A GROUPOID FOR COMMUTATIVE AND NONCOMMUTATIVE OPERATIONS: A STEP TOWARDS QUANTUM/RELATIVITY UNIFICATION

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The unexploited unification of general relativity and quantum physics is a painstaking issue that prevents physicists to properly understanding the whole of Nature. Here we propose a pure mathematical approach that introduces the problem in terms of group theory. Indeed, we build a cyclic groupoid (a nonemptyset with a binary operation defined on it) that encompasses both the theories as subsets, making it possible to join together two of their most dissimilar experimental results, *i.e*., the commutativity detectable in our macroscopic relativistic world and the non-commutativity detectable in the quantum, microscopic world. Further, we provide a feasible physical counterpart able to throw a bridge between relativity and quantum mechanics, namely, the gravitational force. The latter stands for an operator able to reduce the countless orthonormal bases required by quantum mechanics to just one, *i.e*., the relativistic basis of an observer located in a single cosmic area.

Both general relativity (GR) and quantum physics (QF) describe Nature using mathematical structures such as tensors and probability theory (Yilmaz, 1982; Comte, 1996; Fre, 2013). The unification of GR and QF stands for the Holy Grail in physics:indeed, the two issues, despite the fact that both work very well in the description of different features of Nature, seem to be incomparable and not reducible to a unified, self-consistent framework. Several recent efforts have been provided in order to correlate them (*e.g.*, Ghose, 1997; Skalsky, 2010; Fre, 2013; Elitzur, Dolev, Kolenda, 2017; Aerts, 2017), but they tend to paint a somewhat incomplete picture of the milieu described by GR and QF. Indeed, quantum dynamics and special relativity display very different behaviours and features, starting from the very subtending mathematical structures. General relativity is described on a 3+1 dimensional pseudo-Riemannian manifold with tensor fields obeying certain partial differential equations, while quantum field theory is described on an R^4 projective Hilbert space,with operator-valued fields obeying certain Lorentz-invariant partial differential equations and commutation relationships (Tegmark 2008). Therefore, in the Euclidean framework, describable by three spatial dimensions plus time, we achieve vectors equipped with just one basis, while in the projective Hilbert space of quantum mechanics, equipped with infinite dimensions, we achieve vectors with different bases which are sets of equivalence classes. In other words, the macroscopic world of relativity has an orthonormal basis common to all the observables (*i.e*., a set of vectors *B* in Euclidean or Hilbert space such that every vector can be written as a linear combination of vectors from B, while all vectors from B have length 1 and any two of them are orthogonal (Tumulka , 2009)), while the microscopic world of quantum dynamics displays different bases that are the source of the rather counterintuitive physical effects that we experience during experimental observations.
Efforts have been provided in order to correlate the two different manifolds. For example, techniques of deformation quantization, such as the Wigner-Weyl transform, have provided rules in order to strain the "classical" commutative algebra of observables into a quantum non-commutative one (Curtright et al., 2014). In this paper we will focus, rather than on manifolds, on another important difference between the two physical frameworks of GR and QF: the commutability (Goodman, 2003). In the macroscopic world dictated by GR, the binary operation of multiplication is commutative, i.e., changing the order of the operands does not change the result. This means that: ab=ba. On the other side, in QF, the linear operators representing a pair of physical variables do not commute, rather are mutually complementary. This means that cd ≠ dc. Examples of complementary quantum properties which cannot all be observed or measured simultaneously are position and momentum, energy and duration, spin on different axes, wave and particle, entanglement and coherence, and so on (Kalckar et al., 1996). Here we raise the question whether it is

feasible to correlate the two mathematical operations of commutativity and noncommutativity in an unified scheme. Such issue (a set that encompasses both the commutativity and noncommutativity subsets) cannot be tackled through the otherwise successful Fraenkel-Zermelo set, because its axioms do not work in this peculiar context. Therefore, we ask here: is there a loop between commutativity and noncommutativity, that gives rise to a quantifiable graph? If the answer is affirmative, does this graph have a physical interpretation?

A CYCLIC GROUPOID ENCOMPASSES LOOPS BETWEEN COMMUTATIVE AND NONCOMMUTATIVE OPERATIONS

We start from a groupoid A, encompassing commutative operations, and a groupoid B, encompassing non-commutative ones. We require that a single observable must be either macroscopic (commutative, subjected to relativity) or microscopic (non-commutative, subjected to quantum dynamics). Here we demonstrate that a set C does exist, able to encompass both A and B. The procedure is illustrated in **Figure 1**.

