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# Geometry Beyond Algebra

Hard – Hard = Easy

The Theorem of Overlapped Polynomials (TOP)  
and its Application to the Sawa Masayoshi's Sangaku Problem

The Adventure of Solving a Mathematical Challenge Stated in 1821



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Spain - 2011

In memoriam of Tere.

The solution presented here to the geometric Sangaku problem proposed by Masayoshi in 1821 and for which no solutions were known, don't belong to me, because I was inspired by an angel named Tere (Maria Teresa Suárez Fernández).

I offer this work as a tribute to the woman who decided to live beside me until the last day of her life, and now from Heaven she guides me with her Light and protects me with her Love.

I needed a way to tell her how much she means to me, until I can do it by myself, when at last she and I unite in Eternity.

Jesús Álvarez Lobo.

Oviedo, June 14, 2011.

ABSTRACT.

This work presents for the first time a solution to the 1821 unsolved Sawa Masayoshi's problem, giving an *explicit and algebraically exact solution* for the *symmetric case* (particular case  $b = c$ , i.e.,  $\triangle ABC \equiv$  right-angled isosceles triangle), see (1.60) and (1.61).

Despite the isosceles triangle restriction is not necessary, in view of the complexity of the explicit algebraic solution for the symmetric case, one can guessing the impossibility of achieving an explicit relationship for the *asymmetric case* (the more general case:  $\triangle ABC \equiv$  right-angled scalene triangle). For this case is given a proof of *existence and uniqueness of solution* and a *proof of the impossibility* of getting such a relationship, even implicitly, if the sextic equation (2.54) it isn't solvable. Nevertheless, in (2.56) – (2.58) it is shown the way to solve the asymmetric case under the condition that (2.54) be solvable.

Furthermore, it is proved that with a slight modification in the *final set of variables* ( $\mathcal{F}$ ), it is still possible to establish a relation between them, see (2.59) and (2.61), which provides a bridge that connects the primitive relationship by means of *numerical methods*, for every given right-angled triangle  $\triangle ABC$ .

And as the attempt to solve Fermat's conjecture (or Fermat's last theorem), culminated more than three centuries later by Andrew Wiles, led to the development of powerful theories of more general scope, the attempt to solve the Masayoshi's problem has led to the development of the *Theory of Overlapping Polynomials (TOP)*, whose application to this problem reveals a great potential that might be extrapolated to other frameworks.

ACKNOWLEDGEMENTS

The first part of this work was presented at the University of Nagoya (Japan) on 5 August of 2011, by Hidetoshi Fukagawa, worldwide leading expert in Sangaku Geometry. It was also overseen by Tony Rothman (Princeton University), Kirsty Klafton (University of Edinburgh), Emanuele Delucchi (Bremen University, Germany), Peter Wong (Bates College, USA) and Miguel Amengual Covas (member of the commission of the European Mathematical Olympiad). I want to express to all of them my sincere appreciation.

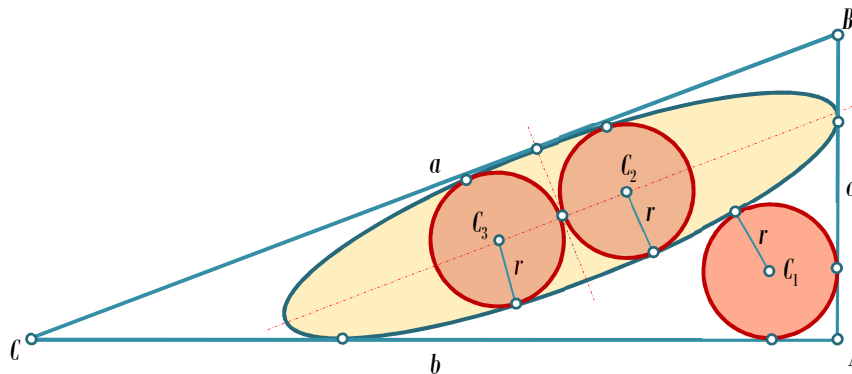
I wish to express my gratitude in the most emphatic form to Alan Horwitz (Penn State University, USA) to assume the unpleasant task of checking my work, as an author authorized in arXiv to endorse me to submit my article.

## Introduction.

Proposed by the Nippon mathematician *Sawa Masayoshi* in 1821, **this problem remained unsolved.**

As shown in the picture below, an ellipse  $\mathcal{E}$  is inscribed in  $\triangle ABC$ , right-angled triangle at  $A$ , with its major axis parallel to the hypotenuse  $a$  of  $\triangle ABC$ . Within the ellipse are inscribed two circumferences of radius  $r$ ,  $\mathcal{C}_2$  and  $\mathcal{C}_3$ , that are tangent to the ellipse and each other at the center of the ellipse. Inside-tangent to the sides  $b$  and  $c$  of the right angle  $A$  and outside-tangent to the ellipse, there is another circumference,  $\mathcal{C}_1$ , with the same radius  $r$ .

Challenge: find a relationship between  $a$ ,  $b$ ,  $c$  and  $r$ .



The Travel Diary of Mathematician Yamaguchi Kanzan. "I have arrived at Tatsuno, city near Himeji, and visited the Syosya temple to record a sangaku proposed by Sawa in 1821. In the evening, visited me and he showed me an unsolved problem and two sangaku problems of the Syosya temple. I have written them down in my diary as follows: . . ." *From the book "Sacred Mathematics. Japanese Temple Geometry"*, by Fukagawa Hidetoshi and Tony Rothman.

According to the brilliant mathematician Paul Erdős, "A math problem that takes more than a century to be solved, it is a number theory problem". And I would add: "If a seemingly simple mathematical problem, because it hides an underlying extreme complexity, takes more than a century to be solved, most likely it is that it will be a number theory problem... or a geometric Sangaku problem, and only an exhaustive analysis could lead to its solution". Jesús Álvarez Lobo.

The Masayoshi's problem kept a surprise: the explicit relationship between the length of one side of  $\triangle ABC$  and the radius of the circumferences or is so extensive and intricate that it cannot be written without fragmentation, or both the path to prove the impossibility of getting a relationship for the asymmetric case and the equations for the alternative case are huge and of an bewildering complexity, despite the valuable aid provided by the TOP. Everybody can believe in the existence of more simply relationships, (that's easy!), but the reality is that, up today, the ones presented here are the unique known.

Some approaches, cut-off paths and other collateral interesting results found through the surveying, has been moved to final appendixes, in order to lighten the reading.

In spite of the proofs are carefully developed, some steps have been omitted for brevity. However, it isn't hard to follow the exhaustive evolving of the work.

## §0. Theory of Overlapped Polynomials.

### 0.1. Definitions.

**Definition 1.** Class  $\mathcal{O}_n(x-r)^\mu$  of Overlapped Polynomials: Set of all polynomials of degree  $n$  in the variable  $x$  which have the root  $x = r$  with multiplicity not lower than  $\mu$ .

**Definition 2.** Monic Transformed of a Polynomial:

$$\mathcal{M}\left(\sum_{i=0}^n a_i x^i\right) = \frac{1}{a_n} \sum_{i=0}^n a_i x^i \quad (0.01)$$

### 0.2. Lemma of Overlapped Polynomials 1 (LOP1).

The difference of the monic transforms of two overlapped polynomials of degree  $n$  in the variable  $x$ , that share the root  $r$  with multiplicity not lower than  $\mu$ , it is a polynomial of degree  $m < n$  that keeps the root  $r$  with multiplicity at least  $\mu$ :

$$S(x), T(x) \in \mathcal{O}_n(x-r)^\mu \Rightarrow \mathcal{M}\{S(x)\} - \mathcal{M}\{T(x)\} \in \mathcal{O}_m(x-r)^\mu, m < n \quad (0.02)$$

**Proof.**

$$\left. \begin{array}{l} \mathcal{M}\{S(x)\} = (x-r)^\mu s(x) \\ \mathcal{M}\{T(x)\} = (x-r)^\mu t(x) \end{array} \right\} \Rightarrow \mathcal{M}\{S(x)\} - \mathcal{M}\{T(x)\} = (x-r)^\mu [s(x) - t(x)] \quad (0.03)$$

where  $s$  and  $t$  are monic polynomials of degree  $n - \mu$ ,

$$\begin{aligned} s(x) &= x^{n-\mu} + \sum_{i=1}^{n-\mu} a_{n-\mu-i} x^{n-\mu-i} \\ t(x) &= x^{n-\mu} + \sum_{i=1}^{n-\mu} b_{n-\mu-i} x^{n-\mu-i} \end{aligned} \quad (0.04)$$

$$\therefore s(x) - t(x) = \sum_{i=1}^{n-\mu} (a_{n-\mu-i} - b_{n-\mu-i}) x^{n-\mu-i} \quad (0.05)$$

Therefore,

$$\mathcal{M}\{S(x)\} - \mathcal{M}\{T(x)\} = (x-r)^\mu \sum_{i=1}^{n-\mu} (a_{n-\mu-i} - b_{n-\mu-i}) x^{n-\mu-i} \in \mathcal{O}_{m < n}(x-r)^\mu \quad (0.06)$$

Be noted that, by Viète-Girard formula, the degree of  $S(x) - T(x)$  is less than  $n - 1$  iff the sum of the roots of  $s(x)$  is equal to the sum of the roots of  $t(x)$ , the degree of  $S(x) - T(x)$  is less than  $n - 2$  iff also the sum of the binary products of the roots of  $s(x)$  is equal to the sum of the binary products of the roots of  $t(x)$ , and so on... (0.07)

### 0.3. Lemma of Overlapped Polynomials 2 (LOP2).

$$S, T \in \mathcal{O}_n(x-r)^\mu \Rightarrow \Delta_{ST}^{\mu-1} \equiv \mathcal{M}\{S^{(\mu-1)}\} - \mathcal{M}\{T^{(\mu-1)}\} \in \mathcal{O}_k(x-r)^1, k \leq n-\mu \quad (0.08)$$

denoting  $S^{(\mu-1)}$ ,  $T^{(\mu-1)}$  the derivatives of order  $\mu-1$  of  $S$  and  $T$ , respectively.

**Proof.**

$S(x)$  and  $T(x)$  can be factored as in (0.03),  $S(x) = (x-r)^\mu s(x)$ ,  $T(x) = (x-r)^\mu t(x)$ , where  $s$  and  $t$  are the monic polynomials defined in (0.03).

Applying the *Leibniz's formula* for the  $\mu-1$ -th derivative of the product to  $S(x)$ ,

$$\{S(x)\}^{(\mu-1)} = \{(x-r)^\mu s\}^{(\mu-1)} = \sum_{i=0}^{\mu-1} \binom{\mu-1}{i} \{(x-r)^\mu\}^{(\mu-1-i)} s^{(i)} \quad (0.09)$$

and similarly for  $T(x)$ ,

$$\{T(x)\}^{(\mu-1)} = \{(x-r)^\mu t\}^{(\mu-1)} = \sum_{i=0}^{\mu-1} \binom{\mu-1}{i} \{(x-r)^\mu\}^{(\mu-1-i)} t^{(i)} \quad (0.10)$$

It is straightforward that the terms of the sums (0.09) and (0.10) are all of degree  $n-\mu+1$  and contain the powers of  $x-r$  with integer exponents between 1 and  $\mu$ , so,

$$\{S(x)\}^{(\mu-1)} = (x-r)p(x) \quad \{T(x)\}^{(\mu-1)} = (x-r)q(x) \quad (0.11)$$

where  $p$  and  $q$  are polynomials of degree  $n-\mu$ ,

$$p(x) = \sum_{k=0}^{n-\mu} p_k x^k \quad q(x) = \sum_{k=0}^{n-\mu} q_k x^k \quad (0.12)$$

Hence,

$$\mathcal{M}\{S^{(\mu-1)}\} = \mathcal{M}\{(x-r)p(x)\} = (x-r)\mathcal{M}\{p(x)\} = \frac{1}{p_{n-\mu}}(x-r)\sum_{i=0}^{n-\mu} p_i x^i \quad (0.13)$$

$$\mathcal{M}\{T^{(\mu-1)}\} = \mathcal{M}\{(x-r)q(x)\} = (x-r)\mathcal{M}\{q(x)\} = \frac{1}{q_{n-\mu}}(x-r)\sum_{i=0}^{n-\mu} q_i x^i \quad (0.14)$$

$$\mathcal{M}\{S^{(\mu-1)}\} - \mathcal{M}\{T^{(\mu-1)}\} = (x-r)\left(\frac{1}{p_{n-\mu}}\sum_{i=0}^{n-\mu} p_i x^i - \frac{1}{q_{n-\mu}}\sum_{i=0}^{n-\mu} q_i x^i\right) \quad (0.15)$$

But the summations of (0.15) are monic polynomials of degree  $n-\mu$ , therefore, applying the LOP1,

$$\Delta_{ST}^{\mu-1} \equiv \mathcal{M}\{S^{(\mu-1)}\} - \mathcal{M}\{T^{(\mu-1)}\} = (x-r)\left(\frac{1}{p_{n-\mu}}\sum_{i=0}^{n-\mu-1} p_i x^i - \frac{1}{q_{n-\mu}}\sum_{i=0}^{n-\mu-1} q_i x^i\right) \in \mathcal{O}_k(x-r)^1, k \leq n-\mu \quad (0.16)$$



#### 0.4. Theorem of Overlapped Polynomials (TOP).

This theorem is an extension of LOP2, removing the restriction of equal degree of the overlapped polynomials involved. Therefore, the TOP represents the widest generalization of the Theory of Overlapped Polynomials.

The theorem is stated as follows:

$$\left. \begin{array}{l} S \in \mathcal{O}_n(x-r)^\sigma \\ T \in \mathcal{O}_m(x-r)^\tau \\ \delta \equiv n-m \in \mathbb{N} \\ \mu \equiv \min\{\sigma, \tau\} \end{array} \right\} \Rightarrow \Delta_{ST}^{\mu-1} \equiv \mathcal{M}\{S^{(\mu-1)}\} - \mathcal{M}\{(x^\delta T)^{(\mu-1)}\} \in \mathcal{O}_k(x-r)^1, k \leq n-\mu \quad (0.17)$$

**Proof.**

$$\left. \begin{array}{l} S \in \mathcal{O}_n(x-r)^\sigma \\ T \in \mathcal{O}_m(x-r)^\tau \\ \delta \equiv n-m \in \mathbb{N} \\ \mu \equiv \min\{\sigma, \tau\} \end{array} \right\} \Rightarrow \begin{cases} S \in \mathcal{O}_n(x-r)^\mu \subseteq \mathcal{O}_n(x-r)^\sigma \\ x^\delta T \in \mathcal{O}_{m+\delta}(x-r)^\mu = \mathcal{O}_n(x-r)^\mu \subseteq \mathcal{O}_n(x-r)^\tau \end{cases} \quad (0.18)$$

where the inclusion relationship are due to the *nested* characterization stated in the definition of the *overlapped classes of polynomials*. Hence, applying the LOP2,

$$S, x^\delta T \in \mathcal{O}_n(x-r)^\mu \Rightarrow \Delta_{ST}^{\mu-1} \equiv \mathcal{M}\{S^{(\mu-1)}\} - \mathcal{M}\{(x^\delta T)^{(\mu-1)}\} \in \mathcal{O}_k(x-r)^1, k \leq n-\mu \quad (0.19)$$

#### 0.4. Applications of the TOP.

In the Masayoshi's problem, to relate the radius of the circumferences inscribed in the ellipse with the radius of the outside tangent circle to the ellipse, is necessary to calculate the coordinates of the tangent point  $T$  as a function only of  $b$ ,  $c$  and  $r$ , but the equations that determine it are complete quartics of a extreme complexity. On the other hand, as should be shown below, it is possible to define three independent quartics in the abscise  $x_T$  of  $T$ . At first glance, it could look like superfluous to state three different condition for the same purpose, but amazingly it is possible to get advantage of this circumstance, what can be crucial in certain conditions.

