simple rather than others. Moreover, a decision can be made with respect to a lot of parameters concerning the fact easily by that. That is why, this algorithm is more generous, we think. Finally, the conclusion of the present work has been stated in Section 6.

2 Preliminaries

In this section, we recall some necessary definitions related to fuzzy set, intuitionistic fuzzy soft matrix, neutrosophic set, neutrosophic soft set, NSM for the sake of completeness.

2.1 Definition [28]

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t - norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative.
(ii) $*$ is continuous.
(iii) $a * 1 = 1 * a = a,\ \forall a \in [0,1]$.
(iv) $a * b \leq c * d$ if $a \leq c,\ b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab, a * b = \min\{a, b\}, a * b = \max\{a + b - 1, 0\}$.

2.2 Definition [28]

A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t - conorm (s - norm) if $\diamond$ satisfies the following conditions :
(i) $\diamond$ is commutative and associative.
(ii) $\diamond$ is continuous.
(iii) $a \diamond 0 = 0 \diamond a = a,\ \forall a \in [0,1]$.
(iv) $a \diamond b \leq c \diamond d$ if $a \leq c,\ b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

2.3 Definition [16]

Let U be an initial universe, E be the set of parameters and $A \subseteq E$. Let, (f_A, E) be an intuitionistic fuzzy soft set over U. Then a subset of $U \times E$ is uniquely defined by $R_A = \{(u, e) : e \in A, u \in f_A(e)\}$ which is called a relation form of (f_A, E). The membership function and non-membership functions are written by $\mu_{R_A} : U \times E \to [0,1]$ and $\nu_{R_A} : U \times E \to [0,1]$ where $\mu_{R_A}(u, e) \in [0,1]$ and $\nu_{R_A}(u, e) \in [0,1]$ are the membership value and non-membership value, respectively of $u \in U$ for each $e \in E$. If $(\mu_{ij}, \nu_{ij}) = (\mu_{R_A}(u_i, e_j), \nu_{R_A}(u_i, e_j))$, we can define a matrix $[(\mu_{ij}, \nu_{ij})]_{m \times n} =$

$$\begin{pmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{pmatrix}$$

which is called an $m \times n$ IFSM of the IFSS (f_A, E) over U. Therefore, we can say that a IFSS (f_A, E) is uniquely characterised by the matrix $[(\mu_{ij}, \nu_{ij})]_{m \times n}$ and both concepts are interchangeable. The set of all $m \times n$ IFS matrices over U will be denoted by IFSM$_{m \times n}$.

2.4 Definition [20]

Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$. That is $T_A, I_A, F_A : X \to]^{-}0, 1^{+}[$. There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$ and so, $^{-}0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^{+}$.

2.5 Definition [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.

2.6 Definition [21]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NSS over U, where $F : A \to NS(U)$ is a mapping.

This concept has been modified by Deli and Broumi [26] as given below.

2.7 Definition [26]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs,

$$N = \{(e, \{< x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0,1]$ are respectively called truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

2.7.1 *Example*

Let $U = \{h_1, h_2, h_3\}$ be a set of houses and $E = \{e_1(\text{beautiful}), e_2(\text{good location}), e_3, (\text{green surrounding})\}$ be a set of parameters describing the nature of houses. Let,

$$f_N(e_1) = \{< h_1,(0.5,0.6,0.3)>,< h_2,(0.4,0.7,0.6)>, < h_3,(0.6,0.2,0.3)>\}$$
$$f_N(e_2) = \{< h_1,(0.6,0.3,0.5)>,< h_2,(0.7,0.4,0.3)>, < h_3,(0.8,0.1,0.2)>\}$$
$$f_N(e_3) = \{< h_1,(0.7,0.4,0.3)>,< h_2,(0.6,0.7,0.2)>, < h_3,(0.7,0.2,0.5)>\}$$

Then $N = \{[e_1, f_N(e_1)], [e_2, f_N(e_2)], [e_3, f_N(e_3)]\}$ is an NSS over (U, E). The tabular representation of the NSS N is given in Table 1.

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

Table 1 : Tabular form of NSS N.

2.8 Definition [26]

1. The complement of a neutrosophic soft set N is denoted by N^o and is defined by :

$$N^o = \{(e, \{< x, F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x) >: x \in U\}) : e \in E\}$$

2. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then N_1 is said to be the neutrosophic soft subset of N_2 if $\forall e \in E$ and $x \in U$

$$T_{f_{N_1}(e)}(x) \leq T_{f_{N_2}(e)}(x), \quad I_{f_{N_1}(e)}(x) \geq I_{f_{N_2}(e)}(x), \quad F_{f_{N_1}(e)}(x) \geq F_{f_{N_2}(e)}(x).$$

We write $N_1 \subseteq N_2$ and then N_2 is the neutrosophic soft superset of N_1.

3. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined by

$$N_3 = \{(e, \{< x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x), I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x), F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)$.

4. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined by :

$$N_4 = \{(e, \{< x, T_{f_{N_4}(e)}(x), I_{f_{N_4}(e)}(x), F_{f_{N_4}(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_{N_4}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x), I_{f_{N_4}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x), F_{f_{N_4}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x)$.

2.9 Definition [26]

1. Let N be a neutrosophic soft set over $N(U)$. Then a subset of $N(U) \times E$ is uniquely defined by : $R_N = \{(f_N(x), x) : x \in E, f_N(x) \in N(U)\}$ which is called a relation form of (N, E). The characteristic function of R_N is written as :

$$\Theta_{R_N} : N(U) \times E \to [0,1] \times [0,1] \times [0,1] \quad \text{by}$$
$$\Theta_{R_N}(u, x) = (T_{f_N(x)}(u), I_{f_N(x)}(u), F_{f_N(x)}(u))$$

where $T_{f_N(x)}(u), I_{f_N(x)}(u), F_{f_N(x)}(u)$ are truth-membership, indeterminacy-membership and falsity-membership of $u \in U$, respectively.

2. Let $U = \{u_1, u_2, \cdots, u_m\}, E = \{x_1, x_2, \cdots, x_n\}$ and N be a neutrosophic soft set over $N(U)$. Then,

R_N	$f_N(x_1)$	$f_N(x_2)$	$\cdots$	$f_N(x_n)$
u_1	$\Theta_{R_N}(u_1, x_1)$	$\Theta_{R_N}(u_1, x_2)$	$\cdots$	$\Theta_{R_N}(u_1, x_n)$
u_2	$\Theta_{R_N}(u_2, x_1)$	$\Theta_{R_N}(u_2, x_2)$	$\cdots$	$\Theta_{R_N}(u_2, x_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\Theta_{R_N}(u_m, x_1)$	$\Theta_{R_N}(u_m, x_2)$	$\cdots$	$\Theta_{R_N}(u_m, x_n)$

If $a_{ij} = \Theta_{R_N}(u_i, x_j)$, we can define a matrix

$$[a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

such that $a_{ij} = (T_{f_N(x_j)}(u_i), I_{f_N(x_j)}(u_i), F_{f_N(x_j)}(u_i)) = (T^a_{ij}, I^a_{ij}, F^a_{ij})$, which is called an $m \times n$ neutrosophic soft matrix (NS-matrix) of the neutrosophic soft set N over $N(U)$.

According to this definition, a neutrosophic soft set N is uniquely characterised by a matrix $[a_{ij}]_{m\times n}$. Therefore, we shall identify any neutrosophic soft set with it's soft NS-matrix and use these two concepts as interchangeable. The set of all $m \times n$ NS-matrix over $N(U)$ will be denoted by $\tilde{N}_{m\times n}$. From now on we shall delete the subscripts $m \times n$ of $[a_{ij}]_{m\times n}$, we use $[a_{ij}]$ instead of $[a_{ij}]_{m\times n}$, since $[a_{ij}] \in \tilde{N}_{m\times n}$ means that $[a_{ij}]$ is an $m \times n$ NS-matrix for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$.

2.10 Definition [26]

Let $[a_{ij}], [b_{ij}] \in \tilde{N}_{m\times n}$. Then,

1. $[a_{ij}]$ is a zero NS-matrix, denoted by $[\tilde{0}]$, if $a_{ij} = (0, 1, 1), \ \forall i, j$.

2. $[a_{ij}]$ is a universal NS-matrix, denoted by $[\tilde{1}]$, if $a_{ij} = (1, 0, 0), \ \forall i, j$.

3. $[a_{ij}]$ is an NS-submatrix of $[b_{ij}]$, denoted by $[a_{ij}]\tilde{\subseteq}[b_{ij}]$, if $T^a_{ij} \leq T^b_{ij}, \ I^a_{ij} \geq I^b_{ij}, \ F^a_{ij} \geq F^b_{ij}, \ \forall i, j$.

4. $[a_{ij}]$ and $[b_{ij}]$ are equal NS- matrices, denoted by $[a_{ij}] = [b_{ij}]$, if $a_{ij} = b_{ij}, \ \forall i, j$.