The set A = {a,b,c}, while the operation <o>: AxA → A.
A <o> b =b <o> a = b reads: “(a,b) maps to b”, and “(b,a) maps to b”.
In general, x <o> y = y <o> X for all elements x,y in A. This means that (A, <o>) is an Abelian groupoid, and the operation o is commutative.

The set B = {d,e,f,g}, while the operation <o>: BxB → B.
d <o> e = e reads: “(d,e) maps to e”, and “(e,d) maps to e”. Further, e <o> f = f reads: “(e,f) maps to f”, but f <o> e =f does not map to f”.
This means that (B, <o>) is a non-Abelian groupoid, and the operation<o> is not commutative.

Sets A and B are contained in C. The set C {a,b,c,d,e,f,g}, while the operation [<o>]: CxC → C.
a [<o>] b = b [<o>] a = b commutes.
a [<o>] d = d [<o>] a = d commutes.
b [<o>] e = e [<o>] b = e commutes.
c [<o>] f = f [<o>] c = f commutes.
e [<o>] f = f, f [<o>] e = e does not commute, and so on.
Therefore, (C, [<o>]) is a non-Abelian groupoid, and the operation [<o>] is not commutative.

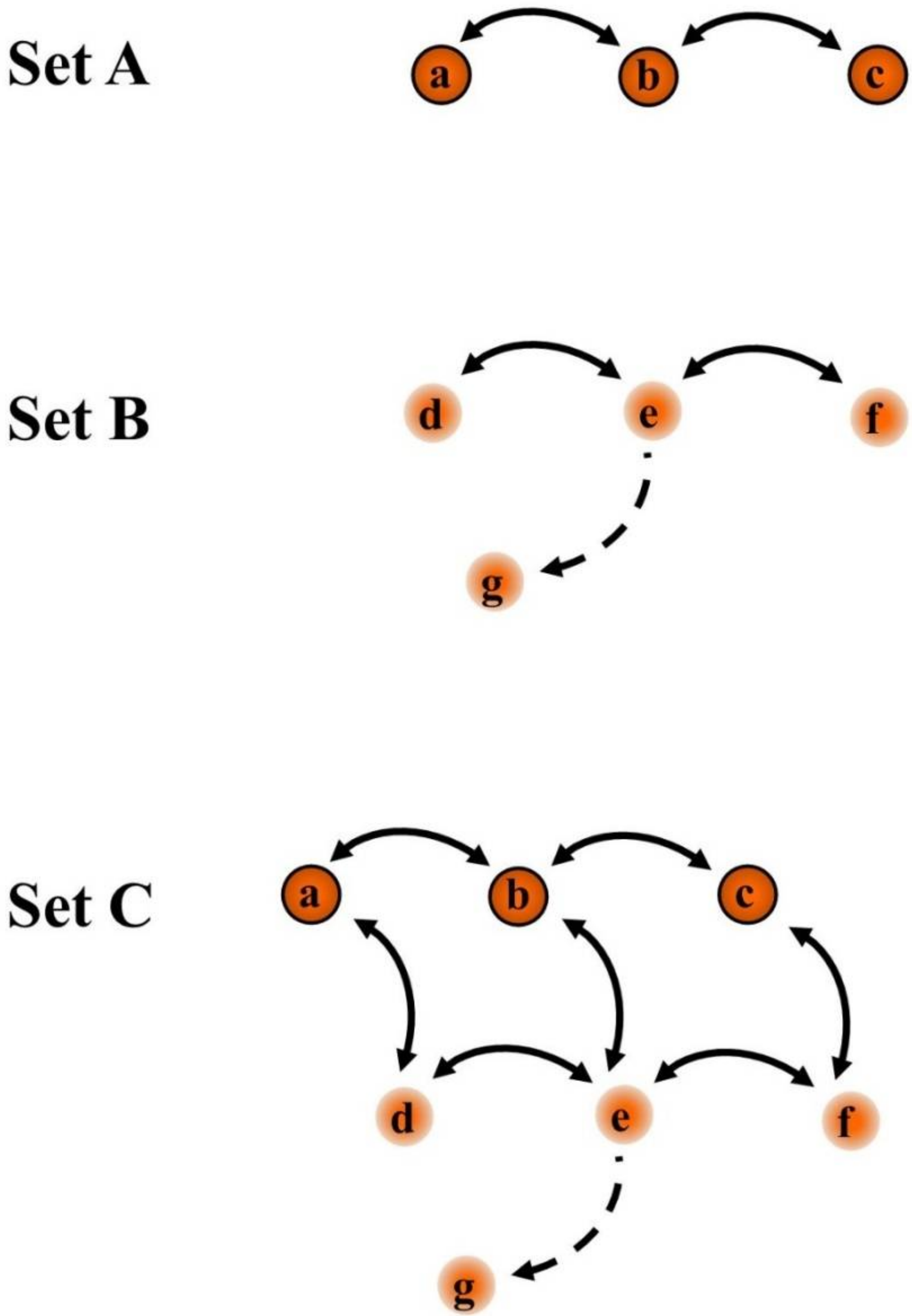

Figure. Three sets in terms of cyclic groupoids. The circles a, a,b and c (encompassed in the set A) stand for physical observables experimentally detectable in our macroscopic relativistic world, equipped with commutative properties. The circles d,e,f and g (encompassed in the set B) stand for physical observables experimentally detectable in our microscopic quantistic world, equipped with noncommutative properties. The set C embeds both the sets A and B. The solid, double-arrowed lines stand for commutative relationships, while the dotted arrows for non-commutative ones.

THE PHYSICAL MEANING OF CYCLIC GRUPOIDS

In order to exit from the realm of the pure mathematics and prove to be useful in the description of Nature, the commutative interactions that allow the unification of the above-mentioned sets A and B into the groupoid C need to display a physical counterpart. In other words, our approach requires a quantifiable physical parameter that stands for the double solid arrows that connects a and d, b and e, c and f (**Figure 2**).
At first, we need to explain why, by a physical point of view, GR displays commutative behaviour, while QF does not.
In the context of QT, before an experimental measurement, the following equation holds:
$A\ \psi_a = a\ \psi_a$