In general, there may be several possibilities, depending on the degree of multiplicity of the root shared and on the depth (order of the derivative) at which the theorem be applied.

The tangent point  $T$  of two curves is twofold and this reasoning led to the development of the *Theory of Overlapping Polynomials* in the generalized form in which it is presented, i.e., considering *multiplicities* of the roots. However, we must consider that multiplicity cannot be inferred from the mere consideration of the geometry of the problem (Algebra is generous and often gives more than is asked!), since in the *transformation of rationalization* (squaring) *strange solutions* can be introduced. At this problem the TOP has been implemented in its simplest form (considering *simple roots*), but even so a unit reduction in the degree of an equation can be crucial to achieve a solution, and will always represent a substantial simplification of the calculations and the solution.





Moreover, the TOP can be applied repeatedly, thereby achieving further reductions in the degree of the equations. For instance, from three polynomial equations of any degree bigger than 2 that share a root  $x = r$ , by successive applications of the TOP, is possible to reduce the degree of the polynomial equation of least degree in two units.

For example, suppose you have two equations of 6<sup>th</sup> degree and one of 5<sup>th</sup> degree for the determination of a certain magnitude  $s$ . At first, the problem should be algebraically unsolvable, but it might go beyond the limits of Algebra by applying the TOP: with a first application, it is we obtained a *quintic* from the two *sextics*, and then, applied to the two *quintics* gives a *quartic* which is already solvable algebraically.

Suppose now that you have three equations of 6<sup>th</sup> grade for the determination of a certain magnitude  $s$ . Since the sextics are, in general, algebraically unsolvable, it looks like that the problem hasn't algebraic solution, but it possible trespass the limits of Algebra by applying the TOP: applied at a pair of sextics (for instance, to the first a second) we get a quintic, and applied to other pair of sextics (for example, first and third) we get another quintic. Finally, the application of the TOP to the pair of quintics derived from the sextics gives us a quartic that is always solvable by radicals.

It is not particularly unusual to find different equations for the determination of a value of a variable under certain conditions, as problems often lets different approaches leading to different equations. Usually the equation will be of the same degree as they have to contain all the necessary information to define the problem, but even if they have different degrees it is possible to apply the TOP.

Let's see a concrete example to clarify the way the TOP is applied.

Suppose you have three polynomials,

$$u(x) \equiv 9x^6 - 24x^5 + 40x^4 - 38x^3 + 33x^2 - 14x + 2 \quad (0.20)$$

$$v(x) \equiv 3x^5 + 11x^4 - 70x^3 + 10x^2 + 67x - 21 \quad (0.21)$$

$$w(x) \equiv 6x^5 - 17x^4 + 14x^3 - 4x + 1 \quad (0.22)$$

that share certain unknown root,  $x = r$ .

Applying the TOP to  $u$  and  $v$ ,

$$\left. \begin{array}{l} u \in \mathcal{O}_6(x-r) \\ v \in \mathcal{O}_5(x-r) \\ \delta \equiv 6-5=1 \\ \mu \equiv 1 \end{array} \right\} \Rightarrow \Delta_{uv}^0 \equiv \mathcal{M}\{u\} - \mathcal{M}\{xv\} \in \mathcal{O}_k(x-r), \quad k \leq 6-1=5 \quad (0.23)$$

i.e.,

$$\Delta_{u(xv)}^0 = \frac{1}{9}u - \frac{1}{6}xv = \frac{1}{6}x^5 + \frac{19}{9}x^4 - \frac{38}{9}x^3 + \frac{13}{3}x^2 - \frac{31}{18}x + \frac{2}{9} \quad (0.24)$$

Now we have three polynomials of 5<sup>th</sup> degree that share the root  $x = r$ , and applying the TOP to the three pairs of quintic equations we will obtain three quartic equations that preserves the same common root.

Let us denote with  $u \bullet w$  the quintic inherited by means of the TOP from the sextic  $u$  and the quintic  $w$ .

$$(u \bullet w) \bullet v \equiv \mathcal{M}\{u \bullet w\} - \mathcal{M}\{v\} = 6u \bullet w - \frac{1}{3}v = 9x^4 - 2x^3 + \frac{68}{3}x^2 - \frac{98}{3}x + \frac{25}{3} \quad (0.25)$$

$$(u \bullet w) \bullet w \equiv \mathcal{M}\{u \bullet w\} - \mathcal{M}\{w\} = 6u \bullet w - \frac{1}{6}w = \frac{31}{2}x^4 - \frac{83}{3}x^3 + 26x^2 - \frac{29}{3}x + \frac{7}{6} \quad (0.26)$$

$$v \bullet w \equiv \Delta_{vw}^0 = \mathcal{M}\{v\} - \mathcal{M}\{w\} = \frac{1}{3}v - \frac{1}{6}w = \frac{13}{2}x^4 - \frac{77}{3}x^3 + \frac{10}{3}x^2 + 23x - \frac{43}{6} \quad (0.27)$$

Now we have three polynomials of 4<sup>th</sup> degree that share the root  $x = r$ , and applying the TOP to the three pairs of quartic equations we will obtain three cubic equations that preserves the root  $x = r$ ,

$$[(u \bullet w) \bullet v][(u \bullet w) \bullet w] \equiv \mathcal{M}\{(u \bullet w) \bullet v\} - \mathcal{M}\{(u \bullet w) \bullet w\} = \frac{436}{279}x^3 + \frac{704}{837}x^2 - \frac{2516}{837}x + \frac{712}{837} \quad (0.28)$$

$$[(u \bullet w) \bullet v][v \bullet w] \equiv \mathcal{M}\{(u \bullet w) \bullet v\} - \mathcal{M}\{v \bullet w\} = \frac{436}{117}x^3 + \frac{704}{351}x^2 - \frac{2516}{351}x + \frac{712}{351} \quad (0.29)$$

$$[(u \bullet w) \bullet w][v \bullet w] \equiv \mathcal{M}\{(u \bullet w) \bullet w\} - \mathcal{M}\{v \bullet w\} = \frac{872}{403}x^3 + \frac{1408}{1209}x^2 - \frac{5032}{1209}x + \frac{1424}{1209} \quad (0.30)$$

If we would try to implement now the TOP to any pair of the above cubic equations, for instance,

$$\mathcal{M}\{[(u \bullet w) \bullet v][(u \bullet w) \bullet w]\} - \mathcal{M}\{[(u \bullet w) \bullet v][v \bullet w]\} \equiv 0 \quad (0.31)$$

we would get the null polynomial because the three equations are equivalent. So, it is impossible to further reduce the degree of the equations.

Solving any of the cubic equations, for instance, get,

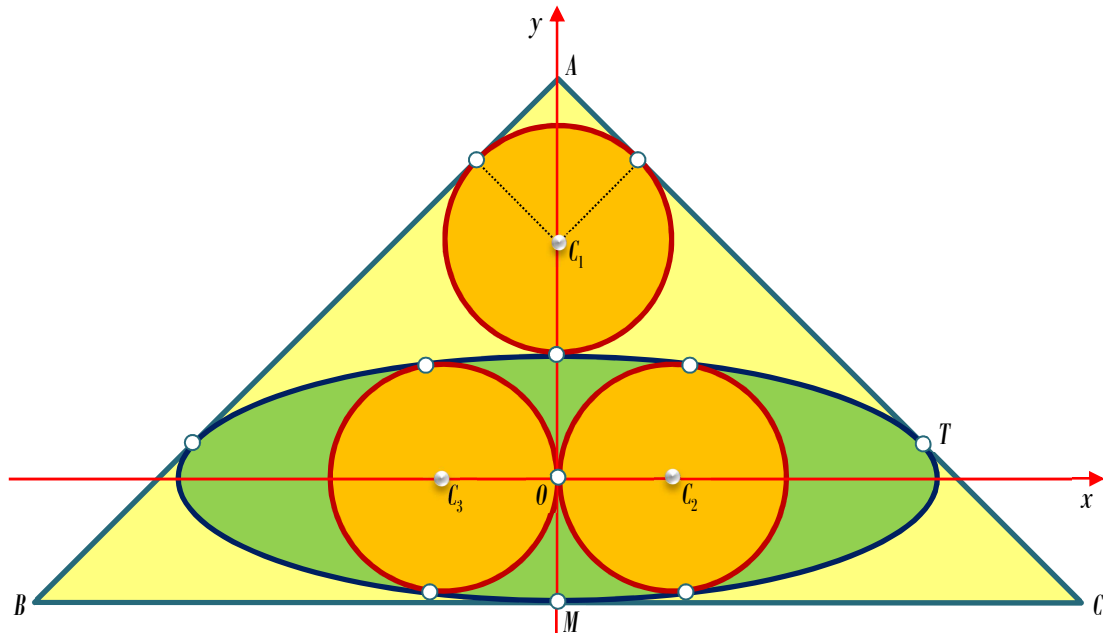
$$x = \pm \frac{\sqrt{86633}}{218} - \frac{95}{218}, \quad x = \frac{1}{3} \quad (0.32)$$

one of whose roots is that which is shared by the three initial equations.

To determine which of the three roots is what is seeking is necessary to use some criteria inferred from the conditions of the problem. In this case, the root shared is  $x = 1/3$ .

§1. Symmetric case:  $b = c$ .

In the orthonormal Cartesian reference system provided in the figure shown in the figure below, the equation of the ellipse is reduced to its canonical form and geometric configuration has axial symmetry on the coordinate axes.



**Equation of the ellipse in canonical form** (center at the origin and major-axis of the ellipse coincident with the  $x$ -axis):

$$\mathcal{E} \equiv \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad (1.01)$$

Let  $\mathcal{z}$  be the straight line defined by  $A$  and  $C$ :

$$\mathcal{z} \equiv y = -x + y_A \quad (1.02)$$

From the intersection of  $\mathcal{z}$  and the ellipse  $\mathcal{E}$ :

$$\mathcal{z} \cap \mathcal{E} \equiv \frac{x^2}{\alpha^2} + \frac{(-x + y_A)^2}{\beta^2} = 1,$$

that can be written as,

$$\left(1 + \frac{\beta^2}{\alpha^2}\right)x^2 - 2y_A x + y_A^2 - \beta^2 = 0 \quad (1.03)$$

However, the necessary and sufficient condition for  $\mathcal{z}$  to be tangent to  $\mathcal{E}$  at  $T$  is that the last equation has twofold roots, or, equivalently, discriminant null:

$$\mathcal{z} \cap \mathcal{E} = T \Leftrightarrow \Delta = (2y_A)^2 - 4\left(1 + \frac{\beta^2}{\alpha^2}\right)(y_A^2 - \beta^2) = 0,$$

or,

$$y_A^2 - \left(1 + \frac{\beta^2}{\alpha^2}\right)(y_A^2 - \beta^2) = 0 \Leftrightarrow -\frac{\beta^2}{\alpha^2}y_A^2 + \left(1 + \frac{\beta^2}{\alpha^2}\right)\beta^2 = 0 \Leftrightarrow y_A = \pm\sqrt{\alpha^2 + \beta^2},$$

and for the positive semi-ellipse,

$$y_A = +\sqrt{\alpha^2 + \beta^2} \quad (1.04)$$

Furthermore, straight by simple observation of the picture are obtained the coordinates of A,

$$A(x_A, y_A) = A(0, \beta + r + r\sqrt{2})$$

as  $\overline{AC_1}$  is the diagonal of a square of side  $r$ . Hence,

$$y_A = (1 + \sqrt{2})r + \beta \quad (1.05)$$

The pair of equations (1.04) and (1.05), that arises in a fortunate choice of the reference system, will be crucial to the resolution of the problem. Equating these two expressions of  $y_A$ , we have:

$$\sqrt{\alpha^2 + \beta^2} = \beta + (1 + \sqrt{2})r \Leftrightarrow \alpha^2 + \beta^2 = \beta^2 + 2\beta(1 + \sqrt{2})r + (1 + \sqrt{2})^2 r^2,$$

that can be expressed as

$$\frac{1}{1 + \sqrt{2}} \alpha^2 = 2\beta r + (1 + \sqrt{2})r^2 \quad (1.06)$$

In order to compact the expression, let us denote

$$p \equiv \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1 \quad (1.07)$$

Then, (1.06) is written as,

$$p\alpha^2 = 2\beta r + \frac{r^2}{p}, \quad (1.08)$$

and from this one,

$$p^2 = 2p \frac{\beta}{\alpha^2} r + \frac{r^2}{\alpha^2} \quad (1.09)$$

**Intersection of  $\mathcal{C}_2$  and  $\mathcal{E}$ :**

$$\left. \begin{array}{l} \mathcal{E} \equiv \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \\ \mathcal{C}_2 \equiv (x-r)^2 + y^2 = r^2 \end{array} \right\} \mathcal{E} \cap \mathcal{C}_2 \equiv \frac{x^2}{\alpha^2} + \frac{r^2 - (x-r)^2}{\beta^2} = 1,$$

quadratic equation in  $x$  that can be written as,

$$\left(1 - \frac{\beta^2}{\alpha^2}\right)x^2 - 2rx + \beta^2 = 0 \quad (1.10)$$

Now, taking into account that, by definition, the eccentricity of the ellipse is

$$\varepsilon = \frac{\gamma}{\alpha} = \frac{\sqrt{\alpha^2 - \beta^2}}{\alpha} = \sqrt{1 - \frac{\beta^2}{\alpha^2}} \quad (1.11)$$

the equation (1.10) can be expressed as,

$$\varepsilon^2 x^2 - 2rx + \beta^2 = 0 \quad (1.12)$$

Moreover, the **condition of tangency** between the circle with center at  $C_2$  and the ellipse is that the contact occurs only between two points of equal abscissa, what will happen iff the discriminant of (1.12), as a quadratic equation in  $x$ , is null, and this condition is translated in the following relationship:

$$\Delta = 0 \Leftrightarrow (2r)^2 - 4\varepsilon^2\beta^2 = 0,$$

that implies,

$$r = \beta\varepsilon \quad (1.13)$$

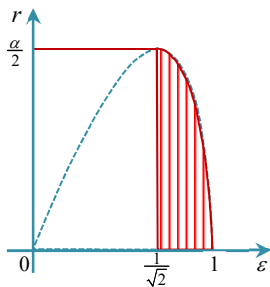
However, should be noted that the maximum radius of the circles inscribed in the semi-ellipse of semi-major axis  $\alpha$  is  $\alpha/2$ . Whereas as a constant,  $r$  can be expressed as function of  $\varepsilon$  as the only variable:

$$\left. \begin{array}{l} r = \beta\varepsilon \\ 1 - \frac{\beta^2}{\alpha^2} = \varepsilon^2 \Leftrightarrow \beta^2 = \alpha^2(1 - \varepsilon^2) \end{array} \right\} \Rightarrow r = \pm\alpha\varepsilon\sqrt{1 - \varepsilon^2},$$

but in the context of the problem,

$$r(\varepsilon) = \alpha\varepsilon\sqrt{1 - \varepsilon^2} \quad (1.14)$$

Let us study the increasing / decreasing of this function:  $r'(\varepsilon) = \alpha \frac{1 - 2\varepsilon^2}{\sqrt{1 - \varepsilon^2}} = 0 \Rightarrow \varepsilon = \pm \frac{1}{\sqrt{2}}$ ,



$$\left\{ \begin{array}{l} r'(\varepsilon) > 0, \quad 0 < \varepsilon < \frac{1}{\sqrt{2}} \\ r'(\varepsilon) < 0, \quad \frac{1}{\sqrt{2}} < \varepsilon < 1 \end{array} \right. \quad (1.15)$$

as can be checked trivially.