5. Complement of $[a_{ij}]$ is denoted by $[a_{ij}]^o$ and is defined as another NS-matrix $[c_{ij}]$ such that $c_{ij} = (F^a_{ij}, 1 - I^a_{ij}, T^a_{ij}), \ \forall i, j$.

3 Neutrosophic soft matrix

In this section, we have introduced some definitions and have included some new operations related to NSM.

3.1 Definition

Let $U = \{u_1, u_2, \cdots, u_m\}$ and $E = \{e_1, e_2, \cdots, e_n\}$ be the universal set of objects and the parametric set, respectively. Suppose, N be a neutrosophic soft set over (U, E) given by $N = \{<(e, f_N(e)) >: e \in E\}$ where

$$f_N(e) = \{< u, (T_{f_N(e)}(u), I_{f_N(e)}(u), F_{f_N(e)}(u)) >: u \in U\}.$$

Thus, $f_N(e)$ corresponds a relation on $\{e\} \times U$ i.e., $f_N(e) = \{(e, u_i) : 1 \le i \le m\}$ for each $e \in E$. It is obviously a symmetric relation. Now, consider a relation R_E on $U \times E$ given by $R_E = \{(u, e) : e \in E, u \in f_N(e)\}$. It is called a relation form of the NSS N over (U, E). The characteristic function of R_E is $\chi_{R_E} : U \times E \to [0, 1] \times [0, 1] \times [0, 1]$ and is defined as : $\chi_{R_E}(u, e) = (T_{f_N(e)}(u), I_{f_N(e)}(u), F_{f_N(e)}(u))$. The tabular representation of R_E is given in Table 2.

	e_1	e_2	$\cdots$	e_n
u_1	$\chi_{R_E}(u_1, e_1)$	$\chi_{R_E}(u_1, e_2)$	$\cdots$	$\chi_{R_E}(u_1, e_n)$
u_2	$\chi_{R_E}(u_2, e_1)$	$\chi_{R_E}(u_2, e_2)$	$\cdots$	$\chi_{R_E}(u_2, e_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\chi_{R_E}(u_m, e_1)$	$\chi_{R_E}(u_m, e_2)$	$\cdots$	$\chi_{R_E}(u_m, e_n)$

Table 2 : Tabular form of R_E

If $a_{ij} = \chi_{R_E}(u_i, e_j)$, then we can define a matrix

$$[a_{ij}]_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij} = (T_{f_N(e_j)}(u_i), I_{f_N(e_j)}(u_i), F_{f_N(e_j)}(u_i)) = (T^a_{ij}, I^a_{ij}, F^a_{ij})$.

Thus, we shall identify any neutrosophic soft set with it's NSM and use these two concepts as interchangeable. Since we consider the full parametric set E, so each NSS N over (U, E) corresponds a unique NSM $[a_{ij}]_{m\times n}$ where cardinality of U and E are m and n, respectively. To get another NSM of the same order over (U, E), we need to define another NSS over (U, E). The set of all NSMs of order $m \times n$ is denoted by $NSM_{m\times n}$. Whenever U and E are fixed, we get all NSMs of unique order i.e., to obtain an NSM of distinct order, at least any of U and E will have to be changed.

3.1.1 *Example*

Consider the Example [2.7.1]. The relation form of the NSS N over the said (U, E) is

	e_1	e_2	e_3
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

Hence, the NSM corresponding to this NSS N over (U, E) is :

$$[a_{ij}]_{3\times 3} = \begin{pmatrix} (0.5, 0.6, 0.3) & (0.6, 0.3, 0.5) & (0.7, 0.4, 0.3) \\ (0.4, 0.7, 0.6) & (0.7, 0.4, 0.3) & (0.6, 0.7, 0.2) \\ (0.6, 0.2, 0.3) & (0.8, 0.1, 0.2) & (0.7, 0.2, 0.5) \end{pmatrix}$$

Next, let $E_1 = \{e_1(\text{cheap}), e_2(\text{moderate}), e_3(\text{high}), e_4(\text{very high})\}$ be another set of parameters describing the cost of houses in U. The relation form of an NSS M over (U, E_1) is written as :

	e_1	e_2	e_3	e_4
h_1	(.4, .5, .5)	(.5, .7, .6)	(.2, .5, .8)	(.5, .6, .4)
h_2	(.6, .4, .7)	(.6, .3, .4)	(.7, .6, .5)	(.8, .4, .3)
h_3	(.7, .3, .4)	(.5, .2, .5)	(.8, .4, .4)	(.1, .6, .6)

Here, the NSM corresponding to the NSS M over (U, E_1) is $[b_{ij}]_{3\times 4} =$

$$\begin{pmatrix} (.4, .5, .5) & (.5, .7, .6) & (.2, .5, .8) & (.5, .6, .4) \\ (.6, .4, .7) & (.6, .3, .4) & (.7, .6, .5) & (.8, .4, .3) \\ (.7, .3, .4) & (.5, .2, .5) & (.8, .4, .4) & (.1, .6, .6) \end{pmatrix}$$

3.2 Definition

Let $A = [a_{ij}] \in NSM_{m\times n}$ where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then,

1. A is called a square NSM if $m = n$ i.e., if the number of rows and the number of columns are equal. The NSS corresponding to this NSM has the same number of objects and parameters.

2. A square NSM $A = [a_{ij}]_{n\times n}$ is called upper triangular NSM if $a_{ij} = (0, 1, 1),\ \forall i > j$ and is called lower triangular NSM if $a_{ij} = (0, 1, 1),\ \forall i < j$.

A is called triangular NSM if it is either neutrosophic soft upper triangular or neutrosophic soft lower triangular matrix.

3. The transpose of a square NSM $A = [a_{ij}]_{n\times n}$ is another square NSM of same order obtained from $[a_{ij}]$ by interchanging it's rows and columns. It is denoted by A^t. Thus $A^t = [a_{ij}]^t = [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t = [(T^a_{ji}, I^a_{ji}, F^a_{ji})]$. The NSS corresponding to A^t becomes a new NSS over the same universe and the same parametric set.

4. A square NSM $A = [a_{ij}]_{n\times n}$ is said to be a symmetric NSM if $A^t = A$ i.e., if $a_{ij} = a_{ji}$ or $(T^a_{ij}, I^a_{ij}, F^a_{ij}) = (T^a_{ji}, I^a_{ji}, F^a_{ji}),\ \forall i, j$.

3.3 Definition

Let $A = [a_{ij}] \in NSM_{m\times n}$, where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then, the scalar multiple of NSM A by a scalar k is defined by $kA = [ka_{ij}]_{m\times n}$ where $0 \le k \le 1$.

3.3.1 *Example*

Let $A = [a_{ij}]_{2\times 3} =$

$$\begin{pmatrix} (0.4, 0.5, 0.5) & (0.5, 0.7, 0.6) & (0.5, 0.6, 0.4) \\ (0.6, 0.4, 0.7) & (0.7, 0.3, 0.4) & (0.8, 0.4, 0.3) \end{pmatrix}$$

be an NSM. Then the scalar multiple of this matrix by $k = 0.5$ is $kA = [ka_{ij}]_{2\times 3} =$

$$\begin{pmatrix} (0.20, 0.25, 0.25) & (0.25, 0.35, 0.30) & (0.25, 0.30, 0.20) \\ (0.30, 0.20, 0.35) & (0.35, 0.15, 0.20) & (0.40, 0.20, 0.15) \end{pmatrix}$$

3.4 Proposition

Let $A = [a_{ij}], B = [b_{ij}] \in NSM_{m\times n}$ where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. For two scalars $s, k \in [0, 1]$,
(i) $s(kA) = (sk)A$. (ii) $s \le k \Rightarrow sA \le kA$. (iii) $A \subseteq B \Rightarrow sA \subseteq sB$.

Proof.

$$\begin{aligned} \text{(i)}\ \ s(kA) &= s[ka_{ij}] = s[(kT^a_{ij}, kI^a_{ij}, kF^a_{ij})] \\ &= [(skT^a_{ij}, skI^a_{ij}, skF^a_{ij})] = sk[(T^a_{ij}, I^a_{ij}, F^a_{ij})] \\ &= sk[a_{ij}] = (sk)A. \end{aligned}$$

(ii) Since $T^a_{ij}, I^a_{ij}, F^a_{ij} \in [0, 1],\ \forall i, j$ so, $sT^a_{ij} \le kT^a_{ij}$, $sI^a_{ij} \le kI^a_{ij}$, $sF^a_{ij} \le kF^a_{ij}$.
Now, $sA = [(sT^a_{ij}, sI^a_{ij}, sF^a_{ij})] \le [(kT^a_{ij}, kI^a_{ij}, kF^a_{ij})] = kA$.

$$\begin{aligned} \text{(iii)}\ \ A \subseteq B &\Rightarrow [a_{ij}] \subseteq [b_{ij}] \\ &\Rightarrow T^a_{ij} \le T^b_{ij}, I^a_{ij} \ge I^b_{ij}, F^a_{ij} \ge F^b_{ij}, \forall i, j \\ &\Rightarrow sT^a_{ij} \le sT^b_{ij}, sI^a_{ij} \ge sI^b_{ij}, sF^a_{ij} \ge sF^b_{ij}, \forall i, j \\ &\Rightarrow s[a_{ij}] \subseteq s[b_{ij}] \\ &\Rightarrow sA \subseteq sB \end{aligned}$$

3.5 Theorem

Let $A = [a_{ij}]_{m\times n}$ be an NSM where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then,
(i) $(kA)^t = k\,A^t$ for $k \in [0, 1]$ being a scalar.
(ii) $(A^t)^t = A$.
(iii) If $A = [a_{ij}]_{n\times n}$ is an upper triangular (lower triangular) NSM, then A^t is lower triangular (upper triangular) NSM.