Where A is the observable, ψ_a is its autostate, i.e., the autovector of the operator associated to A, and a is the autovalue of the operator A, to which correspond one or more autostates ψ_a. All the ψ_a constitute a whole, standing for a wave function that is the superposition of all of them. Every ψ_a stands for a physical state achievable when A is measured. Indeed, after an accurate measurement of A, the system collapses to one of the values of a_n, or of a(f). The probability of a single ψ_a to be "chosen" is correlated with the square moduli of the coefficients C_n, or c(f). In other words, after the measure of A, just a single ψ_a is chosen, according to a probability. In simpler words, observables exist that do not commute: this means that, in QT, an orthonormal basis that is common to all the observables does not occur. Instead, every observable displays a different basis.

On the opposite side, the commutative properties of the relativistic world depend on the fact that an orthonormal basis is common to all the observables. Therefore, it is feasible that, leaving apart the differences in the subtending manifolds, the macroscopic world might be characterized by the choice of just one of the countless bases described by QF. Is there a possible mechanism that allows a "simple" reduction of the bases, as it occurs from QT to GR?
Here the gravitational field comes into play. Indeed, this force is neglectable at microlevels of observation, because, in a quantum context, the other three forces and the intrinsic quantum oscillations prevail. This is the realm of non-commutative interactions. However, at higher levels of observation, where Einstein's general equations of relativity work properly, the gravitational field starts to exert its influence, distorting and constraining the cosmic phase space.
Therefore, we hypothesize that gravity is the missing link between commutative and noncommutative interactions: its first observable occurrence at the cosmic meso-scales allows the passage from a quantum regime to a relativistic one. It does not matter whether the matter/energy curves the space manifold, or the manifold curvature gives rise to matter/energy: the required result here is that the orthonormal bases are modified by GR.
In our framework, the gravitational force stands for an operator able to reduce all the orthonormal bases to just one, the one that is detectable by an observer located in her own single zone of the cosmic space, according to the requirements of GR. The choice of coordinate frames dictated by GR reduces, in the frame where the observer is located, the number of bases to just one. Therefore, the gravitational locations established by GR lead to the erasure of the most of multiple bases of QD, leaving just one basis.