So,  $r$  reaches a *relative and absolute maximum* for  $\varepsilon = \frac{1}{\sqrt{2}}$ , of value  $r\left(\frac{1}{\sqrt{2}}\right) = \alpha \frac{1}{\sqrt{2}} \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\alpha}{2}$ ,

i.e.,

$$r(\varepsilon)_{\max} = r\left(\frac{1}{\sqrt{2}}\right) = \frac{\alpha}{2} \quad (1.16)$$

Hence, the domain of  $r(\varepsilon)$  is,

$$\text{Dom } r(\varepsilon) = \left( \frac{1}{\sqrt{2}}, 1 \right) \quad (1.17)$$

The left end of the domain of  $r(\varepsilon)$  coincides with the minimum value of the eccentricity for which the radius of the circles inscribed in the semi-ellipse reaches a maximum. This limit value can also be determined as follows:

$$r = \beta\varepsilon = \frac{\alpha}{2} \Leftrightarrow \frac{\beta^2}{\alpha^2} = \frac{1}{4\varepsilon^2} \Leftrightarrow 1 - \varepsilon^2 = \frac{1}{4\varepsilon^2},$$

i.e.,

$$4\varepsilon^4 - 4\varepsilon^2 + 1 = 0 \tag{1.18}$$

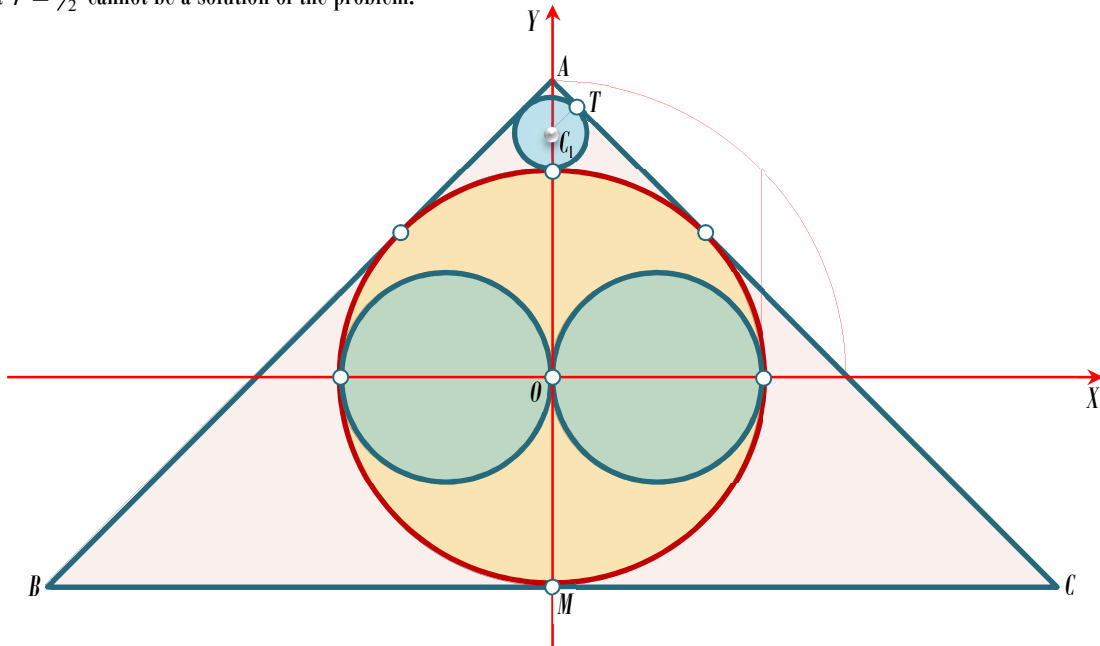
biquadratic equation whose unique real and positive root is  $\varepsilon = \frac{1}{\sqrt{2}}$ .

The circle with center at  $C_2$  inscribed in the right semi-ellipse has maximum radius  $r = \frac{\alpha}{2}$  for  $\varepsilon \leq \frac{1}{\sqrt{2}}$ , as it only has contact with the ellipse at the right apex (point of tangency). If  $\varepsilon = 1$  the ellipse is degenerated in a segment.

In short,

$$r = \begin{cases} \beta\varepsilon & \text{si } \varepsilon \in \left(\frac{1}{\sqrt{2}}, 1\right) \\ \frac{\alpha}{2} & \text{si } \varepsilon \in \left[0, \frac{1}{\sqrt{2}}\right] \end{cases} \tag{1.19}$$

But  $r = \frac{\alpha}{2}$  cannot be a solution of the problem.



Indeed, referring to the figure above, according to (1.04) and (1.05),  $y_A = \overline{OA} = \sqrt{\alpha^2 + \beta^2} = (1 + \sqrt{2})r_1 + \beta$ . Hence,

$$\alpha^2 + \beta^2 = (1 + \sqrt{2})^2 r_1^2 + 2(1 + \sqrt{2})r_1 + \beta^2,$$

$$(1 + \sqrt{2})^2 r_1^2 + 2(1 + \sqrt{2})r_1 - \alpha^2 = 0 \tag{1.20}$$

quadratic equation in  $r_1$ , whose solutions are,

$$r_1 = \pm(\sqrt{2} - 1)(\sqrt{\alpha^2 + 1} \mp 1).$$

Discarding the negative root, that has no sense in this context,

$$r_1 = (\sqrt{2} - 1)(\sqrt{\alpha^2 + 1} - 1) \quad (1.21)$$

and from (1.21), we obtain:

$$r_1 < \frac{1}{2}(\sqrt{\alpha^2 + 1} - 1) < \frac{1}{2}(\sqrt{\alpha^2 + 2\alpha + 1} - 1) = \frac{1}{2}(\sqrt{(\alpha + 1)^2} - 1) = \frac{1}{2}((\alpha + 1) - 1) = \frac{\alpha}{2} = r_2 = r_3.$$

Consequently, the solution of the problem, unique due the monotony of  $r(\varepsilon)$ , is confined to the domain of  $r(\varepsilon)$ , established in (1.17).

Now, eliminating  $r$  in (1.09) through the relationship (1.13), which links the radius of the inside the ellipse circumferences with the semi-major axis of the ellipse and its eccentricity, we obtain:

$$\left. \begin{array}{l} p^2 = 2p \frac{\beta}{\alpha^2} r + \frac{r^2}{\alpha^2} \\ r = \beta \varepsilon \end{array} \right\} \Rightarrow p^2 = 2p \frac{\beta^2}{\alpha^2} \varepsilon + \frac{\beta^2}{\alpha^2} \varepsilon^2 \Leftrightarrow p^2 = 2p(1 - \varepsilon^2)\varepsilon + (1 - \varepsilon^2)\varepsilon^2,$$

from which it is obtained the complete quartic equation in  $\varepsilon$ ,

$$\varepsilon^4 + 2p\varepsilon^3 - \varepsilon^2 - 2p\varepsilon + p^2 = 0 \quad (1.22)$$

By the lineal *Tschirnhausen transformation*,

$$\varepsilon \equiv \xi - \frac{p}{2} \quad (1.23)$$

is eliminated the term of third grade in (1.22) and, after replacing  $p$  by its value (1.07) and simplifying, we get the **depressed quartic**:

$$\xi^4 + \left(3\sqrt{2} - \frac{11}{2}\right)\xi^2 + (4\sqrt{2} - 6)\xi + \frac{33}{16} - \frac{5}{4}\sqrt{2} = 0 \quad (1.24)$$

And denoting the coefficients of this equation by

$$C \equiv 3\sqrt{2} - \frac{11}{2}, \quad D \equiv 4\sqrt{2} - 6, \quad E \equiv \frac{33}{16} - \frac{5}{4}\sqrt{2} \quad (1.25)$$

the equation (1.24) can be written as,

$$\xi^4 + C\xi^2 + D\xi + E = 0 \quad (1.26)$$

Completing a square on the left side and transposing the other terms to the right side, we have,

$$\xi^4 + C\xi^2 + D\xi + E = 0 \Leftrightarrow \xi^4 + 2C\xi^2 + C^2 = C\xi^2 - D\xi + C^2 - E,$$

$$(\xi^2 + C)^2 = C\xi^2 - D\xi + C^2 - E \quad (1.27)$$

Let us transform (1.27) by introducing the variable  $\zeta$  thus,

$$\left[ (\xi^2 + C) + \zeta \right]^2 = \frac{C\xi^2 - D\xi + C^2 - E + 2\zeta(\xi^2 + C) + \zeta^2}{(\xi^2 + C)^2} \quad (1.28)$$

This is the crucial trick, since for some  $\zeta \in \mathbb{R}$  the right side of (1.28), which is quadratic in  $\xi$ , will be a *perfect square*, and the *necessary and sufficient condition* for this is that the *discriminant* of

$$(C + 2\zeta)\xi^2 - D\xi + C^2 - E + 2C\zeta + \zeta^2 = 0 \quad (1.29)$$

is null,

$$\Delta = 0 \Leftrightarrow (-D)^2 - 4(C + 2\zeta)[C^2 - E + 2C\zeta + \zeta^2] = 0 \quad (1.30)$$

cubic equation in  $\zeta$ :

$$\zeta^3 + \frac{5C}{2}\zeta^2 + (2C^2 - E)\zeta + \left\{ \frac{C(C^2 - E)}{2} - \frac{D^2}{8} \right\} = 0 \quad (1.31)$$

Denoting

$$c \equiv \frac{5C}{2}, \quad d \equiv 2C^2 - E, \quad e \equiv \frac{C(C^2 - E)}{2} - \frac{D^2}{8} \quad (1.32)$$

i.e.,

$$c \equiv \frac{5(6\sqrt{2} - 11)}{4}, \quad d \equiv \frac{1511}{16} - \frac{259\sqrt{2}}{4}, \quad e \equiv \frac{5203\sqrt{2}}{32} - \frac{14769}{64} \quad (1.33)$$

we can write (1.31) in a more compact form

$$\zeta^3 + c\zeta^2 + d\zeta + e = 0 \quad (1.34)$$

and applying again a *Tschirnhausen transformation*,

$$\zeta \equiv z - \frac{c}{3} \quad (1.35)$$

is eliminated the quadratic term and we get the **depressed cubic**:

$$z^3 + \left( 4\sqrt{2} - \frac{73}{12} \right) z + \left\{ \frac{22\sqrt{2}}{3} - \frac{1133}{108} \right\} = 0 \quad (1.36)$$

Denoting its coefficients by

$$\gamma \equiv 4\sqrt{2} - \frac{73}{12}, \quad \delta \equiv \frac{22\sqrt{2}}{3} - \frac{1133}{108} \quad (1.37)$$

we can write (1.36) simply as

$$z^3 + \gamma z + \delta = 0 \quad (1.38)$$



To solve (1.38), let us make the change,

$$z \equiv u + v \quad (1.39)$$

Then,  $z^3 + \gamma z + \delta = 0 \Leftrightarrow (u + v)^3 + \gamma(u + v) + \delta = 0 \Leftrightarrow u^3 + v^3 + 3uv(u + v) + \gamma(u + v) + \delta = 0$ , i.e.,

$$u^3 + v^3 + (\gamma + 3uv)(u + v) + \delta = 0 \quad (1.40)$$

If we choose  $u, v$  so that  $\gamma + 3uv = 0$ , we get two equations:

$$u^3 + v^3 = -\delta \quad (1.41)$$

$$u^3 v^3 = -\left(\frac{\gamma}{3}\right)^3 \quad (1.42)$$

Equations that give us the sum and product, respectively, of the cubes of  $u$  and  $v$ , which allows us to construct a quadratic equation whose roots are these cubes:

$$t^2 + \delta t - \left(\frac{\gamma}{3}\right)^3 = 0 \quad (1.43)$$

The solutions of (1.43) are,

$$t = -\frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\gamma}{3}\right)^3} \quad (1.44)$$

With no loss of generality, we can write:

$$u^3 = -\frac{\delta}{2} + \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\gamma}{3}\right)^3}, \quad v^3 = -\frac{\delta}{2} - \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\gamma}{3}\right)^3} \quad (1.45)$$

Thus, by (1.39), a solution of (1.38) is given by (*Cardano - Tartaglia formula*):

$$z = \sqrt[3]{-\frac{\delta}{2} + \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\gamma}{3}\right)^3}} + \sqrt[3]{-\frac{\delta}{2} - \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\gamma}{3}\right)^3}} \in \mathbb{R} \quad (1.46)$$

Note: this root is real for being  $\gamma \geq 0$ .

According to (1.37), the value of the discriminant of (1.38) is,

$$\Delta \equiv \left(\frac{\delta}{2}\right)^2 + \left(\frac{\gamma}{3}\right)^3 = \frac{2639}{108} - \frac{311\sqrt{2}}{18} \quad (1.47)$$

Let us write (1.46) in a more compact form,

$$z = \sqrt[3]{-\frac{\delta}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{\delta}{2} - \sqrt{\Delta}} \quad (1.48)$$

Undoing now the variable change (1.35) and the value of  $c$  given by (1.33),

$$\zeta = \frac{5(11-6\sqrt{2})}{12} + \sqrt[3]{-\frac{\delta}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{\delta}{2} - \sqrt{\Delta}} \quad (1.49)$$

For the value of  $\zeta$  given by (1.49) the quadratic equation (1.29) has a double root (since its discriminant vanishes). Therefore, (1.28) can be expressed as,

$$\left[ (\xi^2 + C) + \zeta \right]^2 = \left\{ \sqrt{C + 2\zeta} \left[ \xi - \frac{D}{2(C + 2\zeta)} \right] \right\}^2 \quad (1.50)$$

Extracting the square root of both sides of (1.50) and simplifying the right side,

$$\xi^2 + C + \zeta = \pm \left( \xi \sqrt{C + 2\zeta} - \frac{D}{2\sqrt{C + 2\zeta}} \right) \quad (1.51)$$

From (1.51) we get two quadratic equations in  $\xi$ ,

$$\xi^2 - \left\{ \sqrt{C + 2\zeta} \right\} \xi + \frac{D}{2\sqrt{C + 2\zeta}} + C + \zeta = 0 \quad (1.52)$$

$$\xi^2 + \left\{ \sqrt{C + 2\zeta} \right\} \xi - \frac{D}{2\sqrt{C + 2\zeta}} + C + \zeta = 0 \quad (1.53)$$

and each of these ones provides a pair of solutions of the depressed quartic (1.26):

$$\xi = \frac{1}{2} \left[ +\sqrt{C + 2\zeta} \pm \sqrt{-\frac{2D}{\sqrt{C + 2\zeta}} - 3C - 2\zeta} \right] \quad (1.54)$$

$$\xi = \frac{1}{2} \left[ -\sqrt{C + 2\zeta} \pm \sqrt{+\frac{2D}{\sqrt{C + 2\zeta}} - 3C - 2\zeta} \right] \quad (1.55)$$

And finally, undoing the change of variable (1.23) and the value of  $D$ , given by (1.25),