Proof.(i) Here $(kA)^t,\ k\,A^t \in NSM_{n\times m}$. Now,

$$\begin{aligned} (kA)^t &= [(kT^a_{ij}, kI^a_{ij}, kF^a_{ij})]^t = [(kT^a_{ji}, kI^a_{ji}, kF^a_{ji})] \\ &= k[(T^a_{ji}, I^a_{ji}, F^a_{ji})] = k[(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t = k\,A^t. \end{aligned}$$

(ii) Here $A^t \in NSM_{n\times m}$ and so $(A^t)^t \in NSM_{m\times n}$. Now,

$$\begin{aligned} (A^t)^t &= ([(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t)^t = [(T^a_{ji}, I^a_{ji}, F^a_{ji})]^t \\ &= [(T^a_{ij}, I^a_{ij}, F^a_{ij})] = A. \end{aligned}$$

(iii) Straight forward.

3.6 Definition

Let $A = [a_{ij}] \in NSM_{m\times n}$, where $m = n$ and $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then, the trace of NSM A is denoted by $tr(A)$ and is defined as $tr(A) = \sum_{i=1}^{m}[T^a_{ii} - (I^a_{ii} + F^a_{ii})]$.

3.6.1 *Example*

Let $A = [a_{ij}]_{3\times 3} =$

$$\begin{pmatrix} (0.5, 0.6, 0.3) & (0.6, 0.3, 0.5) & (0.7, 0.4, 0.3) \\ (0.4, 0.7, 0.6) & (0.7, 0.4, 0.3) & (0.6, 0.7, 0.2) \\ (0.6, 0.2, 0.3) & (0.8, 0.1, 0.2) & (0.7, 0.2, 0.5) \end{pmatrix}$$

be an NSM. Then $tr(A) = (0.5-0.6-0.3)+(0.7-0.4-0.3)+(0.7-0.2-0.5) = -0.4$

3.7 Proposition

Let $A = [a_{ij}] \in NSM_{n\times n}$, where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. If $k \in [0, 1]$ is a scalar, then $tr(kA) = k\,tr(A)$.

Proof. $tr(kA) = \sum_{i=1}^{n}[kT^a_{ii} - (kI^a_{ii} + kF^a_{ii})] = k\sum_{i=1}^{n}[T^a_{ii} - (I^a_{ii} + F^a_{ii})] = k\,tr(A)$.

3.8 Max-Min Product of NSMs

Two NSMs A and B are said to be conformable for the product $A \otimes B$ if the number of columns of the NSM A be equal to the number of rows of the NSM B and this product becomes also an NSM. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $A \otimes B = [c_{ik}]_{m\times p}$ where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij}), b_{jk} = (T^b_{jk}, I^b_{jk}, F^b_{jk})$ and $c_{ik} = (\max_j \min(T^a_{ij}, T^b_{jk}), \min_j \max(I^a_{ij}, I^b_{jk}), \min_j \max(F^a_{ij}, F^b_{jk}))$. Clearly, $B \otimes A$ can not be defined here.

3.8.1 *Example*

$$\text{Let}\quad A = [a_{ij}]_{3\times 2} = \begin{pmatrix} (0.5, 0.6, 0.3) & (0.6, 0.3, 0.5) \\ (0.4, 0.7, 0.6) & (0.7, 0.4, 0.3) \\ (0.6, 0.2, 0.3) & (0.8, 0.1, 0.2) \end{pmatrix}$$

and $B = [b_{jk}]_{2\times 3} =$

$$\begin{pmatrix} (0.4, 0.5, 0.5) & (0.5, 0.7, 0.6) & (0.5, 0.6, 0.4) \\ (0.6, 0.4, 0.7) & (0.7, 0.3, 0.4) & (0.8, 0.4, 0.3) \end{pmatrix}$$

be two NSMs. Then, $A \otimes B = [c_{ik}]_{3\times 3} =$

$$\begin{pmatrix} (0.6, 0.4, 0.5) & (0.6, 0.3, 0.5) & (0.6, 0.4, 0.4) \\ (0.6, 0.4, 0.6) & (0.7, 0.4, 0.4) & (0.7, 0.4, 0.3) \\ (0.6, 0.4, 0.5) & (0.7, 0.3, 0.4) & (0.8, 0.4, 0.3) \end{pmatrix}$$

One calculation is provided herewith for convenience of $A \otimes B$.

$$\begin{aligned} T^c_{21} &= \max_j\{\min(T^a_{21}, T^b_{11}), \min(T^a_{22}, T^b_{21})\} \\ &= \max\{\min(0.4, 0.4), \min(0.7, 0.6)\} = 0.6 \\ I^c_{21} &= \min_j\{\max(I^a_{21}, I^b_{11}), \max(I^a_{22}, I^b_{21})\} \\ &= \min\{\max(0.7, 0.5), \max(0.4, 0.4)\} = 0.4 \\ F^c_{21} &= \min_j\{\max(F^a_{21}, F^b_{11}), \max(F^a_{22}, F^b_{21})\} \\ &= \min\{\max(0.6, 0.5), \max(0.3, 0.7)\} = 0.6 \end{aligned}$$

Thus, $c_{21} = (0.6, 0.4, 0.6)$ and so on.

3.9 Theorem

Let $A = [a_{ij}]_{m\times n}, B = [b_{jk}]_{n\times p}$ be two NSMs where a_{ij}=$(T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then, $(A \otimes B)^t = B^t \otimes A^t$

Proof. Let $A \otimes B = [c_{ik}]_{m\times p}$. Then $(A \otimes B)^t = [c_{ki}]_{p\times m}, A^t$ = $[a_{ji}]_{n\times m}, B^t = [b_{kj}]_{p\times n}$ and so the order of $(B^t \otimes A^t)$ is $(p\times m)$. Now,

$$\begin{aligned} & (A \otimes B)^t \\ = & [(T^c_{ki}, I^c_{ki}, F^c_{ki})]_{p\times m} \\ = & [(\max_j \min(T^b_{kj}, T^a_{ji}),\ \min_j \max(I^b_{kj}, I^a_{ji}), \\ & \min_j \max(F^b_{kj}, F^a_{ji}))]_{p\times m} \\ = & [(T^b_{kj}, I^b_{kj}, F^b_{kj})]_{p\times n} \otimes [(T^a_{ji}, I^a_{ji}, F^a_{ji})]_{n\times m} = B^t \otimes A^t. \end{aligned}$$

4 Operators of NSMs

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})]$, $B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})] \in NSM_{m\times n}$. Then,

(i) **Union** $A \cup B = C$ where $T^c_{ij} = T^a_{ij} \diamond T^b_{ij}$, $I^c_{ij} = I^a_{ij} * I^b_{ij}$, $F^c_{ij} = F^a_{ij} * F^b_{ij}$, $\forall i, j$.

(ii) **Intersection** $A \cap B = C$ where $T^c_{ij} = T^a_{ij} * T^b_{ij}$, $I^c_{ij} = I^a_{ij} \diamond I^b_{ij}$, $F^c_{ij} = F^a_{ij} \diamond F^b_{ij}$, $\forall i, j$.

(iii)**Arithmetic mean** $A \circledast B = C$ where $T^c_{ij} = \frac{T^a_{ij}+T^b_{ij}}{2}$, $I^c_{ij} = \frac{I^a_{ij}+I^b_{ij}}{2}$, $F^c_{ij} = \frac{F^a_{ij}+F^b_{ij}}{2}$, $\forall i, j$.

(iv) **Weighted arithmetic mean** $A \circledast^w B = C$ where $T^c_{ij} = \frac{w_1 T^a_{ij}+w_2 T^b_{ij}}{w_1+w_2}$, $I^c_{ij} = \frac{w_1 I^a_{ij}+w_2 I^b_{ij}}{w_1+w_2}$, $F^c_{ij} = \frac{w_1 F^a_{ij}+w_2 F^b_{ij}}{w_1+w_2}$, $\forall i, j$ and $w_1, w_2 > 0$.

(v) **Geometric mean** $A \circledcirc B = C$ where $T^c_{ij} = \sqrt{T^a_{ij} \cdot T^b_{ij}}$, $I^c_{ij} = \sqrt{I^a_{ij} \cdot I^b_{ij}}$, $F^c_{ij} = \sqrt{F^a_{ij} \cdot F^b_{ij}}$, $\forall i, j$.