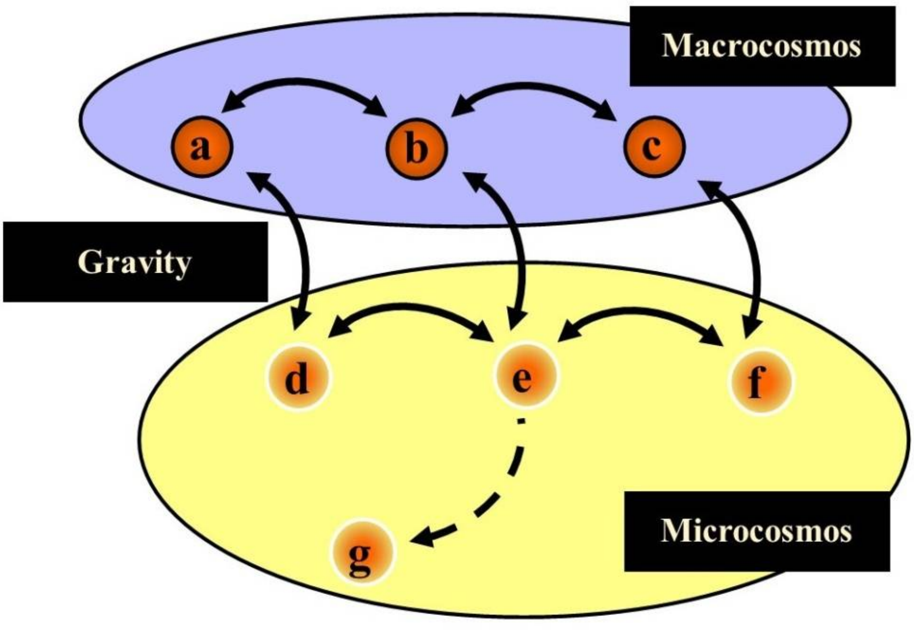

Figure 2. The possible physical counterparts of the mathematical framework described by the set C illustrated in Figure 1. See text for further details.

CONCLUSIONS

Several efforts have been made in order to unify GR and QT, but, until now, the results are missing. Many investigations aimed to provide transformation rules between the two manifolds of GR and QT. To make an example, Feffman et al. (2015) assessed the geometric Whitney problem on how a Riemannian manifold can be constructed to approximate a metric space. Especially, deformation quantization, such as the above-mentioned Wigner-Weyl transform, is able to achieve a complete phase space formulation of quantum mechanics, completely equivalent to the Hilbert-space operator representation, with star-multiplications that are isomorphically parallel operator multiplications (Curtright et al., 2014). By expressing quantum mechanics in the same phase space as that of classical mechanics, the map achieved through the Wigner-Weyl transform facilitates recognition of quantum mechanics as a deformation of the classical one. However, such techniques are limited to the description of just a mere representation change from Hilbert space to phase space, because they are not able to build a successful quantization scheme, namely a method to produce a quantum theory out of a classical one. Concerning the very last experimental efforts, even the recent, promising finding of simultaneous observation of quantization and interference patterns (Piazza et al., 2015) stands for a technical device, rather than a proper conceptual framework able to unify GR and QT.

Some Authors view QT as a deformation of the combinatorial or Hamiltonian quantisation of three dimensional gravity in the Chern-Simons formulation. In this approach, quantum groups replace the local isometry groups, and non-commutative spacetimes replace the classical model spacetimes (Schroers, 2011). Furthermore, Kalau and Malze (1995) derived an action for gravity in the framework of non-commutative geometry by using the Wodzicki residue. Furthermore, they achieved a gravity action for commutative geometry which is the usual Einstein-Hilbert action. They also provided a non-commutative extension given by the tensor product of the algebra of smooth functions on a manifold and a finite dimensional matrix algebra.
Special relativity has been tackled also by the noncommutative stanpoint. Girelli and Livine (2004) argued that Deformed Special Relativity is obtained by imposing a maximal energy to Special Relativity and by deforming the Poincarè symmetry in order to accommodate this requirement. Variations of the same procedure lead to a non-commutative space structure that preserves conservation laws. In turn, the Very Special Relativity's (VSR) approach demonstrated that that the subgroup of the Poincaré group is sufficient to describe the spacetime symmetries of observed physical phenomena. Indeed, Das et al. (2011) introduced a novel non-commutative spacetime structure that enjoys the symmetries of deformed VSR: this trick allowed them to build a point particle Lagrangian that lives in a non-commutative phase space.

In contrast with the above-mentioned efforts, here we provide a purely mathematical framework, where the unification of the symplectic phase space of the macroscopic relativistic world and the noncommutative phase space of the microscopic quantistic world it is not strictly required. Indeed, our mathematical approach allows to join, in a single group theoretic framework, both the commutative and non-commutative operations that are the tenets of , respectively, our macroscopic relativistic world and the microscopic quantum realm. Then we hypothesize the possible physical counterpart: the gravity field. In topological words, gravity stands for a connexion (to make an example, an Ehresmann connexion) that is able to throw a bridge between two manifolds, that, although deeply different, share however something in common: they are equipped with opposite properties (commutativity and non-commutativity).
Concerning the required physical counterparts, we noticed that the commutative operations stand for the choice of the single basis required by relativity, while noncommutativity for the countless bases required by QT. Therefore, the gravitational field constrains an observer to be located in a single zone of the cosmic phase space, thus allowing her to detect observables as relativistic vectors equipped with a single basis.

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