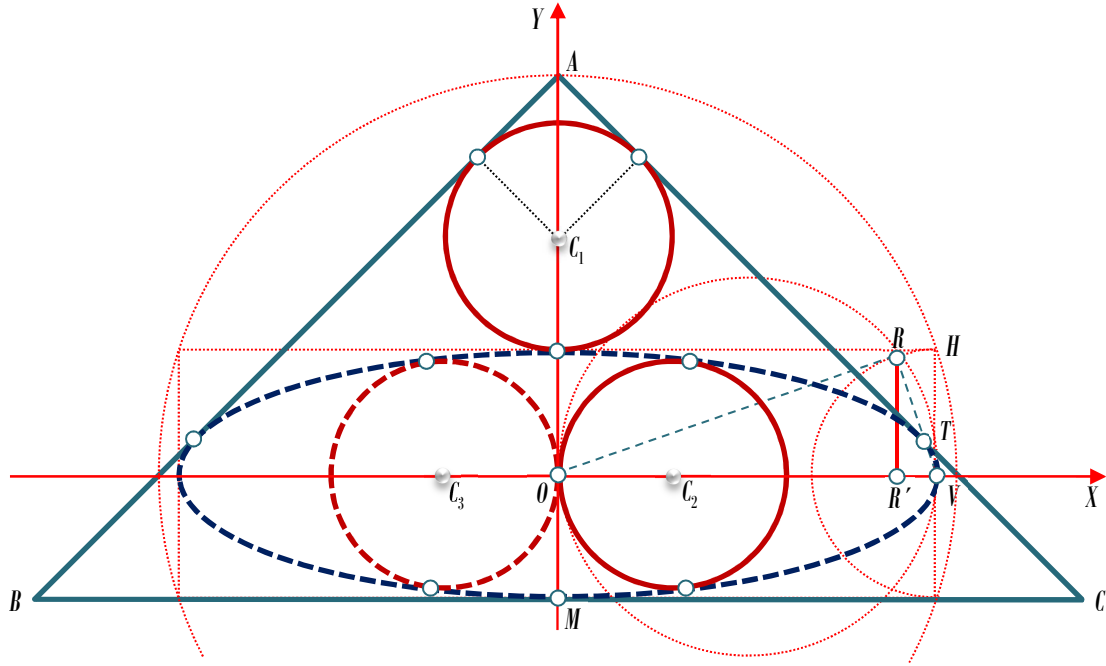
$$\varepsilon = \frac{1}{2} \left[ 1 - \sqrt{2} + \sqrt{C + 2\zeta} \pm \sqrt{+\frac{4(3 - \sqrt{2})}{\sqrt{C + 2\zeta}} - 3C - 2\zeta} \right] \Rightarrow \begin{cases} \varepsilon \approx 0.9476620415 \\ \varepsilon \approx 0.1762492425 \end{cases} \quad (1.56)$$

$$\varepsilon = \frac{1}{2} \left[ 1 - \sqrt{2} - \sqrt{C + 2\zeta} \pm \sqrt{-\frac{4(3 - \sqrt{2})}{\sqrt{C + 2\zeta}} - 3C - 2\zeta} \right] \Rightarrow \varepsilon \approx -0.9761692044 \pm i 0.2726245316 \quad (1.57)$$

where the values of  $C$ ,  $\delta$ ,  $\Delta$  and  $\zeta$  are given by (1.25), (1.37), (1.47) and (1.49), respectively.

Obviously, in the context of the problem the last two roots has no sense, and the second, by the discussion made above, it is not valid. So, the only acceptable solution is the first, i.e.,  $r_1 = r_2 = r_3 \Leftrightarrow \varepsilon = 0.94766204\dots$

**Relationship between  $b$  and  $r$ .**



Straight from the above picture, we get the height  $\overline{AM}$  of the triangle  $\triangle CAB$ :

$$\overline{AM} = 2\beta + r + r\sqrt{2} = 2\beta + r(1 + \sqrt{2}) \quad (1.58)$$

Since  $\triangle AMC$  is an isosceles right triangle, the *Pythagorean Theorem* gives:

$$b = \overline{AC} = \overline{AM}\sqrt{2} = [2\beta + r(1 + \sqrt{2})]\sqrt{2} = \left[2\frac{r}{\varepsilon} + r(1 + \sqrt{2})\right]\sqrt{2} = r\left[2 + \left(1 + \frac{2}{\varepsilon}\right)\sqrt{2}\right],$$

i.e.,

$$\frac{b}{r} = 2 + \left(1 + \frac{2}{\varepsilon}\right)\sqrt{2} \quad (1.59)$$

Therefore, an **explicit and algebraically exact relation between  $b$  and  $r$**  is

$$\frac{b}{r} = 2 + \sqrt{2} + \frac{4\sqrt{2}}{1 - \sqrt{2} + \sqrt{C + 2\zeta} + \sqrt{\frac{4(3 - 2\sqrt{2})}{\sqrt{C + 2\zeta}} - 3C - 2\zeta}} \quad (1.60)$$

where,

$$\left. \begin{array}{l} C \equiv 3\sqrt{2} - \frac{11}{2} \\ \delta \equiv \frac{22\sqrt{2}}{3} - \frac{1133}{108} \\ \Delta \equiv \frac{2639}{108} - \frac{311}{9\sqrt{2}} \end{array} \right\} \zeta \equiv 5\left(\frac{11}{12} - \frac{1}{\sqrt{2}}\right) - \sum_{k=1}^2 \sqrt[3]{\frac{\delta}{2} + (-1)^k \sqrt{\Delta}} \quad (1.61)$$

**Graphic representation of  $r$ .**

For a geometrically exact representation of the circle inscribed in the right semi-ellipse, its radius can be determined as shown in the above drawing, where  $R$  is the intersection point of the circle with center at the midpoint of  $OV$  and diameter  $\alpha$  and the circle with center at  $V$  and diameter  $\beta$ . Then,

$$\overline{RR'} \perp \overline{OV} \wedge \boxed{\overline{RR'} = r} \quad (1.62)$$

Indeed,  $\triangle VRO$  is rectangle in  $R$ , and verified:

$$\left. \begin{array}{l} \overline{VO} = \alpha \\ \overline{VR} = \beta \\ \overline{RO} = \gamma \\ \triangle RR'O \sim \triangle VRO \end{array} \right\} \Rightarrow \frac{\overline{RR'}}{\overline{RO}} = \frac{\overline{VR}}{\overline{VO}} \Leftrightarrow \overline{RR'} = \overline{VR} \cdot \frac{\overline{RO}}{\overline{VO}} = \beta \frac{\gamma}{\alpha} = \beta \varepsilon = r \Leftrightarrow \boxed{\overline{RR'} = r} \quad (1.63)$$

Approximate value of the ratio  $\frac{b}{r}$  with 3000 decimal digits:

$$\frac{b}{r} \approx 6.3988504908313964106495157287278877674472204061565403354592771240691523730333207501758409383551522944$$

1023878174854651640468456996687068973239665149295906421075063249576063491593420540113369980768041554  
 2261184792106475817952240280238929292177543863534381751892191381811825172495071286753477187798972172  
 5157840494054209712156351268956764262678951334582589824139061257397271562674430007374644245088738121  
 5752937661619158810371550353830298896131654523806268076416303046466932699733493430716676084742789500  
 9321302367533275855874744625607455712042044600666498673003616785995329545391667895140857356972483175  
 5294249333962306618211976144376198184981958938277166998343743252036900095310115948025152210958909222  
 1513138042777850780756984606806710439625975680568147193395352556329568874837557216897273332382622548  
 0371780358763405280416802464854623121859352005127742007353166957344482980485700299651594172045582147  
 4873077788539077870006266513811650783029431840886716647639507358724606203091986829847162154480347058  
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 7830384749181436162719413011813276117818766252675997809270813269283682912210223918787358216454740996  
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 6458808747162628561407010698819283256778346445819210020468153735707096776364173437639413662033632733  
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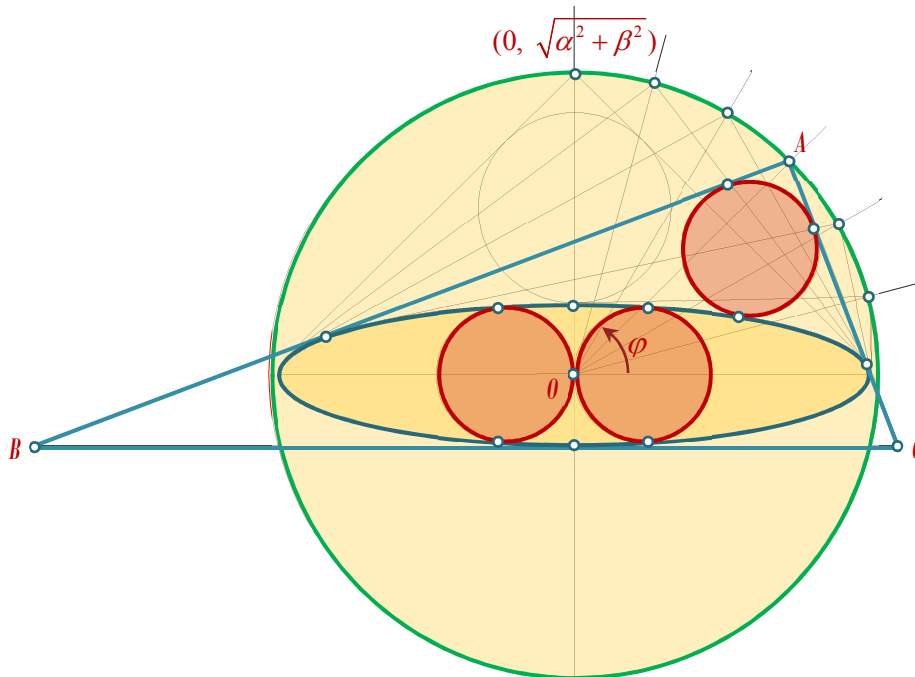


## §2. Asymmetric case: $b \neq c$ .

### 2.1. Existence and uniqueness of solution.

By observation of the below picture can be seen straightforward that *the solution is unique* in each quadrant, since the radius of the circles inscribed in the ellipse decreases as the eccentricity of the ellipse increases, according to (1.15) and (1.17), while the radius of the outside circle grows. Therefore, can only be a value of the eccentricity for which the radius of the three circles are equal. Hence, there is a bijection  $\varepsilon \leftrightarrow r$ .

On the other hand, the eccentricity increases when decreasing the angle  $\varphi$  and, consequently, the radius  $r$  increases as  $\varphi$  increases. There is, therefore, also a bijection  $\varphi \leftrightarrow r$ , that applies  $(90^\circ, 0) \leftrightarrow (r_{\max}, 0)$ .



The radius for the three circles inscribed reaches a *maximum* when the value of eccentricity is the one obtained for the particular case studied before ( $\varphi = 90^\circ \Rightarrow \triangle ABC$  is a *right-angled isosceles triangle*).

But now the  $\triangle ABC$  has no axial symmetry, so it is expected a significant complication in the calculations. The outline of several approaches (shown in appendixes B, C, D and F) from which outcrops an intractable complexity, allows glimpsing the impossibility of obtaining an explicit algebraic solution.

Despite that the extremely complexity of the relationship gotten for the symmetric case, doesn't represents a mathematical proof of the impossibility of getting relationships for the asymmetric case, indeed it is a proof for the intuition and common sense: the asymmetry increases the complication, both in the outlines as in the difficult of the equations. Nevertheless, the aforementioned impossibility will be rigorously translated to an algebraic condition: the solubility of a sextic, (2.33).



## 2.2. Outlining the path to the solution: What we need for defining algebraically the geometric conditions?.

First of all, we must see the problem in a *bird view*, looking for the necessary equations for translate the geometric picture into an algebraic description.

In the study of the symmetric case we had yet determined the equation (1.13), that describe the *condition of tangency* imposed to the circumferences  $\mathcal{C}_2$  and  $\mathcal{C}_3$ , and this relationship is also valid for the asymmetric case, since it doesn't depend on  $\triangle ABC$ .

Other condition is that the ellipse  $\mathcal{E}$  has to be tangent to the three sides of  $\triangle ABC$ . The *condition of tangency* among  $\mathcal{E}$  and the hypotenuse  $a$  is trivial for being parallel to  $a$  the major axis of  $\mathcal{E}$ . The condition of tangency among  $\mathcal{E}$  and the other two sides,  $b$  and  $c$ , will be determine cutting  $\mathcal{E}$  with the lines that contains  $b$  and  $c$ , and imposing that the intersection verifies in a single point (the intersection of a line and an ellipse is determined by a quadratic equation, and the condition of tangency implies that its discriminant is null).

The *condition of tangency* between  $\mathcal{C}_1$  and  $b$  and  $c$  is given simply imposing that the center  $C_1$  of  $\mathcal{C}_1$  belongs to the bisector line of the right-angle  $A$ .

The *condition of tangency* between  $\mathcal{C}_1$  and  $\mathcal{E}$  can be given imposing that, at an undefined point  $T$  of  $\mathcal{C}_1$  and  $\mathcal{E}$ , both curves share the tangent line, and this will occur iff the derivatives of  $\mathcal{C}_1$  and  $\mathcal{E}$  take the same value at  $T$  (i.e., tangents equal slope).

The equation of  $\mathcal{E}$  involves two parameters,  $\alpha$  and  $\beta$ , and the relationship (1.13) a third parameter,  $\varepsilon$ . Therefore, we need, at least, four independent equations for removing these three parameters. But by the definition of *eccentricity of the ellipse* we have an additional relationship between the three parameters involved, and by mean of this one and (1.13) the remaining equation can be easily reduced to a mono - parametric form, for instance, to an  $\varepsilon$  - parametric for. It is necessary to find a third relationship, an  $\varepsilon$  - parametric equation independent of the coordinates of  $T$ , for expressing  $\varepsilon$  in the set of the final variables,  $\mathcal{F} = \{a, b, c, r\}$ . Let's denote this hypothetic relationship as  $\varepsilon_r$ .

Substituting now  $\varepsilon_r$  into the equation derived of the *condition of tangency* between  $\mathcal{C}_1$  and  $\mathcal{E}$ , we'll find an expression for one coordinate of  $T$ , (for instance, the abscise  $x_T$ ) in terms of the FVs, and using the equation of  $\mathcal{C}_1$  we could find the other coordinate of  $T$  (for instance, the ordinate  $y_T$ ) in terms of the FVs.

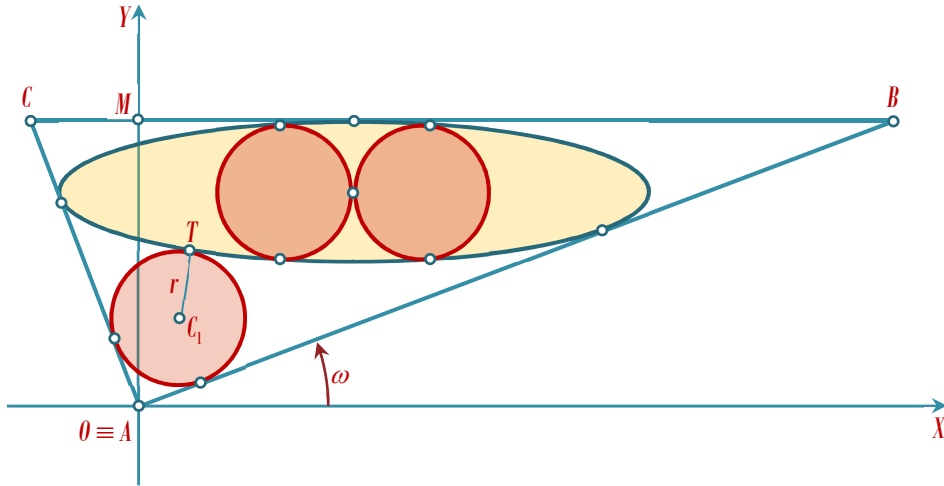
Finally, equaling to  $r$  the distance between  $C_1$  and  $T$ , we would find the searched relationship among the variables of  $\mathcal{F}$ .