(vi) **Weighted geometric mean** $A \circledcirc^w B = C$ where

$$T^c_{ij} = \sqrt[(w_1+w_2)]{(T^a_{ij})^{w_1} \cdot (T^b_{ij})^{w_2}},$$
$$I^c_{ij} = \sqrt[(w_1+w_2)]{(I^a_{ij})^{w_1} \cdot (I^b_{ij})^{w_2}},$$
$$F^c_{ij} = \sqrt[(w_1+w_2)]{(F^a_{ij})^{w_1} \cdot (F^b_{ij})^{w_2}},\ \forall i, j \text{ and } w_1, w_2 > 0.$$

(vii) **Harmonic mean** $A \boxdot B = C$ where $T^c_{ij} = \frac{2T^a_{ij}T^b_{ij}}{T^a_{ij}+T^b_{ij}}$, $I^c_{ij} = \frac{2I^a_{ij}I^b_{ij}}{I^a_{ij}+I^b_{ij}}$, $F^c_{ij} = \frac{2F^a_{ij}F^b_{ij}}{F^a_{ij}+F^b_{ij}}$, $\forall i, j$.

(viii) **Weighted harmonic mean** $A \boxdot^w B = C$ where $T^c_{ij} = \frac{w_1+w_2}{\frac{w_1}{T^a_{ij}}+\frac{w_2}{T^b_{ij}}}$, $I^c_{ij} = \frac{w_1+w_2}{\frac{w_1}{I^a_{ij}}+\frac{w_2}{I^b_{ij}}}$, $F^c_{ij} = \frac{w_1+w_2}{\frac{w_1}{F^a_{ij}}+\frac{w_2}{F^b_{ij}}}$, $\forall i, j$ and $w_1, w_2 > 0$.

4.1 Proposition

Let $A = [a_{ij}], B = [b_{ij}] \in NSM_{m\times n}$, where a_{ij} = $(T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then,
(i) $(A \cup B)^t = A^t \cup B^t$, $(A \cap B)^t = A^t \cap B^t$.
(ii) $(A \circledast B)^t = A^t \circledast B^t$, $(A \circledast^w B)^t = A^t \circledast^w B^t$.
(iii) $(A \circledcirc B)^t = A^t \circledcirc B^t$, $(A \circledcirc^w B)^t = A^t \circledcirc^w B^t$.
(iv) $(A \boxdot B)^t = A^t \boxdot B^t$, $(A \boxdot^w B)^t = A^t \boxdot^w B^t$.

Proof. (i) Here $A \cup B, (A \cup B)^t, A^t, B^t, A^t \cup B^t \in NSM_{m\times n}$. Now,

$$\begin{aligned} (A \cup B)^t &= [(T^a_{ij} \diamond T^b_{ij}, I^a_{ij} * I^b_{ij}, F^a_{ij} * F^b_{ij})]^t \\ &= [(T^a_{ji} \diamond T^b_{ji}, I^a_{ji} * I^b_{ji}, F^a_{ji} * F^b_{ji})] \\ &= [(T^a_{ji}, I^a_{ji}, F^a_{ji})] \cup [(T^b_{ji}, I^b_{ji}, F^b_{ji})] \\ &= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t \cup [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^t \\ &= A^t \cup B^t. \end{aligned}$$

Next $A \cap B, (A \cap B)^t, A^t \cap B^t \in NSM_{m\times n}$. Now,

$$\begin{aligned} (A \cap B)^t &= [(T^a_{ij} * F^b_{ij}, I^a_{ij} \diamond (1 - I^b_{ij}), F^a_{ij} \diamond T^b_{ij})]^t \\ &= [(T^a_{ji} * F^b_{ji}, I^a_{ji} \diamond (1 - I^b_{ji}), F^a_{ji} \diamond T^b_{ji})] \\ &= [(T^a_{ji}, I^a_{ji}, F^a_{ji})] \cap [(T^b_{ji}, I^b_{ji}, F^b_{ji})] \\ &= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t \cap [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^t \\ &= A^t \cap B^t. \end{aligned}$$

Remaining others can be proved in the similar manner.

4.2 Proposition

Let $A = [a_{ij}], B = [b_{ij}]$ are upper triangular (lower triangular) NSMs of same order. Then (i) $A \cup B$, $A \cap B$ (ii) $A \circledast B$, $A \circledast^w B$ (iii) $A \circledcirc B$, $A \circledcirc^w B$ all are upper triangular (lower triangular) NSMs.

Proof. Straight forward.

4.3 Theorem

Let $A = [a_{ij}], B = [b_{ij}]$ be two symmetric NSMs of same order. Then,

(i) $A \cup A^t, A \cup B, A \cap B, A \circledast B, A \circledast^w B, A \circledcirc B, A \circledcirc^w B, A \boxdot B, A \boxdot^w B$ are so.

(ii) $A \otimes B$ is symmetric iff $A \otimes B = B \otimes A$.
(iii) $A \otimes A^t$, $A^t \otimes A$ both are symmetric.

Proof. Here $A^t = A$ and $B^t = B$ as both are symmetric NSMs. Clearly $A \cup A^t, A \cup B, A \cap B, A \circledast B, A \circledast^w B, A \circledcirc B, A \circledcirc^w B, A \boxdot B, A \boxdot^w B, A \otimes B, B \otimes A, A \otimes A^t, A^t \otimes A$ all are well defined as both the NSMs are same order and square. Now,

(i) These are left to the reader.
(ii) $(A \otimes B)^t = B^t \otimes A^t = B \otimes A = A \otimes B$.
(iii) $(A \otimes A^t)^t = (A^t)^t \otimes A^t = A \otimes A^t$ and $(A^t \otimes A)^t = A^t \otimes (A^t)^t = A^t \otimes A$.

4.4 Proposition

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})] \in NSM_{m\times n}$. Then,
(i) $(A \cup B)^o = A^o \cap B^o$, $(A \cap B)^o = A^o \cup B^o$.
(ii) $(A \circledast B)^o = A^o \circledast B^o$, $(A \circledast^w B)^o = A^o \circledast^w B^o$.

Proof. (i) Here $(A \cup B)^o, A^o \cap B^o \in NSM_{m\times n}$. Now,

$$\begin{aligned}
(A \cup B)^o &= [(T^a_{ij} \diamond T^b_{ij}, I^a_{ij} * I^b_{ij}, F^a_{ij} * F^b_{ij})]^o \\
&= [(F^a_{ij} * F^b_{ij}, 1 - (I^a_{ij} * I^b_{ij}), T^a_{ij} \diamond T^b_{ij})] \\
&= [(F^a_{ij} * F^b_{ij}, (1 - I^a_{ij}) \diamond (1 - I^b_{ij}), T^a_{ij} \diamond T^b_{ij})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \cap [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \cap [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o \\
&= A^o \cap B^o.
\end{aligned}$$

Next, $(A \cap B)^o, A^o \cup B^o \in NSM_{m\times n}$. Now,

$$\begin{aligned}
(A \cap B)^o &= [(T^a_{ij} * T^b_{ij}, I^a_{ij} \diamond I^b_{ij}, F^a_{ij} \diamond F^b_{ij})]^o \\
&= [(F^a_{ij} \diamond F^b_{ij}, 1 - (I^a_{ij} \diamond I^b_{ij}), T^a_{ij} * T^b_{ij})] \\
&= [(F^a_{ij} \diamond F^b_{ij}, (1 - I^a_{ij}) * (1 - I^b_{ij}), T^a_{ij} * T^b_{ij})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \cup [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \cup [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o \\
&= A^o \cup B^o.
\end{aligned}$$

Note : Here, $(1 - I^a_{ij}) \diamond (1 - I^b_{ij}) = 1 - (I^a_{ij} * I^b_{ij})$ and $(1 - I^a_{ij}) * (1 - I^b_{ij}) = 1 - (I^a_{ij} \diamond I^b_{ij})$ hold for dual pairs of non-parameterized t-norms and s-norms e.g., $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$ and $a \diamond b = \min\{a + b, 1\}$ etc.

(ii) Here $(A \circledast B)^o, A^o \circledast B^o \in NSM_{m\times n}$.

$$\begin{aligned}
(A \circledast B)^o &= [(\frac{T^a_{ij} + T^b_{ij}}{2}, \frac{I^a_{ij} + I^b_{ij}}{2}, \frac{F^a_{ij} + F^b_{ij}}{2})]^o \\
&= [(\frac{F^a_{ij} + F^b_{ij}}{2}, 1 - \frac{I^a_{ij} + I^b_{ij}}{2}, \frac{T^a_{ij} + T^b_{ij}}{2})] \\
&= [(\frac{F^a_{ij} + F^b_{ij}}{2}, \frac{(1 - I^a_{ij}) + (1 - I^b_{ij})}{2}, \frac{T^a_{ij} + T^b_{ij}}{2})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \circledast [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \circledast [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o \\
&= A^o \circledast B^o.
\end{aligned}$$

Next, for $w_1, w_2 > 0$, we have,

$$\begin{aligned}
&(A \circledast^w B)^o \\
&= [(\frac{w_1 T^a_{ij} + w_2 T^b_{ij}}{w_1 + w_2}, \frac{w_1 I^a_{ij} + w_2 I^b_{ij}}{w_1 + w_2}, \frac{w_1 F^a_{ij} + w_2 F^b_{ij}}{w_1 + w_2})]^o \\
&= [(\frac{w_1 F^a_{ij} + w_2 F^b_{ij}}{w_1 + w_2}, 1 - \frac{w_1 I^a_{ij} + w_2 I^b_{ij}}{w_1 + w_2}, \frac{w_1 T^a_{ij} + w_2 T^b_{ij}}{w_1 + w_2})] \\
&= [(\frac{w_1 F^a_{ij} + w_2 F^b_{ij}}{w_1 + w_2}, \frac{w_1(1 - I^a_{ij}) + w_2(1 - I^b_{ij})}{w_1 + w_2}, \\
&\quad \frac{w_1 T^a_{ij} + w_2 T^b_{ij}}{w_1 + w_2})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \circledast^w [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \circledast^w [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o = A^o \circledast^w B^o.
\end{aligned}$$

4.5 Proposition (Commutative law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})] \in NSM_{m\times n}$. Then,
(i) $A \cup B = B \cup A$, $A \cap B = B \cap A$ (ii) $A \circledast B = B \circledast A$, $A \circledast^w B = B \circledast^w A$ (iii) $A \circledcirc B = B \circledcirc A$, $A \circledcirc^w B = B \circledcirc^w A$ (iv) $A \boxdot B = B \boxdot A$, $A \boxdot^w B = B \boxdot^w A$.

Proof. Obvious

4.6 Proposition (Associative law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})], C = [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \in NSM_{m\times n}$. Then,
(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (iii) $(A \circledast B) \circledast C \neq A \circledast (B \circledast C)$ (iv) $(A \circledcirc B) \circledcirc C \neq A \circledcirc (B \circledcirc C)$ (v) $(A \boxdot B) \boxdot C \neq A \boxdot (B \boxdot C)$.

Proof. (i) Clearly $(A \cup B) \cup C, A \cup (B \cup C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
&(A \cup B) \cup C \\
&= [(T^a_{ij} \diamond T^b_{ij}, I^a_{ij} * I^b_{ij}, F^a_{ij} * F^b_{ij})] \cup [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \\
&= [((T^a_{ij} \diamond T^b_{ij}) \diamond T^c_{ij}, (I^a_{ij} * I^b_{ij}) * I^c_{ij}, (F^a_{ij} * F^b_{ij}) * F^c_{ij})] \\
&= [(T^a_{ij} \diamond (T^b_{ij} \diamond T^c_{ij}), I^a_{ij} * (I^b_{ij} * I^c_{ij}), F^a_{ij} * (F^b_{ij} * F^c_{ij}))] \\
&= A \cup (B \cup C)
\end{aligned}$$

Similarly, the other results can be verified.

4.7 Proposition (Distributive law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})], C = [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \in NSM_{m\times n}$. Then,
(i) $A \cap (B \circledast C) = (A \cap B) \circledast (A \cap C)$, $(A \circledast B) \cap C =$

$(A \cap C) \circledast (B \cap C)$.
(ii) $A \cup (B \circledast C) = (A \cup B) \circledast (A \cup C)$, $(A \circledast B) \cup C = (A \cup C) \circledast (B \cup C)$.

Proof. (i) Here $A \cap (B \circledast C)$, $(A \cap B) \circledast (A \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& A \cap (B \circledast C) \\
= \; & [(T^a_{ij}, I^a_{ij}, F^a_{ij})] \cap [(\frac{T^b_{ij}+T^c_{ij}}{2}, \frac{I^b_{ij}+I^c_{ij}}{2}, \frac{F^b_{ij}+F^c_{ij}}{2})] \\
= \; & [(T^a_{ij} * \frac{T^b_{ij}+T^c_{ij}}{2}, I^a_{ij} \diamond \frac{I^b_{ij}+I^c_{ij}}{2}, F^a_{ij} \diamond \frac{F^b_{ij}+F^c_{ij}}{2})] \\
= \; & [(\frac{T^a_{ij} * T^b_{ij} + T^a_{ij} * T^c_{ij}}{2}, \frac{I^a_{ij} \diamond I^b_{ij} + I^a_{ij} \diamond I^c_{ij}}{2}, \\
& \frac{F^a_{ij} \diamond F^b_{ij} + F^a_{ij} \diamond F^c_{ij}}{2})] \\
= \; & [(T^a_{ij} * T^b_{ij}, I^a_{ij} \diamond I^b_{ij}, F^a_{ij} \diamond F^b_{ij})] \\
& \circledast [(T^a_{ij} * T^c_{ij}, I^a_{ij} \diamond I^c_{ij}, F^a_{ij} \diamond F^c_{ij})] \\
= \; & (A \cap B) \circledast (A \cap C)
\end{aligned}$$

Next $(A \circledast B) \cap C$, $(A \cap C) \circledast (B \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& (A \circledast B) \cap C \\
= \; & [(\frac{T^a_{ij}+T^b_{ij}}{2}, \frac{I^a_{ij}+I^b_{ij}}{2}, \frac{F^a_{ij}+F^b_{ij}}{2})] \cap [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \\
= \; & [(\frac{T^a_{ij}+T^b_{ij}}{2} * T^c_{ij}, \frac{I^a_{ij}+I^b_{ij}}{2} \diamond I^c_{ij}, \frac{F^a_{ij}+F^b_{ij}}{2} \diamond F^c_{ij})] \\
= \; & [(\frac{T^a_{ij} * T^c_{ij} + T^b_{ij} * T^c_{ij}}{2}, \frac{I^a_{ij} \diamond I^c_{ij} + I^b_{ij} \diamond I^c_{ij}}{2}, \\
& \frac{F^a_{ij} \diamond F^c_{ij} + F^b_{ij} \diamond F^c_{ij}}{2})] \\
= \; & [(T^a_{ij} * T^c_{ij}, I^a_{ij} \diamond I^c_{ij}, F^a_{ij} \diamond F^c_{ij})] \\
& \circledast [(T^b_{ij} * T^c_{ij}, I^b_{ij} \diamond I^c_{ij}, F^b_{ij} \diamond F^c_{ij})] \\
= \; & (A \cap C) \circledast (B \cap C)
\end{aligned}$$

In a similar way, the remaining can be established.

4.8 Proposition (Distributive law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})], C = [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \in NSM_{m\times n}$.
If $a * b = \min\{a,b\}$ and $a \diamond b = \max\{a,b\}$, then

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(iii)$A \circledast (B \cup C) = (A \circledast B) \cup (A \circledast C)$, $(A \cup B) \circledast C = (A \circledast C) \cup (B \circledast C)$.
$A \circledast (B \cap C) = (A \circledast B) \cap (A \circledast C)$, $(A \cap B) \circledast C = (A \circledast C) \cap (B \circledast C)$.

(iv) $A \circledcirc (B \cup C) = (A \circledcirc B) \cup (A \circledcirc C)$, $(A \cup B) \circledcirc C = (A \circledcirc C) \cup (B \circledcirc C)$.
$A \circledcirc (B \cap C) = (A \circledcirc B) \cap (A \circledcirc C)$, $(A \cap B) \circledcirc C = (A \circledcirc C) \cap (B \circledcirc C)$.

(v) $A \boxdot (B \cup C) = (A \boxdot B) \cup (A \boxdot C)$, $(A \cup B) \boxdot C = (A \boxdot C) \cup (B \boxdot C)$.
$A \boxdot (B \cap C) = (A \boxdot B) \cap (A \boxdot C)$, $(A \cap B) \boxdot C = (A \boxdot C) \cap (B \boxdot C)$.

Proof. We shall here prove (i), (iv) and (v) only. The others can be proved in the similar fashion.