Nevertheless, we'll see that the hardest difficulty is to find an explicit relationship  $\varepsilon_r$ , that would be the key for the door to the solution, but, as it will be shown later,  $\varepsilon_r$  is, in fact, the key for proving that it is impossible to find a relation in the set  $\mathcal{F} = \{a, b, c, r\}$ , neither explicit nor implicit, since the equation relating  $r$  to  $\varepsilon$  is of 6th grade in  $\varepsilon$ , without any especial relation of symmetry in its coefficients, so unsolvable by radicals.

However, taking as final set of variables  $\mathcal{F}' = \{a, b, c, \varepsilon\}$ , it is indeed possible establish a relationship, because the equation relating  $r$  to  $\varepsilon$  is of 4th grade in  $r$ , therefore solvable algebraically. Getting a relationship in  $\mathcal{F}' = \{a, b, c, \varepsilon\}$  allows to get another one in  $\mathcal{F} = \{a, b, c, r\}$  for a given right-angled triangle  $\triangle ABC$ , by means of numeric methods.

### 2.3. Switching the reference system.

The previous reference system provides the simplest expression for the equation of the ellipse (reduced to its canonical form), but instead, the equations for the remaining elements result too complex (in appendix B we show its expressions). Let us take now the Cartesian reference system showed in the figure below, obtained from the previous one by a rotation and a translation of axes: rotation of  $180^\circ$  (keeping the parallelism of the coordinate's axes with respect to the axis of the ellipse) and translation of the origin to the vertex A.



### 2.3. Equations in the new reference system.

To relate the new coordinates with the old, just switch  $x$  by  $-x + h$ , and the primitives coordinates of  $C_1$  with the new ones. The slope  $m$  is invariant under the transformation performed ( $180$  degree turn and translation). In this new scenery, the equation of the bisector of A, and the equations for the coordinates of  $C_1$ , are significantly simplify.

The slope of the bisector of A is,

$$\tan\left(\omega + \frac{\pi}{4}\right) = \frac{\tan \omega + \tan \frac{\pi}{4}}{1 - \tan \omega \tan \frac{\pi}{4}} = \frac{1 + \tan \omega}{1 - \tan \omega},$$

where  $\tan \omega$  is the slope of the side AB. Setting,

$$m \equiv \tan \omega \tag{2.01}$$

the equation of the bisector of A is as simply as

$$y = \frac{1+m}{1-m}x \tag{2.02}$$

The new coordinates of  $C_1$  can be obtained as intersection of a circumference with center at the origin and radius equal to the diagonal of a square of side  $r$ . Assuming  $m > 0$ ,

$$\left. \begin{aligned} y &= \frac{1+m}{1-m}x \\ x^2 + y^2 &= (r\sqrt{2})^2 \end{aligned} \right\} \Rightarrow \left[ 1 + \left( \frac{1+m}{1-m} \right)^2 \right] x^2 = 2r^2$$

$$\left. \begin{aligned} x_1 &= \frac{|1-m|r}{\sqrt{1+m^2}} \\ y_1 &= \frac{(1+m)r}{\sqrt{1+m^2}} \end{aligned} \right\} \tag{2.03}$$



and expressing  $m$  as a function of the sides of  $\triangle ABC$ ,

$$\boxed{x_1 = \frac{|c-b|r}{a}} \quad \boxed{y_1 = \frac{(c+b)r}{a}} \quad (2.04)$$

#### 2.4. Centre of the ellipse: analytic and geometric determination.

The equation of the ellipse with center at  $E(x_0, y_0)$  is,

$$\boxed{\mathcal{E} \equiv \frac{(x-x_0)^2}{\alpha^2} + \frac{(y-y_0)^2}{\beta^2} = 1} \quad (2.05)$$

and the equation of the line determined by the vertex of  $A$  and  $B$ ,

$$\boxed{y = mx} \quad (2.06)$$

The intersection of this line and the ellipse is given by,

$$\frac{(x-x_0)^2}{\alpha^2} + \frac{(mx-y_0)^2}{\beta^2} = 1 \Leftrightarrow (x-x_0)^2 + (mx-y_0)^2 - \alpha^2\beta^2 = 0,$$

or,

$$\boxed{(m^2\alpha^2 + \beta^2)x^2 - 2(\beta^2x_0 + m\alpha^2y_0)x + \beta^2x_0^2 + \alpha^2y_0^2 - \alpha^2\beta^2 = 0} \quad (2.07)$$

but the line (2.06) will be tangent to the ellipse (2.05) iff the discriminant of the quadratic equation in  $x$  (2.07) is null,

$$\boxed{\Delta = [-2(\beta^2x_0 + m\alpha^2y_0)]^2 - 4(m^2\alpha^2 + \beta^2)(\beta^2x_0^2 + \alpha^2y_0^2 - \alpha^2\beta^2) = 0} \quad (2.08)$$

and simplifying,

$$\boxed{m^2(x_0^2 - \alpha^2) - 2mx_0y_0 + y_0^2 - \beta^2 = 0} \quad (2.09)$$

Substituting in (2.09)  $m$  by  $-1/m$  we obtain a similar expression for the condition of tangency of the line determined by the vertices  $A$  and  $C$  with the ellipse (2.05):

$$\boxed{x_0^2 - \alpha^2 + 2mx_0y_0 + m^2(y_0^2 - \beta^2) = 0} \quad (2.10)$$

Adding equations (2.09) and (2.10) side by side, and factoring, we have,

$$\boxed{(m^2 + 1)(x_0^2 + y_0^2 - \alpha^2 - \beta^2) = 0} \quad (2.11)$$

that implies,

$$\boxed{\alpha^2 + \beta^2 = x_0^2 + y_0^2} \quad (2.12)$$

This condition is also obtained considering that the vertex  $A$  belongs to the *director circumference* (see Appendix B).

From (2.12),

$$\alpha^2 = x_0^2 + y_0^2 - \beta^2 \quad (2.13)$$

and substituting  $\alpha$  in (2.09) or (2.10) by the right side of (2.13),

$$(y_0^2 - \beta^2)(1 - m^2) = 2mx_0y_0 \quad (2.14)$$

Into this equation hides an interesting geometric relationship, which comes to light when expressed thus,

$$\frac{2m}{1 - m^2} = \frac{y_0^2 - \beta^2}{x_0y_0} \quad (2.15)$$

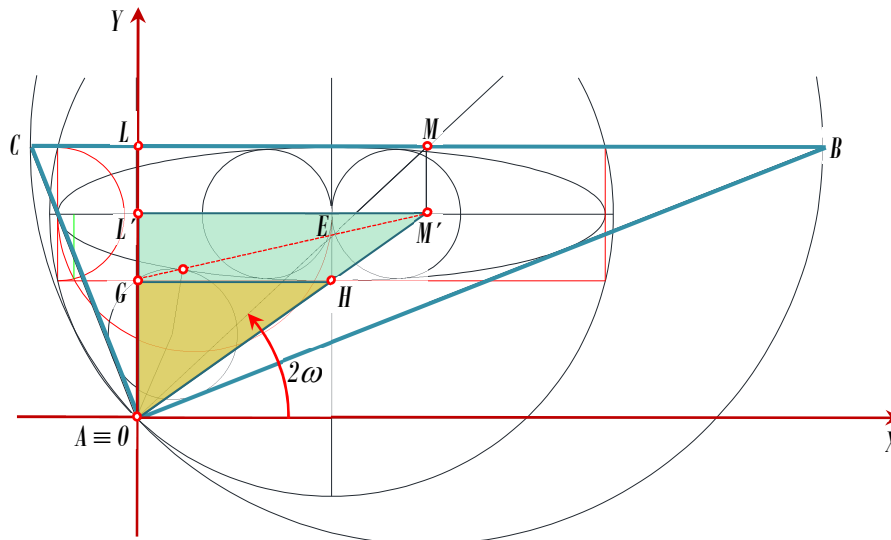
since,

$$\frac{2m}{1 - m^2} = \tan 2\omega \quad (2.16)$$

Using (2.16) and writing right now (2.14) in the shape,

$$\frac{x_0}{y_0 - \beta} = \frac{(y_0 + \beta) \cot 2\omega}{y_0} \quad (2.17)$$

follows a useful geometrical construction to determine the center of the ellipse in terms of  $\beta$ :



$$\frac{\overline{GH}}{\overline{OG}} = \frac{\overline{L'M'}}{\overline{OL'}} = \frac{\overline{LM}}{\overline{OL}} \Leftrightarrow \frac{x_0}{y_0 - \beta} = \frac{(y_0 + \beta) \cot 2\omega}{y_0} \quad (2.18)$$

But  $\triangle OBM'$  and  $\triangle OCM'$  are isosceles because both triangles have two equal angles; hence,  $M$  is the middle point of the hypotenuse of  $\triangle ABC$ , and the vertex  $H$  of the ellipse is simply the intersection of  $AM'$  with the parallel to  $BC$  by  $G$ , where  $\overline{GL} = 2\beta$  and  $M'$  is the projection of  $M$  over the major axis of the ellipse.

From (2.17), taking into account that  $y_0 + \beta$  is the height of  $\triangle ABC$  relative to its hypotenuse, and can be expressed as,

$$y_0 + \beta = \frac{bc}{a} \quad (2.19)$$

since  $m$  is the slope of  $c$  in  $\triangle ABC$  and  $m = \tan \hat{B} = b/c$ .

Solving for  $x_0$  in (2.17) and replacing  $y_0 + \beta$  by the value given by (2.19), since  $\tan \omega = b/c$ ,

$$x_0 = [(y_0 + \beta) \cot 2\omega] \cdot \frac{y_0 - \beta}{y_0} = \frac{bc}{a} \cdot \frac{c^2 - b^2}{2bc} \left(1 - \frac{\beta}{y_0}\right),$$

$$x_0 = \frac{c^2 - b^2}{2a} \left(1 - \frac{\beta}{y_0}\right) \quad (2.20)$$

Note that,  $\frac{c^2 - b^2}{2a} = \frac{-\frac{b^2}{a} + \frac{c^2}{a}}{2} = \frac{x_C + x_B}{2} = x_M$ , that is, the abscise of the middle point of  $\overline{BC}$ .

Let's denote,

$$k \equiv \frac{c^2 - b^2}{2a} \quad (2.21)$$

$$h \equiv \frac{bc}{a} \quad (2.22)$$

**Theorem.** The locus of the centers of the ellipses with minor semi-axis of length  $\beta$  inscribed in a right-angled triangle  $\triangle ABC$ , with major-axis parallel to the hypotenuse, in a Cartesian reference system with origin at the vertex  $A$  of the right-angle  $\beta$  and abscissa axis below the hypotenuse and parallel to it, belong to the **hyperbole**,

$$y = \frac{\beta}{1 - \frac{x}{k}} \quad (2.23)$$

Removing  $y_0$  in (2.20) by means of (2.19), we obtain the  $\beta$ - parametric equations of the center of the ellipse:

$$\begin{cases} x_0 = k \left[ 1 - \left( \frac{h}{\beta} - 1 \right)^{-1} \right] \\ y_0 = h - \beta \end{cases} \quad (2.23)$$

and by eliminating  $\beta$  through (1.13) we get the  $\varepsilon$ - parametric equations of the center of the ellipse:

$$\begin{cases} x_0 = k \left( 1 - \frac{r}{h\varepsilon - r} \right) \\ y_0 = h - \frac{r}{\varepsilon} \end{cases} \quad (2.24)$$

Notice the simplified relationship,

$$h^2 + k^2 = \left( \frac{a}{2} \right)^2 \quad (2.25)$$

## 2.5. Coordinates of the tangent point $T$ between $\mathcal{C}_1$ and $\mathcal{E}$

For determining the point of tangency between the  $\mathcal{C}_1$  and  $\mathcal{E}$ , as shown below, it is possible to raise three independent equations. At first glance, defining in three different ways the same object (unknown  $x$ ) might seem unnecessary work, and therefore superfluous. However, is quite the opposite, since the implementation of the TOP (which displays here some of its potential) reduces the three quartic to a simple quadratic equation which preserves the root of interest (the abscissa of  $T$ ) shared by the three quartics. The TOP acts synergistically on the equations, so that the joint action of the equations is much more effective than just the sum of their individual actions. Below are defined the three quartics.

### 2.5.1. Quartic for $x_1$ derived from the intersection of $\mathcal{C}_1$ and $\mathcal{E}$

$$\left. \begin{aligned} \frac{(x-x_0)^2}{\alpha^2} + \frac{(y-y_0)^2}{\beta^2} = 1 &\Leftrightarrow y = y_0 \pm \beta \sqrt{1 - \frac{(x-x_0)^2}{\alpha^2}} \\ (x-x_1)^2 + (y-y_1)^2 = r^2 &\Leftrightarrow y = y_1 \pm \sqrt{r^2 - (x-x_1)^2} \end{aligned} \right\} \Rightarrow y_0 \pm \beta \sqrt{1 - \frac{(x-x_0)^2}{\alpha^2}} = y_1 \pm \sqrt{r^2 - (x-x_1)^2}.$$

Taking into account that the contact take place between the *lower semi-ellipse* and the *upper semi-circumference*, the signs of the radicals must be  $-$  in the equation of the ellipse and  $+$  in the equation of the circumference:

$$y_0 - \beta \sqrt{1 - \frac{(x-x_0)^2}{\alpha^2}} = y_1 + \sqrt{r^2 - (x-x_1)^2} \quad (2.26)$$