(i) Here $A \cap (B \cup C)$, $(A \cap B) \cup (A \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& A \cap (B \cup C) \\
= \; & [(T^a_{ij}, I^a_{ij}, F^a_{ij})] \cap [(\max\{T^b_{ij}, T^c_{ij}\}, \min\{I^b_{ij}, I^c_{ij}\}, \\
& \min\{F^b_{ij}, F^c_{ij}\})] \\
= \; & [(\min\{T^a_{ij}, \max\{T^b_{ij}, T^c_{ij}\}\}, \max\{I^a_{ij}, \min\{I^b_{ij}, I^c_{ij}\}\}, \\
& \max\{F^a_{ij}, \min\{F^b_{ij}, F^c_{ij}\}\})] \\
= \; & [(\max\{\min\{T^a_{ij}, T^b_{ij}\}, \min\{T^a_{ij}, T^c_{ij}\}\}, \min\{\max\{I^a_{ij}, I^b_{ij}\}, \\
& \max\{I^a_{ij}, I^c_{ij}\}\}, \min\{\max\{F^a_{ij}, F^b_{ij}\}, \max\{F^a_{ij}, F^c_{ij}\}\})] \\
= \; & [(\min\{T^a_{ij}, T^b_{ij}\}, \max\{I^a_{ij}, I^b_{ij}\}, \max\{F^a_{ij}, F^b_{ij}\})] \\
& \cup [(\min\{T^a_{ij}, T^c_{ij}\}, \max\{I^a_{ij}, I^c_{ij}\}, \max\{F^a_{ij}, F^c_{ij}\})] \\
= \; & (A \cap B) \cup (A \cap C)
\end{aligned}$$

Next $(A \cup B) \cap C, (A \cap C) \cup (B \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& (A \cup B) \cap C \\
= \; & [(\max\{T^a_{ij}, T^b_{ij}\}, \min\{I^a_{ij}, I^b_{ij}\}, \min\{F^a_{ij}, F^b_{ij}\})] \\
& \cap [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \\
= \; & [(\min\{\max\{T^a_{ij}, T^b_{ij}\}, T^c_{ij}\}, \max\{\min\{I^a_{ij}, I^b_{ij}\}, I^c_{ij}\}, \\
& \max\{\min\{F^a_{ij}, F^b_{ij}\}, F^c_{ij}\})] \\
= \; & [(\max\{\min\{T^a_{ij}, T^c_{ij}\}, \min\{T^b_{ij}, T^c_{ij}\}\}, \min\{\max\{I^a_{ij}, I^c_{ij}\}, \\
& \max\{I^b_{ij}, I^c_{ij}\}\}, \min\{\max\{F^a_{ij}, F^c_{ij}\}, \max\{F^b_{ij}, F^c_{ij}\}\})] \\
= \; & [(\min\{T^a_{ij}, T^c_{ij}\}, \max\{I^a_{ij}, I^c_{ij}\}, \max\{F^a_{ij}, F^c_{ij}\})] \\
& \cup [(\min\{T^b_{ij}, T^c_{ij}\}, \max\{I^b_{ij}, I^c_{ij}\}, \max\{F^b_{ij}, F^c_{ij}\})] \\
= \; & (A \cap C) \cup (B \cap C)
\end{aligned}$$

(iv) Here $A \circledcirc (B \cup C), (A \circledcirc B) \cup (A \circledcirc C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& A \circledcirc (B \cup C) \\
= \; & [(T^a_{ij}, I^a_{ij}, F^a_{ij})] \circledcirc [(\max\{T^b_{ij}, T^c_{ij}\}, \min\{I^b_{ij}, I^c_{ij}\}, \\
& \min\{F^b_{ij}, F^c_{ij}\})] \\
= \; & [(\sqrt{T^a_{ij} \cdot \max\{T^b_{ij}, T^c_{ij}\}}, \sqrt{I^a_{ij} \cdot min\{I^b_{ij}, I^c_{ij}\}}, \\
& \sqrt{F^a_{ij} \cdot min\{F^b_{ij}, F^c_{ij}\}})] \\
= \; & [(\max\{\sqrt{T^a_{ij} \cdot T^b_{ij}}, \sqrt{T^a_{ij} \cdot T^c_{ij}}\}, \min\{\sqrt{I^a_{ij} \cdot I^b_{ij}}, \\
& \sqrt{I^a_{ij} \cdot I^c_{ij}}\}, \min\{\sqrt{F^a_{ij} \cdot F^b_{ij}}, \sqrt{F^a_{ij} \cdot F^c_{ij}}\})]
\end{aligned}$$

$$= [(\sqrt{T^a_{ij}\cdot T^b_{ij}}, \sqrt{I^a_{ij}\cdot I^b_{ij}}, \sqrt{F^a_{ij}\cdot F^b_{ij}})] \cup[(\sqrt{T^a_{ij}\cdot T^c_{ij}}, \sqrt{I^a_{ij}\cdot I^c_{ij}}, \sqrt{F^a_{ij}\cdot F^c_{ij}})] = (A \circledcirc B)\cup(A\circledcirc C)$$

Next $(A\cup B)\circledcirc C, (A\circledcirc C)\cup(B\circledcirc C)\in NSM_{m\times n}$. Now,

$$(A\cup B)\circledcirc C = [(\max\{T^a_{ij},T^b_{ij}\},\min\{I^a_{ij},I^b_{ij}\},\min\{F^a_{ij},F^b_{ij}\})] \circledcirc[(T^c_{ij},I^c_{ij},F^c_{ij})]$$
$$= [(\sqrt{\max\{T^a_{ij},T^b_{ij}\}\cdot T^c_{ij}}, \sqrt{\min\{I^a_{ij},I^b_{ij}\}\cdot I^c_{ij}}, \sqrt{\min\{F^a_{ij},F^b_{ij}\}\cdot F^c_{ij}},)]$$
$$= [(\max\{\sqrt{T^a_{ij}\cdot T^c_{ij}}, \sqrt{T^b_{ij}\cdot T^c_{ij}}\}, \min\{\sqrt{I^a_{ij}\cdot I^c_{ij}}, \sqrt{I^b_{ij}\cdot I^c_{ij}}\}, \min\{\sqrt{F^a_{ij}\cdot F^c_{ij}}, \sqrt{F^b_{ij}\cdot F^c_{ij}}\})]$$
$$= [(\sqrt{T^a_{ij}\cdot T^c_{ij}}, \sqrt{I^a_{ij}\cdot I^c_{ij}}, \sqrt{F^a_{ij}\cdot F^c_{ij}})] \cup[(\sqrt{T^b_{ij}\cdot T^c_{ij}}, \sqrt{I^b_{ij}\cdot I^c_{ij}}, \sqrt{F^b_{ij}\cdot F^c_{ij}})] = (A\circledcirc C)\cup(B\circledcirc C)$$

(v) Here $A\boxdot(B\cup C), (A\boxdot B)\cup(A\boxdot C)\in NSM_{m\times n}$. Now,

$$A\boxdot(B\cup C) = [(T^a_{ij},I^a_{ij},F^a_{ij})]\boxdot [(\max\{T^b_{ij},T^c_{ij}\},\min\{I^b_{ij},I^c_{ij}\},\min\{F^b_{ij},F^c_{ij}\})]$$
$$= [(\frac{2\cdot T^a_{ij}\cdot\max\{T^b_{ij},T^c_{ij}\}}{T^a_{ij}+\max\{T^b_{ij},T^c_{ij}\}}, \frac{2\cdot I^a_{ij}\cdot\min\{I^b_{ij},I^c_{ij}\}}{I^a_{ij}+\min\{I^b_{ij},I^c_{ij}\}}, \frac{2\cdot F^a_{ij}\cdot\min\{F^b_{ij},F^c_{ij}\}}{F^a_{ij}+\min\{F^b_{ij},F^c_{ij}\}})]$$
$$= [(\max\{\frac{2T^a_{ij}T^b_{ij}}{T^a_{ij}+T^b_{ij}}, \frac{2T^a_{ij}T^c_{ij}}{T^a_{ij}+T^c_{ij}}\}, \min\{\frac{2I^a_{ij}I^b_{ij}}{I^a_{ij}+I^b_{ij}}, \frac{2I^a_{ij}I^c_{ij}}{I^a_{ij}+I^c_{ij}}\}, \min\{\frac{2F^a_{ij}F^b_{ij}}{F^a_{ij}+F^b_{ij}}, \frac{2F^a_{ij}F^c_{ij}}{F^a_{ij}+F^c_{ij}}\})]$$
$$= [(\frac{2T^a_{ij}T^b_{ij}}{T^a_{ij}+T^b_{ij}}, \frac{2I^a_{ij}I^b_{ij}}{I^a_{ij}+I^b_{ij}}, \frac{2F^a_{ij}F^b_{ij}}{F^a_{ij}+F^b_{ij}})] \cup[(\frac{2T^a_{ij}T^c_{ij}}{T^a_{ij}+T^c_{ij}}, \frac{2I^a_{ij}I^c_{ij}}{I^a_{ij}+I^c_{ij}}, \frac{2F^a_{ij}F^c_{ij}}{F^a_{ij}+F^c_{ij}})] = (A\boxdot B)\cup(A\boxdot C)$$

Next $(A\cup B)\boxdot C, (A\boxdot C)\cup(B\boxdot C)\in NSM_{m\times n}$. Now,

$$(A\cup B)\boxdot C = [(\max\{T^a_{ij},T^b_{ij}\},\min\{I^a_{ij},I^b_{ij}\},\min\{F^a_{ij},F^b_{ij}\})] \boxdot[(T^c_{ij},I^c_{ij},F^c_{ij})]$$
$$= [(\frac{2\cdot\max\{T^a_{ij},T^b_{ij}\}\cdot T^c_{ij}}{\max\{T^b_{ij},T^c_{ij}\}+T^c_{ij}}, \frac{2\cdot\min\{I^a_{ij},I^b_{ij}\}\cdot I^c_{ij}}{\min\{I^b_{ij},I^c_{ij}\}+I^c_{ij}}, \frac{2\cdot\min\{F^a_{ij},F^b_{ij}\}\cdot F^c_{ij}}{\min\{F^b_{ij},F^c_{ij}\}+F^c_{ij}})]$$
$$= [(\max\{\frac{2T^a_{ij}T^c_{ij}}{T^a_{ij}+T^c_{ij}}, \frac{2T^b_{ij}T^c_{ij}}{T^b_{ij}+T^c_{ij}}\}, \min\{\frac{2I^a_{ij}I^c_{ij}}{I^a_{ij}+I^c_{ij}}, \frac{2I^b_{ij}I^c_{ij}}{I^b_{ij}+I^c_{ij}}\}, \min\{\frac{2F^a_{ij}F^c_{ij}}{F^a_{ij}+F^c_{ij}}, \frac{2F^b_{ij}F^c_{ij}}{F^b_{ij}+F^c_{ij}}\})]$$
$$= [(\frac{2T^a_{ij}T^c_{ij}}{T^a_{ij}+T^c_{ij}}, \frac{2I^a_{ij}I^c_{ij}}{I^a_{ij}+I^c_{ij}}, \frac{2F^a_{ij}F^c_{ij}}{F^a_{ij}+F^c_{ij}})] \cup[(\frac{2T^b_{ij}T^c_{ij}}{T^b_{ij}+T^c_{ij}}, \frac{2I^b_{ij}I^c_{ij}}{I^b_{ij}+I^c_{ij}}, \frac{2F^b_{ij}F^c_{ij}}{F^b_{ij}+F^c_{ij}})] = (A\boxdot C)\cup(B\boxdot C)$$

4.9 Proposition (Idempotent law)

Let $A=[(T^a_{ij},I^a_{ij},F^a_{ij})]\in NSM_{m\times n}$. Then,
(i) $A\circledast^w A = A$ (ii) $A\circledcirc^w A = A$ (iii) $A\boxdot^w A = A$.

Proof. For all i,j and $w_1,w_2>0$ we have,

(i) $A\circledast^w A = [(\frac{w_1T^a_{ij}+w_2T^a_{ij}}{w_1+w_2}, \frac{w_1I^a_{ij}+w_2I^a_{ij}}{w_1+w_2}, \frac{w_1F^a_{ij}+w_2F^a_{ij}}{w_1+w_2},)] = [(T^a_{ij},I^a_{ij},F^a_{ij})]=A$.

(ii) $A\circledcirc^w A = [(\sqrt[(w_1+w_2)]{(T^a_{ij})^{w_1}\cdot(T^a_{ij})^{w_2}}, \sqrt[(w_1+w_2)]{(I^a_{ij})^{w_1}\cdot(I^a_{ij})^{w_2}}, \sqrt[(w_1+w_2)]{(F^a_{ij})^{w_1}\cdot(F^a_{ij})^{w_2}})]$
$= [(\sqrt[(w_1+w_2)]{(T^a_{ij})^{w_1+w_2}}, \sqrt[(w_1+w_2)]{(I^a_{ij})^{w_1+w_2}}, \sqrt[(w_1+w_2)]{(F^a_{ij})^{w_1+w_2}})] = [(T^a_{ij},I^a_{ij},F^a_{ij})]=A$.

(iii) $A\boxdot^w A = [(\frac{w_1+w_2}{\frac{w_1}{T^a_{ij}}+\frac{w_2}{T^a_{ij}}}, \frac{w_1+w_2}{\frac{w_1}{I^a_{ij}}+\frac{w_2}{I^a_{ij}}}, \frac{w_1+w_2}{\frac{w_1}{F^a_{ij}}+\frac{w_2}{F^a_{ij}}})] = [(T^a_{ij},I^a_{ij},F^a_{ij})]=A$.

5 Neutrosophic soft matrix theory in decision making (score function algorithm)

5.1 Definition

1. Let $A=[a_{ij}]_{m\times n}$ be an NSM where $a_{ij}=(T^a_{ij},I^a_{ij},F^a_{ij})$. Then the value of the matrix A is denoted by $V(A)$ and is defined as : $V(A)=[v^a_{ij}]_{m\times n}$ where $v^a_{ij}=T^a_{ij}-I^a_{ij}-F^a_{ij},\ \forall i,j$.

2. The score of two NSMs A and B is defined as $S(A,B)=[s_{ij}]_{m\times n}$ where $s_{ij}=v^a_{ij}+v^b_{ij}$. So, $S(A,B)=V(A)+V(B)$.

3. The total score for each object in U is $\Sigma^n_{j=1}s_{ij}$.

5.2 Properties of Score Function

Value matrices are classical real matrices which follow all properties of classical real matrices. The score function is basically a real matrix in classical sense derived from two or more value matrices. So score functions obey all properties of real matrices.

5.3 Methodology

Suppose, N number of decision makers wish to select an object jointly from m number of objects i.e., universal set U with respect to n number of features i.e., parametric set E. Each decision maker forms an NSS over (U, E) and corresponding to each NSS, each get an NSM of order $m \times n$. It needs to compute the value matrix corresponding to each matrix. Then the score matrix and finally, the total score of each object will be calculated.

5.3.1 Algorithm

Step 1 : Construct the NSMs from the given NSSs.
Step 2 : Calculate the value matrices of corresponding NSMs.
Step 3 : Compute the score matrix from value matrices and the total score for each object in U.
Step 4 : Find the object of maximum score and it is the optimal solution.
Step 5 : If score is maximum for more than one object, then find $\Sigma_{j=1}^{n}(s_{ij})^k$, $k \geq 2$ successively. Choose the object of maximum score and hereby the optimal solution.

5.3.2 Case study 1 (application in class room)

Three students $\{s_1, s_2, s_3\}$ from class - x in a school have been shortened to win the best student award in an academic session. A team of three teachers $\{T_1, T_2, T_3\}$ has been formed by the Head Master of that school for this purpose. Final selection is based on the set of parameters $\{e_1, e_2, e_3, e_4, e_5\}$ indicating the quality of student, participation in school cultural programme, class room interactions, maintenance of discipline in class room, daily attendance, respectively. Teachers have given their valuable opinions by the following NSSs separately i.e., first NSS given by first teacher and so on.

$M = \{f_M(e_1), f_M(e_2), f_M(e_3), f_M(e_4), f_M(e_5)\}$ where

$$
\begin{aligned}
f_M(e_1) &= \{< s_1, (0.7, 0.2, 0.6) >, < s_2, (0.6, 0.3, 0.5) >, \\
&\quad < s_3, (0.8, 0.3, 0.5) >\} \\
f_M(e_2) &= \{< s_1, (0.4, 0.6, 0.7) >, < s_2, (0.7, 0.6, 0.3) >, \\
&\quad < s_3, (0.5, 0.5, 0.4) >\} \\
f_M(e_3) &= \{< s_1, (0.5, 0.5, 0.3) >, < s_2, (0.7, 0.4, 0.4) >, \\
&\quad < s_3, (0.6, 0.4, 0.6) >\} \\
f_M(e_4) &= \{< s_1, (0.6, 0.6, 0.5) >, < s_2, (0.5, 0.8, 0.6) >, \\
&\quad < s_3, (0.4, 0.7, 0.4) >\}
\end{aligned}
$$

$N = \{f_N(e_1), f_N(e_2), f_N(e_3), f_N(e_4), f_N(e_5)\}$ where

$$
\begin{aligned}
f_N(e_1) &= \{< s_1, (0.6, 0.4, 0.5) >, < s_2, (0.7, 0.4, 0.2) >, \\
&\quad < s_3, (0.9, 0.4, 0.2) >\} \\
f_N(e_2) &= \{< s_1, (0.5, 0.5, 0.6) >, < s_2, (0.8, 0.5, 0.1) >, \\
&\quad < s_3, (0.6, 0.7, 0.5) >\} \\
f_N(e_3) &= \{< s_1, (0.7, 0.3, 0.4) >, < s_2, (0.8, 0.5, 0.3) >, \\
&\quad < s_3, (0.5, 0.6, 0.7) >\} \\
f_N(e_4) &= \{< s_1, (0.7, 0.5, 0.3) >, < s_2, (0.6, 0.7, 0.5) >, \\
&\quad < s_3, (0.5, 0.5, 0.5) >\} \\
f_N(e_5) &= \{< s_1, (0.6, 0.4, 0.6) >, < s_2, (0.6, 0.3, 0.7) >, \\
&\quad < s_3, (0.8, 0.3, 0.3) >\}\}
\end{aligned}
$$