Streamlining, developing and collecting terms (used the DERIVE computer algebra system), there arises a quartic equation, that we denote by  $u(x) = 0$ , showing the following dissuasive-looking: (2.27)

$$u(x) = \alpha^2 \epsilon^2 (h \epsilon - r)^4 x^4 + 4 \alpha^2 \epsilon^2 (r-h \epsilon)^3 (a k \epsilon^2 - 1) (h \epsilon - 2 r) r + (b-c) (r-h \epsilon)^3 x^3 + 2 (a^2 (h \epsilon^2 - 2) + 4 h^2 r \epsilon^2 + 3 (2-\epsilon) + h \epsilon^2 (r^2 (\epsilon^2 + 5 \epsilon - 12) - k \epsilon^2 (\epsilon - 1) (3 \epsilon - 2))) + 2 h r \epsilon^2 (2 k \epsilon^2 (\epsilon - 1) (3 \epsilon - 2) - r^2 (\epsilon + \epsilon^4) - r^2 (4 k \epsilon^2 (\epsilon - 1) (3 \epsilon - 2) + r^2 (2-\epsilon))) + 2 a r \epsilon (r-h \epsilon) (b (h \epsilon^2 (\epsilon - 2) - 2 h \epsilon (k \epsilon (\epsilon - 1) + r (\epsilon - 2))) + r (4 k \epsilon (\epsilon - 1) + r (\epsilon - 2))) + c (h \epsilon^2 (\epsilon - 2) + 2 h \epsilon (k \epsilon (\epsilon - 1) + r (2-\epsilon))) - r (4 k \epsilon (\epsilon - 1) + r (2-\epsilon))) - 4 r \epsilon^2 (b^2 - b c \epsilon + c^2) (h \epsilon - r)^2 (h \epsilon - r)^2 x^2 + 4 \alpha^2 \epsilon^2 (h \epsilon - 2 r) (h \epsilon - 4 h r \epsilon + h \epsilon^2 (r (\epsilon + 5) - k \epsilon^2 (\epsilon - 1))) + 2 h r (2 k \epsilon^2 (\epsilon - 1) - r^2 (\epsilon + 1)) - r^2 \epsilon (4 k \epsilon (\epsilon - 1) - r^2) + a r (b (h \epsilon^2 - 2 h \epsilon (k \epsilon (\epsilon - 1) + r (2-\epsilon))) + h \epsilon^2 (k \epsilon^2 (\epsilon - 1) + 8 k r \epsilon (\epsilon - 1) + r (\epsilon + 7))) - 2 h r \epsilon (2 k \epsilon^2 (\epsilon - 1) + 5 k r \epsilon (\epsilon - 1) + r^2 (\epsilon + 3)) + r (4 k \epsilon^2 (\epsilon - 1) + 4 k r \epsilon (\epsilon - 1) + r^2 (\epsilon + 2))) - c (4 h \epsilon^2 + 2 h \epsilon (k \epsilon (\epsilon - 1) - 2 r) + h \epsilon^2 (k \epsilon^2 (\epsilon - 1) + 8 k r \epsilon (\epsilon - 1) + r (\epsilon + 7))) - 2 h r \epsilon (2 k \epsilon^2 (\epsilon - 1) + 5 k r \epsilon (\epsilon - 1) + r^2 (\epsilon + 3)) + r^2 (\epsilon + 3)) + r^2 (4 k \epsilon^2 (\epsilon - 1) + 4 k r \epsilon (\epsilon - 1) + r^2 (\epsilon + 2))) (h \epsilon - r)^2 - 2 r \epsilon^2 (b^2 (h \epsilon - r)^2 + 2 b c k \epsilon (1-\epsilon) (h \epsilon - 2 r) - c^2 (h \epsilon - r)^2) (h \epsilon - r)^2 x + 2 (h \epsilon - 8 h^2 r \epsilon - 2 h \epsilon^2 (k \epsilon^2 (\epsilon - 1) + r (\epsilon - 13))) + 4 h^2 r \epsilon^3 (4 k \epsilon^2 (\epsilon - 1) + r^2 (3 \epsilon - 11)) + h \epsilon^2 (k \epsilon^2 (\epsilon - 1) - 2 k r \epsilon^2 (\epsilon + 24 \epsilon - 25) + r^2 (5 \epsilon - 32 \epsilon + 41)) - 4 h r \epsilon^2 (2 k \epsilon^2 (\epsilon - 1) - k r \epsilon^2 (3 \epsilon + 16 \epsilon - 19) + r^2 (5 \epsilon - 12 \epsilon + 5)) + 2 h r^2 (12 k \epsilon^2 (\epsilon - 1) - k r \epsilon^2 (13 \epsilon + 15 \epsilon - 28) + r^2 (15 \epsilon - 21 \epsilon + 2)) - 4 h r^3 \epsilon (8 k \epsilon^2 (\epsilon - 1) - 2 k r \epsilon^2 (3 \epsilon - 2) + 5 r^2 (\epsilon - 1)) + r^4 (16 k \epsilon^2 (\epsilon - 1) + 8 k r \epsilon^2 (1-\epsilon) + r^2 (5 \epsilon - 4)) + 4 a r (b+c) (r-h \epsilon)^3 (h \epsilon - 4 h r \epsilon + h \epsilon^2 (r (\epsilon + 5) - k \epsilon^2 (\epsilon - 1))) + 2 h r (2 k \epsilon^2 (\epsilon - 1) - r^2 (\epsilon + 1)) - r^2 \epsilon (4 k \epsilon^2 (\epsilon - 1) + r^2 (\epsilon + 1)) + r^2 (2 h \epsilon - 4 h r \epsilon + r^2 (2-\epsilon)) (h \epsilon - r)^2 + 2 b c \epsilon (h \epsilon - 4 h r \epsilon + h \epsilon^2 (5 r - k \epsilon^2 (\epsilon - 1))) + 2 h r (2 k \epsilon^2 (\epsilon - 1) - r^2) + 4 k r \epsilon^2 (1-\epsilon) + c^2 (2 h \epsilon - 4 h r \epsilon + r^2 (2-\epsilon)) (h \epsilon - r)^2) (h \epsilon - r)^2 = 0$$

### 2.5.2. Quartic for $x_1$ which is obtained by equating the derivatives of $\mathcal{C}_1$ and $\mathcal{E}$ in its explicit form.

Derivatives of the upper canonic *semi-ellipse*  $\mathcal{E}$  and the lower *semi-circumference*  $\mathcal{C}_1$ :

$$\frac{(x-x_0)^2}{\alpha^2} + \frac{(y-y_0)^2}{\beta^2} = 1 \Leftrightarrow y = y_0 \pm \beta \sqrt{1 - \frac{(x-x_0)^2}{\alpha^2}} \Rightarrow y' = \mp \beta \frac{\frac{x-x_0}{\alpha^2}}{\sqrt{1 - \frac{(x-x_0)^2}{\alpha^2}}},$$

$$y' = -\frac{\beta}{\alpha} \frac{1}{\sqrt{\left(\frac{\alpha}{x_0 - x}\right)^2 - 1}} \quad (2.28)$$

$$(x-x_1)^2 + (y-y_1)^2 = r^2 \Leftrightarrow y = y_1 \pm \sqrt{r^2 - (x-x_1)^2} \Rightarrow y' = \mp \frac{x-x_1}{\sqrt{r^2 - (x-x_1)^2}},$$

$$y' = -\frac{1}{\sqrt{\left(\frac{r}{x-x_1}\right)^2 - 1}} \quad (2.29)$$

the derivatives of both curves at the tangent point  $T$  represent the slope of the common tangent line; hence, they have to be equals:

$$-\frac{\beta}{\alpha} \frac{1}{\sqrt{\left(\frac{\alpha}{x_0-x}\right)^2 - 1}} = -\frac{1}{\sqrt{\left(\frac{r}{x-x_1}\right)^2 - 1}} \Leftrightarrow \frac{\beta}{\alpha} \sqrt{\left(\frac{r}{x-x_1}\right)^2 - 1} = \sqrt{\left(\frac{\alpha}{x_0-x}\right)^2 - 1}$$

$$\left(\frac{\beta}{\alpha}\right)^2 \left[\left(\frac{r}{x-x_1}\right)^2 - 1\right] = \left(\frac{\alpha}{x_0-x}\right)^2 - 1 \quad (2.30)$$

where  $\alpha^2$  can be expressed in terms of  $r$  and  $\varepsilon$ ,

$$\alpha^2 = \frac{r^2}{\varepsilon^2(1-\varepsilon^2)} \quad (2.31)$$

From (2.30) and (2.31),

$$1 - \left(\frac{\beta}{\alpha}\right)^2 = \left(\frac{\alpha}{x_0-x}\right)^2 - \left(\frac{\beta}{\alpha}\right)^2 \left(\frac{r}{x-x_1}\right)^2 \Leftrightarrow \varepsilon^2 = \frac{r^2}{\varepsilon^2(1-\varepsilon^2)(x_0-x)^2} - \frac{r^2(1-\varepsilon^2)}{(x-x_1)^2}$$

or,

$$\varepsilon^2(1-\varepsilon^2)(x_0-x)^2 \left[ \varepsilon^2(x-x_1)^2 + r^2(1-\varepsilon^2) \right] - r^2(x-x_1)^2 = 0 \quad (2.32)$$

Developing and collecting terms in  $x$  we obtain the following quartic denoted  $v(x) = 0$ : (2.33)

$$v(x) = a^2 r^2 \varepsilon^2 (1-\varepsilon^2)^2 (h \varepsilon - r)^2 x^4 + 2 a^2 \varepsilon^2 (1-\varepsilon^2) (a k (h \varepsilon - 2 r) + r (b-c) (r-h \varepsilon)) (h \varepsilon - r)^3 x^3 + (a^2 (k \varepsilon (h \varepsilon - 1) - r (\varepsilon - 2 \varepsilon + \varepsilon - 1)) + 2 h r \varepsilon (r^2 (h \varepsilon - 2 \varepsilon + \varepsilon - 1) - 2 k \varepsilon^2 (\varepsilon - 1)) + r^2 (4 k \varepsilon (h \varepsilon - 1) - r (\varepsilon - 2 \varepsilon + \varepsilon - 1))) + 4 a k r \varepsilon^2 (1-\varepsilon^2) (b-c) (h \varepsilon - r) (h \varepsilon - 2 r) + r^2 \varepsilon^2 (b-c)^2 (\varepsilon - 1) (h \varepsilon - r)^2) x^2 + r^2 (a^2 k r \varepsilon^2 (\varepsilon - 1) (h \varepsilon - r) (h \varepsilon - 2 r) + a (b-c) (h \varepsilon (k \varepsilon (h \varepsilon - 1) + r) - 2 h r \varepsilon (2 k \varepsilon (h \varepsilon - 1) + r) + 4 k \varepsilon (h \varepsilon - 1) + r)) + k r \varepsilon^2 (1-\varepsilon^2) (b-c)^2 (h \varepsilon - r) (h \varepsilon - 2 r)) x - (a^2 k \varepsilon^2 (\varepsilon - 1) (h \varepsilon - 2 r) - (b-c)^2 (h \varepsilon (k \varepsilon (h \varepsilon - 1) + r) - 2 h r \varepsilon (2 k \varepsilon (h \varepsilon - 1) + r) + 4 k \varepsilon (h \varepsilon - 1) + r))) = 0$$

### 2.5.3. Quartic for $x_1$ got by equating the derivatives of $\mathcal{C}_1$ and $\mathcal{E}$ in its implicit form.

Deriving implicitly the equations of both curves,

$$\mathcal{E} \equiv \frac{(x-x_0)^2}{\alpha^2} + \frac{(y-y_0)^2}{\beta^2} = 1 \Rightarrow \cancel{\mathcal{Z}}(x-x_0) + \frac{\cancel{\mathcal{Z}}(y-y_0)}{\beta^2} y' = 0 \Rightarrow y' = -\frac{\beta^2}{\alpha^2} \cdot \frac{x-x_0}{y-y_0} \quad (2.34)$$

$$\mathcal{C}_1 \equiv (x-x_1)^2 + (y-y_1)^2 = r^2 \Rightarrow \cancel{\mathcal{Z}}(x-x_1) + \cancel{\mathcal{Z}}(y-y_1) y' = 0 \Rightarrow y' = -\frac{x-x_1}{y-y_1} \quad (2.35)$$





Using (2.46) instead of (2.27), the algorithm to obtain the quadratic equation by iteration of the TOP would be: (2.47)

$$\left. \begin{array}{l} u \in \mathcal{C}_4(x - x_T)^2 \Rightarrow u' \in \mathcal{C}_3(x - x_T) \\ v, w \in \mathcal{C}_4(x - x_T) \Rightarrow \Delta_{vw} \equiv \mathcal{M}\{v\} - \mathcal{M}\{w\} \in \mathcal{C}_3(x - x_T) \end{array} \right\} \Rightarrow \Delta_{u'vw} \equiv \mathcal{M}\{u'\} - \mathcal{M}\{\Delta_{vw}\} \in \mathcal{C}_2(x - x_T)$$

Thus, we have the abscise  $x_T$  as a function of  $a, b, c, r, \varepsilon$ . Replacing  $x_T$  in the equation of  $\mathcal{C}_1$  it is obtained the corresponding ordinate  $y_T$ , also as a function of  $a, b, c, r, \varepsilon$ .

## 2.6. Defining the problem.

Let's gather now a complete set of independent equations to define the problem, translating geometrical meanings to algebraic descriptions.

- ❶ Condition of tangency of  $\mathcal{C}_2$  and  $\mathcal{C}_3$  with each other and with  $\mathcal{E}$ . Fusing (1.13) and (1.17):

$$r = \beta\varepsilon, \quad \varepsilon \in \left( \frac{1}{\sqrt{2}}, 1 \right) \tag{2.48}$$

- ❷ Condition of tangency among the ellipse  $\mathcal{E}$  and the sides  $a, b$  and  $c$  of  $\triangle ABC$ . Unifying (2.12) and (2.24):

$$\alpha^2 + \beta^2 = k^2 \left( 1 - \frac{r}{h\varepsilon - r} \right)^2 + \left( h - \frac{r}{\varepsilon} \right)^2 \tag{2.49}$$

where  $h$  and  $k$  are constants that depend only of  $a, b, c$  according to (2.21) and (2.22).

- ❸ Condition of tangency among the outside-circumference  $\mathcal{C}_1$  and the ellipse  $\mathcal{E}$ .

There are three ways of setting this condition, imposing that:  $\mathcal{C}_1$  and  $\mathcal{E}$  share the tangent point  $T$ , or the normal line at  $T$  contains the center  $C_1$  of  $\mathcal{C}_1$ , or  $\mathcal{C}_1$  and  $\mathcal{E}$  share the tangent line at  $T$ . It will be use (2.44) and the equation of  $\mathcal{C}_1$ .

- ❹ Condition of tangency between the outside-circumference  $\mathcal{C}_1$  and the sides  $b$  and  $c$  of  $\triangle ABC$ .

$$x_1 = \frac{|c - b|r}{a} \quad y_1 = \frac{(c + b)r}{a} \tag{2.04}$$

- ❺ Condition of that the distance between the center  $C_1$  of  $\mathcal{C}_1$  and the tangent point  $T$  is  $r$ .

$$(x_T - x_1)^2 + (y_T - y_1)^2 = r^2 \tag{2.50}$$

- ❻ Relationship among the 3 parameters involved in the problem (lengths of the semi-axis of  $\mathcal{E}$  and its eccentricity):

$$\varepsilon = \sqrt{1 - \frac{\beta^2}{\alpha^2}} \tag{1.11}$$

The condition of equal radius for the three circumferences it is yet included implicitly in the above equations since are used the same variable  $r$  for all them.

This is the simplest system of equations for describing the Masayoshi's problem, but subjacent to its apparent simplicity is hidden an extremely complexity that outcrops when trying to removing the involved parameters, as shown below.



## 2.7. Removing the parameters.

From (1.11) and (2.48),

$$\alpha^2 = \frac{r^2}{\varepsilon^2(1-\varepsilon^2)}, \quad \varepsilon \in \left( \frac{1}{\sqrt{2}}, 1 \right) \quad (2.51)$$

and from (2.48),

$$\beta = \frac{r}{\varepsilon}, \quad \varepsilon \in \left( \frac{1}{\sqrt{2}}, 1 \right) \quad (2.52)$$

Now only remains to eliminate the parameter  $\varepsilon$ , and there is just a way to try it: by means of (2.49), (2.51) and (2.52). Removing  $\alpha$  and  $\beta$  in (2.12) by means of (2.49) and (2.51), and applying (2.24) to the right side of (2.12),

$$\frac{r^2}{\varepsilon^2(1-\varepsilon^2)} + \left( \frac{r}{\varepsilon} \right)^2 = k^2 \left( 1 - \frac{r}{h\varepsilon - r} \right)^2 + \left( h - \frac{r}{\varepsilon} \right)^2 \quad (2.53)$$

and after a tedious series of operations, expanding and simplifying by means of (2.21), (2.22) and (2.25), it is obtained,

$$\left. \begin{array}{l} Q \equiv \frac{r}{a} \\ R \equiv \frac{r}{h} \end{array} \right\} \left\{ \varepsilon^6 - 4R\varepsilon^5 + \{4[Q^2 + R^2] - 1\} \varepsilon^4 + 4R[1 - 2Q^2] \varepsilon^3 - 4R^2\varepsilon^2 + 4Q^2R^2 = 0 \right. \quad (2.54)$$

**Here it is, eureka!: the asymmetric case is solvable iff the sextic (2.54) is solvable algebraically.**

There is no further way to eliminate  $\varepsilon$ , since there are no more independent equations that interrelate these parameters than those that have been used yet. It is, therefore, likely impossible to eliminate the parameter  $\varepsilon$  and, consequently, therefore, it is most likely that nor be possible to get a relationship between the variables  $a$ ,  $b$ ,  $c$  and  $r$  that doesn't involve any of the parameters,  $\alpha$ ,  $\beta$  or  $\varepsilon$ .