$P = \{f_P(e_1), f_P(e_2), f_P(e_3), f_P(e_4), f_P(e_5)\}$ where

$$
\begin{aligned}
f_P(e_1) &= \{< s_1, (0.8, 0.3, 0.3) >, < s_2, (0.8, 0.5, 0.3) >, \\
&\quad < s_3, (1.0, 0.4, 0.2) >\} \\
f_P(e_2) &= \{< s_1, (0.6, 0.4, 0.5) >, < s_2, (0.7, 0.6, 0.2) >, \\
&\quad < s_3, (0.8, 0.5, 0.4) >\} \\
f_P(e_3) &= \{< s_1, (0.8, 0.4, 0.1) >, < s_2, (0.7, 0.5, 0.5) >, \\
&\quad < s_3, (0.6, 0.7, 0.3) >\} \\
f_P(e_4) &= \{< s_1, (0.6, 0.6, 0.2) >, < s_2, (0.8, 0.6, 0.4) >, \\
&\quad < s_3, (0.7, 0.3, 0.6) >\} \\
f_P(e_5) &= \{< s_1, (0.8, 0.4, 0.2) >, < s_2, (0.6, 0.4, 0.3) >, \\
&\quad < s_3, (0.7, 0.5, 0.4) >\}\}
\end{aligned}
$$

The above three NSSs are represented by the NSMs A, B and C, respectively, as following :

$$
\begin{pmatrix}
(.7,.2,.6) & (.4,.6,.7) & (.5,.5,.3) & (.6,.6,.5) & (.8,.3,.4) \\
(.6,.3,.5) & (.7,.6,.3) & (.7,.4,.4) & (.5,.8,.6) & (.7,.2,.6) \\
(.8,.3,.5) & (.5,.5,.4) & (.6,.4,.6) & (.4,.7,.4) & (.9,.1,.2)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(.6,.4,.5) & (.5,.5,.6) & (.7,.3,.4) & (.7,.5,.3) & (.6,.4,.6) \\
(.7,.4,.2) & (.8,.5,.1) & (.8,.5,.3) & (.6,.7,.5) & (.6,.3,.7) \\
(.9,.4,.2) & (.6,.7,.5) & (.5,.6,.7) & (.5,.5,.5) & (.8,.3,.3)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(.8,.3,.3) & (.6,.4,.5) & (.8,.4,.1) & (.6,.6,.2) & (.8,.4,.2) \\
(.8,.5,.4) & (.7,.6,.2) & (.7,.5,.5) & (.8,.6,.4) & (.6,.4,.3) \\
(1,.4,.2) & (.8,.5,.4) & (.6,.7,.3) & (.7,.3,.6) & (.7,.5,.4)
\end{pmatrix}
$$

Then the corresponding value matrices are :

$$
V(A) = \begin{pmatrix}
-.1 & -.9 & -.3 & -.5 & 0.1 \\
-.2 & -.2 & -.1 & -.9 & -.1 \\
0.0 & -.4 & -.4 & -.7 & 0.6
\end{pmatrix}
$$

$$
V(B) = \begin{pmatrix}
-.3 & -.6 & 0.0 & -.1 & -.4 \\
0.1 & 0.2 & 0.0 & -.6 & -.4 \\
0.3 & -.6 & -.8 & -.5 & 0.2
\end{pmatrix}
$$

$$
V(C) = \begin{pmatrix}
0.2 & -.3 & 0.3 & -.2 & 0.2 \\
-.1 & -.1 & -.3 & -.2 & -.1 \\
0.4 & -.1 & -.4 & -.2 & -.2
\end{pmatrix}
$$

The score matrix is :

$$S(A,B,C)=\begin{pmatrix} -0.2 & -1.8 & 00.0 & -0.8 & -0.1 \\ -0.2 & -0.1 & -0.4 & -1.7 & -0.6 \\ 00.7 & -1.1 & -1.6 & -1.4 & 00.6 \end{pmatrix}$$

and the total score $=\begin{pmatrix} -2.9 \\ -3.0 \\ -2.8 \end{pmatrix}$

Hence, the student s_3 will be selected for the best student award from class-x in that academic session.

5.3.3 Case study 2 (application in security management)

An important discussion on internal security management has been arranged by the order of Home Minister. Two officers have mate in that discussion to analyse and arrange the security management in five mega-cities e.g., Delhi(D), Mumbai(M), Kolkata(K), Chennai(C), Bengaluru(B). The priority of management is given to the cities based on the set of parameters $\{a,b,c\}$ indicating their geographical position(e.g., having international boarder line, having sea coast etc), population density, past history of terrorist attack, respectively. Following NSSs refer the opinions of two officers individually regarding that matter.

$N_1=\{f_{N_1}(a), f_{N_1}(b), f_{N_1}(c)\}$ where

$$\begin{aligned} f_{N_1}(a) &= \{< D,(0.9,0.4,0.5)>,< M,(0.8,0.5,0.4)>, \\ &< K,(0.7,0.6,0.6)>,< C,(0.6,0.4,0.7)>, \\ &< B,(0.5,0.3,0.8)>\} \\ f_{N_1}(b) &= \{< D,(0.8,0.5,0.5)>,< M,(0.9,0.3,0.3)>, \\ &< K,(0.7,0.6,0.5)>,< C,(0.6,0.7,0.8)>, \\ &< B,(0.6,0.8,0.5)>\} \\ f_{N_1}(c) &= \{< D,(0.7,0.5,0.4)>,< M,(0.9,0.3,0.2)>, \\ &< K,(0.5,0.6,0.7)>,< C,(0.7,0.4,0.6)>, \\ &< B,(0.6,0.3,0.4)>\} \end{aligned}$$

$N_2=\{f_{N_2}(a), f_{N_2}(b), f_{N_2}(c)\}$ where

$$\begin{aligned} f_{N_2}(a) &= \{< D,(1.0,0.5,0.4)>,< M,(0.9,0.4,0.5)>, \\ &< K,(0.7,0.7,0.5)>,< C,(0.6,0.5,0.3)>, \\ &< B,(0.6,0.7,0.4)>\} \\ f_{N_2}(b) &= \{< D,(0.9,0.4,0.5)>,< M,(0.9,0.2,0.3)>, \\ &< K,(0.8,0.5,0.4)>,< C,(0.7,0.7,0.6)>, \\ &< B,(0.6,0.8,0.7)>\} \\ f_{N_2}(c) &= \{< D,(0.8,0.3,0.2)>,< M,(0.9,0.2,0.1)>, \\ &< K,(0.4,0.5,0.6)>,< C,(0.5,0.6,0.6)>, \\ &< B,(0.7,0.4,0.3)>\} \end{aligned}$$

These two NSSs are represented by the NSMs A and B, respectively, as following :

$$A=\begin{pmatrix} (0.9,0.4,0.5) & (0.8,0.5,0.5) & (0.7,0.5,0.4) \\ (0.8,0.5,0.4) & (0.9,0.3,0.3) & (0.9,0.3,0.2) \\ (0.7,0.6,0.6) & (0.7,0.6,0.5) & (0.5,0.6,0.7) \\ (0.6,0.4,0.7) & (0.6,0.7,0.8) & (0.7,0.4,0.6) \\ (0.5,0.3,0.8) & (0.6,0.8,0.5) & (0.6,0.3,0.4) \end{pmatrix}$$

$$B=\begin{pmatrix} (1.0,0.5,0.4) & (0.9,0.4,0.5) & (0.8,0.3,0.2) \\ (0.9,0.4,0.5) & (0.9,0.2,0.3) & (0.9,0.2,0.1) \\ (0.7,0.7,0.5) & (0.8,0.5,0.4) & (0.4,0.5,0.6) \\ (0.6,0.5,0.3) & (0.7,0.7,0.6) & (0.5,0.6,0.6) \\ (0.6,0.7,0.4) & (0.6,0.8,0.7) & (0.7,0.4,0.3) \end{pmatrix}$$

Then the corresponding value matrices are :

$$V(A)=\begin{pmatrix} 0.0 & -.2 & -.2 \\ -.1 & 0.3 & 0.4 \\ -.5 & -.4 & -.8 \\ -.5 & -.9 & -.3 \\ -.6 & -.7 & -.1 \end{pmatrix}$$

$$V(B)=\begin{pmatrix} 0.1 & 0.0 & 0.3 \\ 0.0 & 0.4 & 0.6 \\ -.5 & -.1 & -.7 \\ -.2 & -.6 & -.7 \\ -.5 & -.9 & 0.0 \end{pmatrix}$$

The score matrix and the total score for selection are :

$$S(A,B)=\begin{pmatrix} 00.1 & -0.2 & 00.1 \\ -0.1 & 00.7 & 01.0 \\ -1.0 & -.5 & -1.5 \\ -0.7 & -1.5 & -1.0 \\ -1.1 & -1.6 & -0.1 \end{pmatrix}$$

$$\text{Total score}=\begin{pmatrix} 00.0 \\ 01.6 \\ -3.0 \\ -3.2 \\ -2.8 \end{pmatrix}$$

Hence, the priority of security management should be given in descending order to Mumbai, Delhi, Bangaluru, Kolkata and Chennai.

6 Conclusion

In this paper, some definitions regarding neutrosophic soft matrices have been brought and some new operators have been included, illustrated by suitable examples. Moreover, application of neutrosophic soft matrix theory in decision making problems have been made. We expect, this paper will promote the future study on different algorithms in several other decision making problems.

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