See appendix C for the analysis of solubility of this equation.

## 2.8. Final relationship between $a$ , $b$ , $c$ and $r$ .

If would be possible to solve the sextic (2.54), denoting the right root (that corresponding to the eccentricity of  $\mathcal{E}$ ) by

$$\varepsilon_r \equiv \varepsilon(a, b, c, r) \quad (2.55)$$

the coordinates of  $T$  would be also functions of  $a$ ,  $b$ ,  $c$ ,  $r$ , and (2.50) would state an *implicit relationship* among  $r$  and  $a$ ,  $b$ ,  $c$ :

$$[x_T(r) - x_1(r)]^2 + [y_T(r) - y_1(r)]^2 = r^2 \quad (2.56)$$

where  $x_T$  is one of the two roots of (2.44),  $y_T$  is the function obtained substituting  $x_T$  in the equation of  $\mathcal{E}_1$ ,  $x_1$  and  $y_1$  the functions defined in (2.04).

## 2.9. Approximate solution by iteration.

Writing (2.56) in the form,

$$r = \sqrt{[x_T(r) - x_1(r)]^2 + [y_T(r) - y_1(r)]^2} \equiv \rho(a, b, c, \varepsilon_r, r) \quad (2.57)$$

the algorithm for  $n$  iterations would be,

$$\begin{cases} r_1 = \rho(r_0) \\ r_2 = \rho(r_1) \\ \vdots \\ r_n = \rho(r_{n-1}) \end{cases} \quad (2.58)$$

where  $r_0$  is an *initial value*, arbitrarily chosen, but in the *range of convergence* of the algorithm. The value of  $n$  depends on the desired accuracy and the algorithm terminates when the values of  $r$  become stable for a certain number of digits.

Obviously, if is known the value of  $r$  it is trivial to define one (or any number) relationships between the sides of the triangle and  $r$ .

## 2.9. An alternative way.

However, by rearranging the equation (2.54) according to the powers of  $r$ , we get a complete quartic in  $r$ ,

$$\left. \begin{array}{l} \sum_{p=0}^4 a_p \varepsilon^p = 0 \\ a_0 \equiv -\left(\frac{bc}{2}\right)^2 \varepsilon^4 (1 - \varepsilon^2) \\ a_1 \equiv abc \varepsilon^3 (1 - \varepsilon^2) \\ a_2 \equiv [(a^2 + h^2) \varepsilon^2 - a^2] \varepsilon^2 \\ a_3 \equiv -2h \varepsilon^3 \\ a_4 \equiv 1 \end{array} \right\} \quad (2.59)$$

equation which always has algebraic solution. It would, therefore, be necessary to slightly modify the conditions of the statement so that the problem had a solution, simply by changing the variable  $r$  by the parameter  $\varepsilon$ .

Applying the formula (A.02) given in the appendix A, its roots have the following explicit algebraic expression:

$$\left. \begin{array}{l} r = \frac{1}{2} \left[ -\frac{a_3}{2} + \sqrt{\frac{C}{3\sqrt[3]{2}} + E} \pm \sqrt{\frac{a_3^2}{2} - \frac{4a_2}{3} - \frac{C}{3\sqrt[3]{2}} - D + F} \right] \\ r = \frac{1}{2} \left[ -\frac{a_3}{2} - \sqrt{\frac{C}{3\sqrt[3]{2}} + E} \pm \sqrt{\frac{a_3^2}{2} - \frac{4a_2}{3} - \frac{C}{3\sqrt[3]{2}} - D - F} \right] \end{array} \right\} \begin{array}{l} A \equiv 2a_2^3 + 9[3(a_1^2 + a_0 a_3^2) - (8a_0 + a_1 a_3) a_2] \\ B \equiv a_2^2 + 3(4a_0 - a_1 a_3) \\ C \equiv \sqrt[3]{A + \sqrt{A^2 - 4B^3}} \\ D \equiv \frac{\sqrt[3]{2}}{3} \frac{B}{C} \\ E \equiv D + \frac{a_3^2}{4} - \frac{2a_2}{3} \\ F \equiv \frac{a_2 a_3 - 2a_1 - \frac{a_3^3}{4}}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} \end{array}$$

Substituting in (A.02) the generic coefficients of the quartic by the specifics ones given by (2.59),

$$\left. \begin{array}{l} a_0 \equiv -\left(\frac{bc}{2}\right)^2 \varepsilon^4 (1 - \varepsilon^2) \\ a_1 \equiv abc \varepsilon^3 (1 - \varepsilon^2) \\ a_2 \equiv [(a^2 + h^2) \varepsilon^2 - a^2] \varepsilon^2 \\ a_3 \equiv -2h \varepsilon^3 \\ a_4 \equiv 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} A \equiv 2a_2^3 + 9[3(a_1^2 + a_0 a_3^2) - (8a_0 + a_1 a_3) a_2] \\ B \equiv a_2^2 + 3(4a_0 - a_1 a_3) \\ C \equiv \sqrt[3]{A + \sqrt{A^2 - 4B^3}} \\ D \equiv \frac{\sqrt[3]{2}}{3} \frac{B}{C} \\ E \equiv D + \frac{a_3^2}{4} - \frac{2a_2}{3} \\ F \equiv \frac{a_2 a_3 - 2a_1 - \frac{a_3^3}{4}}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} \end{array} \right\} \quad (2.60)$$

And after some boring simplifications, we obtain as value of the constants  $A, B, \dots, F$ ,

$$\left\{ \begin{array}{l} A \equiv \left\{ 2 \left[ (a^2 + h^2) \varepsilon^2 - a^2 \right]^3 + 9b^2 c^2 (1 - \varepsilon^2) \left\{ 3(a^2 - a^2 \varepsilon^2 - h^2 \varepsilon^4) + 2(1 + \varepsilon^2) \left[ (a^2 + h^2) \varepsilon^2 - a^2 \right] \right\} \right\} \varepsilon^6 \\ B \equiv \left\{ \left[ (a^2 + h^2) \varepsilon^2 - a^2 \right]^2 - 3b^2 c^2 (1 - \varepsilon^2) (1 - 2\varepsilon^2) \right\} \varepsilon^4 \\ C \equiv \sqrt[3]{A + \sqrt{A^2 - 4B^3}} \\ D \equiv \frac{\sqrt[3]{2} B}{3 C} \\ E \equiv D + h^2 \varepsilon^6 - \frac{2 \left[ (a^2 + h^2) \varepsilon^2 - a^2 \right] \varepsilon^2}{3} \\ F \equiv \frac{2 \left\{ h^3 \varepsilon^6 - h \left[ (a^2 + h^2) \varepsilon^2 - a^2 \right] \varepsilon^2 - abc(1 - \varepsilon^2) \right\} \varepsilon^3}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} \end{array} \right. \quad (2.61)$$

### 2.10. A numeric example.

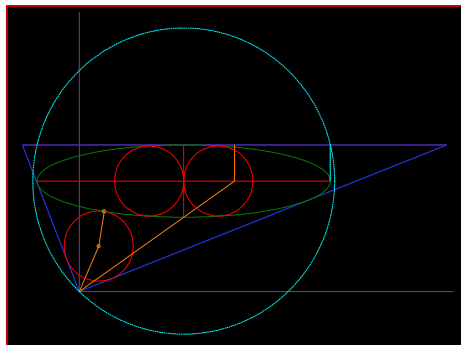
Given the following lengths for the sides of  $\triangle ABC$ ,

$$\triangle ABC \left\{ \begin{array}{l} a \equiv 2.8939431 \\ b \equiv 1.0591663 \\ c \equiv 2.6931530 \end{array} \right. \quad (2.62)$$

the values for the remaining parameters involved in the problem, with a 4-decimal precision, are:

$$\left\{ \begin{array}{l} \alpha = 1 \\ \beta = 0.2431 \end{array} \right\} \Rightarrow \varepsilon = 0.9700$$

$$\left\{ \begin{array}{l} r = 0.2358 \\ x_0 = 0.7125 \\ y_0 = 0.7425 \\ x_1 = 0.1331 \\ y_1 = 0.3057 \\ x_T = 0.1698 \\ y_T = 0.5386 \end{array} \right. \quad (2.63)$$



(2.64)

### Appendix A.

An alternative representation for the explicit relationship given by (1.60) and (1.61). Expressing (1.22) as,

$$\varepsilon^4 + a_3\varepsilon^3 + a_2\varepsilon^2 + a_1\varepsilon + a_0 = 0 \tag{A.01}$$

its roots have the following explicit algebraic expression, (A.02):

$$\left. \begin{aligned} \varepsilon &= \frac{1}{2} \left[ -\frac{a_3}{2} + \sqrt{\frac{C}{3\sqrt[3]{2}} + E} \pm \sqrt{\frac{a_3^2}{2} - \frac{4a_2}{3} - \frac{C}{3\sqrt[3]{2}} - D + F} \right] \\ \varepsilon &= \frac{1}{2} \left[ -\frac{a_3}{2} - \sqrt{\frac{C}{3\sqrt[3]{2}} + E} \pm \sqrt{\frac{a_3^2}{2} - \frac{4a_2}{3} - \frac{C}{3\sqrt[3]{2}} - D - F} \right] \end{aligned} \right\} \begin{cases} A \equiv 2a_2^3 + 9[3(a_1^2 + a_0a_3^2) - (8a_0 + a_1a_3)a_2] \\ B \equiv a_2^2 + 3(4a_0 - a_1a_3) \\ C \equiv \sqrt[3]{A + \sqrt{A^2 - 4B^3}} \\ D \equiv \frac{\sqrt[3]{2}}{3} \frac{B}{C} \\ E \equiv D + \frac{a_3^2}{4} - \frac{2a_2}{3} \\ F \equiv \frac{a_2a_3 - 2a_1 - \frac{a_3^3}{4}}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} \end{cases}$$

Substituting the generic coefficients of the quartic by the specifics ones of our equation, we obtain the value of the constants A, B, ..., as a function of the constant p defined by (1.07):

$$\left. \varepsilon^4 + 2p\varepsilon^3 - \varepsilon^2 - 2p\varepsilon + p^2 = 0 \right\} \begin{cases} a_3 = 2p \\ a_2 = -1 \\ a_1 = -2p \\ a_0 = p^2 \end{cases} \Rightarrow \begin{cases} A \equiv 108p^4 + 144p^2 - 2 \\ B \equiv 24p^2 + 1 \\ C \equiv \sqrt[3]{A + \sqrt{A^2 - 4B^3}} \\ D \equiv \frac{\sqrt[3]{2}}{3} \frac{B}{C} \\ E \equiv D + p^2 + \frac{2}{3} \\ F \equiv \frac{2p(1-p^2)}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} \end{cases} \tag{A.03}$$

$$p = \sqrt{2} - 1 \Rightarrow \begin{cases} D = \frac{\sqrt[3]{2}}{3} \frac{B}{C} \\ E = D + 2(2 - \sqrt{2}) - \frac{1}{3} \\ F \equiv \frac{4(3 - 2\sqrt{2})}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} \end{cases} \left\{ \begin{aligned} B &= 73 - 48\sqrt{2} \\ C &= \sqrt[3]{2} \left[ 11(103 - 72\sqrt{2}) + 12\sqrt{3}(2639 - 1866\sqrt{2}) \right] \end{aligned} \right. \tag{A.04}$$

Substituting the values given by (1.61) into the explicit expressions for the solutions of the quartic, (1.59), we obtain:

$$\varepsilon = \frac{1}{2} \left[ 1 - \sqrt{2} + \sqrt{\frac{C}{3\sqrt[3]{2}} + E} \pm \sqrt{4(2 - \sqrt{2}) - \frac{2}{3} - \frac{C}{3\sqrt[3]{2}} - D + F} \right] \Rightarrow \begin{cases} \varepsilon \approx 0.9476620415 \\ \varepsilon \approx 0.1762492425 \end{cases} \quad (\text{A.05})$$

$$\varepsilon = \frac{1}{2} \left[ 1 - \sqrt{2} - \sqrt{\frac{C}{3\sqrt[3]{2}} + E} \pm \sqrt{4(2 - \sqrt{2}) - \frac{2}{3} - \frac{C}{3\sqrt[3]{2}} - D + F} \right] \Rightarrow \varepsilon \approx -0.9761692044 \pm i0.2726245316 \quad (\text{A.06})$$

and,

$$\frac{b}{r} = 2 + \sqrt{2} + \frac{4\sqrt{2}}{1 - \sqrt{2} + \sqrt{\frac{C}{3\sqrt[3]{2}} + E} + \sqrt{4(2 - \sqrt{2}) - \frac{2}{3} - \frac{C}{3\sqrt[3]{2}} - D + F}} \quad (\text{A.07})$$

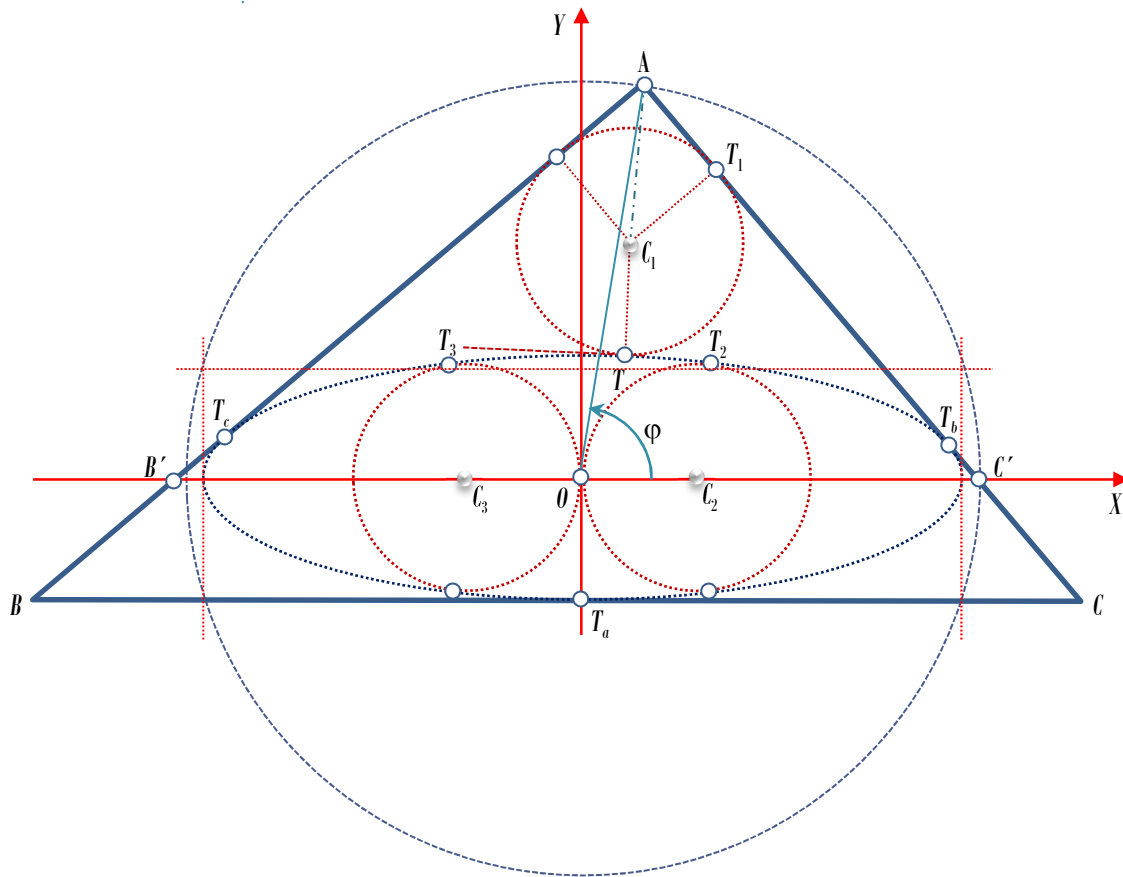
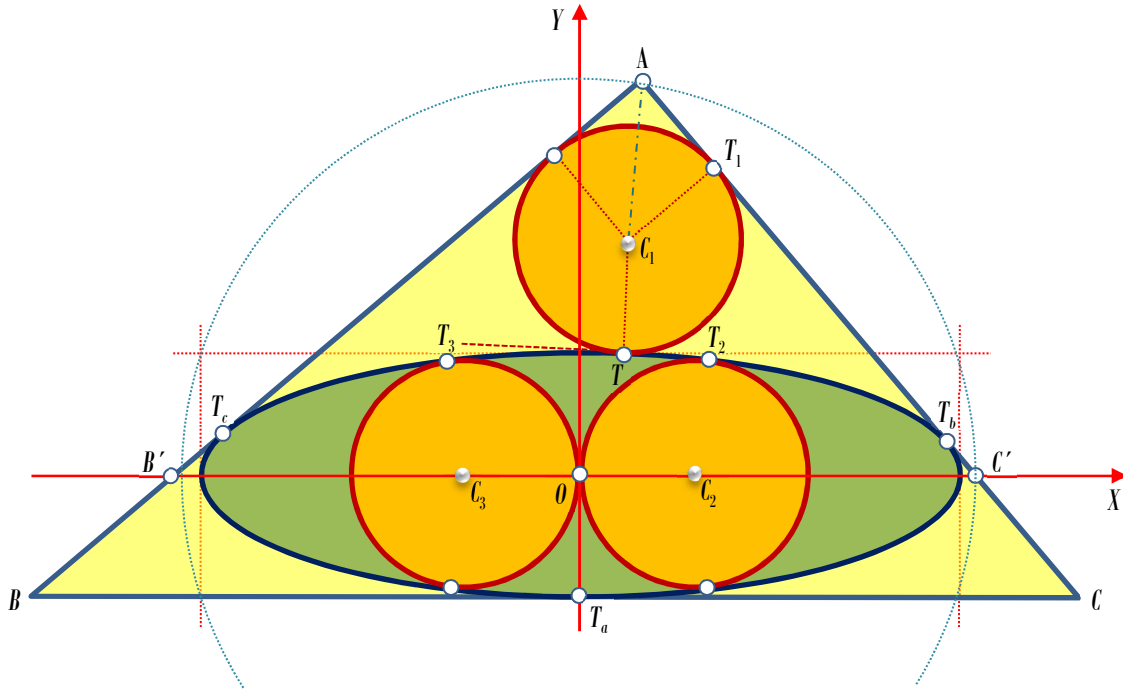
$$\begin{aligned} D = \frac{\sqrt[3]{2}}{3} \frac{B}{C} & \begin{cases} B = 73 - 48\sqrt{2} \\ C = \sqrt[3]{2} \left[ 11(103 - 72\sqrt{2}) + 12\sqrt{3(2639 - 1866\sqrt{2})} \right] \end{cases} \\ E = D + 2(2 - \sqrt{2}) - \frac{1}{3} & \\ F = \frac{4(3 - 2\sqrt{2})}{\sqrt{\frac{C}{3\sqrt[3]{2}} + E}} & \end{aligned} \quad (\text{A.08})$$

To shorten the writing of this formula (whose length exceeds the width of this page for a font size readable), have been split into  $A, B, \dots, F$  parts. This not only represents a greater compactness of the expression but also an economy of calculations because some of these parts is repeated within the expression. Otherwise, writing this steps fully requires too much typing, so it is skipped, but it can be easily verified.

**Appendix B.**

**Developing the equations for the asymmetric case in the reference system used in the symmetric case.**

Let's consider the same reference system used in the solution of the symmetric case, as shown in the pictures below.



**B.1. Explicit equation of the tangent to the ellipse with slope  $m$  and ordinate at the origin  $n$ .**

Let  $t \equiv y = mx + n$  a line that intersects the ellipse  $e$ , (1.01):

$$\left. \begin{array}{l} e \equiv \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \\ t \equiv y = mx + n \end{array} \right\} t \cap e \equiv \frac{x^2}{\alpha^2} + \frac{(mx+n)^2}{\beta^2} = 1,$$

i.e.,

$$(\alpha^2 m^2 + \beta^2)x^2 + 2mn\alpha^2 x + \alpha^2(n^2 - \beta^2) = 0 \quad (\text{B.01})$$

The *necessary and sufficient condition* for  $t$  to be tangent to the ellipse is that there is only one point of contact between the line and the curve, which is equivalent to equation (B.01) has only a root (double), ie, its discriminant must be zero:

$$\Delta = (2mn\alpha^2)^2 - 4(\alpha^2 m^2 + \beta^2)\alpha^2(n^2 - \beta^2) = 0 \quad (\text{B.02})$$

Operating and simplifying (B.02), we obtain:

$$n = \pm \sqrt{\alpha^2 m^2 + \beta^2} \quad (\text{B.03})$$

Thus, the **equation of the tangent of slope  $m$  to the ellipse** (1.01), for  $y > 0$  (positive semi-ellipse), is

$$t \equiv y = mx + \sqrt{\alpha^2 m^2 + \beta^2} \quad (\text{B.04})$$

**B.2. Locus of intersection points of pairs of orthogonal tangents: *director circle*.**

Let us consider a couple of tangents to the positive semi-ellipse in canonical form, generic and orthogonal each other, (B.04) and

$$s \equiv y = -\frac{1}{m}x + \sqrt{\frac{\alpha^2}{m^2} + \beta^2} \quad (\text{B.05})$$

Intersection point of (B.04) and (B.05):

$$s \cap t \equiv \left\{ \begin{array}{l} y = -\frac{1}{m}x + \sqrt{\frac{\alpha^2}{m^2} + \beta^2} \\ y = mx + \sqrt{\alpha^2 m^2 + \beta^2} \end{array} \right\} \Rightarrow x = \frac{\sqrt{\alpha^2 m^{-2} + \beta^2} - \sqrt{\alpha^2 m^2 + \beta^2}}{m + m^{-1}} \quad (\text{B.06})$$

Let  $A(x_A, y_A) \equiv s \cap t$ .

Multiplying by  $m$  the numerator and denominator of (B.06), we have:

$$x_A = \frac{\sqrt{\alpha^2 + \beta^2 m^2} - m\sqrt{\alpha^2 m^2 + \beta^2}}{1 + m^2} \quad (\text{B.07})$$



$$\because A(x_A, y_A) \in t \equiv y = mx + n, y_A = m \frac{\sqrt{\alpha^2 + \beta^2 m^2} - m\sqrt{\alpha^2 m^2 + \beta^2}}{1 + m^2} + \sqrt{\alpha^2 m^2 + \beta^2}, \text{ i.e.,}$$

$$y_A = \frac{m\sqrt{\alpha^2 + \beta^2 m^2} + \sqrt{\alpha^2 m^2 + \beta^2}}{1 + m^2} \quad (\text{B.08})$$

Let us compact the notation through the variable changes,  $u = \sqrt{\alpha^2 + \beta^2 m^2}$ ,  $v = \sqrt{\alpha^2 m^2 + \beta^2}$ , and calculate the sum of the squares of the coordinates of A, given by (B.07) and (B.08):

$$x_A^2 + y_A^2 = \left( \frac{u - mv}{1 + m^2} \right)^2 + \left( \frac{mu + v}{1 + m^2} \right)^2 = \frac{(1 + m^2)(u^2 + v^2)}{(1 + m^2)^2} = \frac{u^2 + v^2}{1 + m^2},$$

and undoing the variable changes,

$$x_A^2 + y_A^2 = \frac{u^2 + v^2}{1 + m^2} = \frac{(\sqrt{\alpha^2 + \beta^2 m^2})^2 + (\sqrt{\alpha^2 m^2 + \beta^2})^2}{1 + m^2} = \frac{(1 + m^2)(\alpha^2 + \beta^2)}{1 + m^2},$$

i.e.,

$$x_A^2 + y_A^2 = \alpha^2 + \beta^2 \quad (\text{B.09})$$

The locus of the points from which can be drawn pairs of orthogonal tangents to the canonical ellipse is the circle centered at the origin of coordinates and radius  $R = \sqrt{\alpha^2 + \beta^2}$ . This circle is called the **director circle**.

### B.3. Coordinates of the tangent point $T_c$ of the line with slope $m$ and the ellipse.

Solving the equation (B.01), under the tangency condition established in (B.02) and the value of  $n$  according to (B.04), we get the abscissa  $x_T$  of the tangent point  $T_c$ :

$$\left. \begin{array}{l} (\alpha^2 m^2 + \beta^2)x^2 + 2mn\alpha^2 x + \alpha^2(n^2 - \beta^2) = 0 \\ \Delta = 0 \\ n = \sqrt{\alpha^2 m^2 + \beta^2} \end{array} \right\} \Rightarrow x_T = \frac{-2\alpha^2 m \sqrt{\alpha^2 m^2 + \beta^2}}{2(\alpha^2 m^2 + \beta^2)},$$

$$\text{or, } x_T = \frac{-\alpha^2 m}{\sqrt{\alpha^2 m^2 + \beta^2}}.$$

$$\text{By replacing } x_T \text{ into } t \equiv y = mx + n, \text{ we obtain, } y_T = \frac{-\alpha^2 m^2}{\sqrt{\alpha^2 m^2 + \beta^2}} + \sqrt{\alpha^2 m^2 + \beta^2} = \frac{\beta^2}{\sqrt{\alpha^2 m^2 + \beta^2}}.$$

Gathering the above results, we have the following expression for the **coordinates of  $T_c$** :

$$T \left( -\frac{\alpha^2 m}{\sqrt{\alpha^2 m^2 + \beta^2}}, \frac{\beta^2}{\sqrt{\alpha^2 m^2 + \beta^2}} \right) \quad (\text{B.10})$$

And from (1.11), as a function of the eccentricity of the ellipse,

$$x_T = -\frac{m}{1 - \varepsilon^2} \cdot \frac{r}{\varepsilon \sqrt{1 + \frac{m^2}{1 - \varepsilon^2}}}, y_T = \frac{r}{\varepsilon \sqrt{1 + \frac{m^2}{1 - \varepsilon^2}}} \quad (\text{B.11})$$

#### B.4. Equation of the bisector of the right angle A.

$$\left. \begin{aligned} c \equiv y = m(x - x_A) + y_A &\Leftrightarrow \frac{mx - y - mx_A + y_A}{\sqrt{1+m^2}} = 0 \\ b \equiv y = -\frac{1}{m}(x - x_A) + y_A &\Leftrightarrow \frac{x + my - x_A - my_A}{\sqrt{1+m^2}} = 0 \end{aligned} \right\} \Leftrightarrow mx - y - mx_A + y_A = \pm(x + my - x_A - my_A)$$

$$mx - y - mx_A + y_A = +(x + my - x_A - my_A) \Leftrightarrow \boxed{(1-m)x + (1+m)y - (1-m)x_A - (1+m)y_A = 0}$$

$$mx - y - mx_A + y_A = -(x + my - x_A - my_A) \Leftrightarrow \boxed{(1+m)x - (1-m)y - (1+m)x_A + (1-m)y_A = 0}$$

For A in the first quadrant its bisector has positive slope, so its implicit equation is,

$$\boxed{(1+m)x - (1-m)y - (1+m)x_A + (1-m)y_A = 0}$$

and explicitly,

$$y = \frac{1+m}{1-m}x + y_A - \frac{1+m}{1-m}x_A \quad (\text{B.12})$$

#### Another way to deduce the equation of the bisector.

The slope of the bisector is

$$\mu = \operatorname{tg} \left( \widehat{B} + \frac{\pi}{4} \right) = \frac{\operatorname{tg} \widehat{B} + \operatorname{tg} \frac{\pi}{4}}{1 - \operatorname{tg} \widehat{B} \cdot \operatorname{tg} \frac{\pi}{4}} = \frac{1+m}{1-m}.$$

Hence, the equation of the bisector is:

$$y = \mu(x - x_A) + y_A = \frac{1+m}{1-m}x - \frac{1+m}{1-m}x_A + y_A.$$

Substituting  $x_A$  by the expression (B.07) and  $y_A$  by the right side of (B.04), taking into account that  $A \in t$ ,

$$\begin{aligned} y &= \frac{1+m}{1-m}x - \frac{1+m}{1-m} \cdot x_A + mx_A + n = \frac{1+m}{1-m}x + \left( m - \frac{1+m}{1-m} \right) \cdot x_A + \sqrt{\alpha^2 m^2 + \beta^2} = \\ &= \frac{1+m}{1-m}x - \frac{1+m^2}{1-m} \cdot \frac{\sqrt{\alpha^2 + \beta^2 m^2} - m\sqrt{\alpha^2 m^2 + \beta^2}}{1+m^2} + \sqrt{\alpha^2 m^2 + \beta^2}, \end{aligned}$$

i.e.,

$$y = \frac{1+m}{1-m}x + \frac{\sqrt{\alpha^2 m^2 + \beta^2} - \sqrt{\alpha^2 + \beta^2 m^2}}{1-m} \quad (\text{B.13})$$

or,

$$y = \frac{1+m}{1-m}x + \frac{r}{1-m} \cdot \frac{\sqrt{1+m^2 - \varepsilon^2} - \sqrt{1+m^2 - m^2 \varepsilon^2}}{\varepsilon \sqrt{1 - \varepsilon^2}} \quad (\text{B.14})$$

### B.5. Coordinates of $C_1$ .

$C_1$  is determined by the intersection of the bisector of  $A$ , (2.12), and the circumference with center  $A$  and radius  $r\sqrt{2}$  :

$$\left. \begin{aligned} (x-x_A)^2 + (y-y_A)^2 = 2r^2 &\Rightarrow y_1 = y_A + \sqrt{2r^2 - (x_1 - x_A)^2} \\ y_1 = \frac{1+m}{1-m}x_1 + y_A - \frac{1+m}{1-m}x_A \end{aligned} \right\} \Leftrightarrow \sqrt{2r^2 - (x_1 - x_A)^2} = \frac{1+m}{1-m}x_1 - \frac{1+m}{1-m}x_A$$

or,

$$2r^2 - (x_1 - x_A)^2 = \left(\frac{1+m}{1-m}\right)^2 (x_1 - x_A)^2,$$

hence,

$$2r^2 = \left[\left(\frac{1+m}{1-m}\right)^2 + 1\right] (x_1 - x_A)^2 \Leftrightarrow r^2 = \frac{1+m^2}{(1-m)^2} (x_1 - x_A)^2, \text{ if } m > 0$$

from which,

$$x_1 = x_A - \frac{1-m}{\sqrt{1+m^2}} r \tag{B.15}$$

and substituting this expression for  $x_1$  into (B.12),

$$y_1 = y_A - \frac{1+m}{\sqrt{1+m^2}} r \tag{B.16}$$

Replacing in (B.15) and (2.16) the coordinates of  $A$  given by (B.07) and (B.08), respectively,

$$(x_1, y_1) = \left( \frac{\sqrt{\alpha^2 + \beta^2 m^2} - m\sqrt{\alpha^2 m^2 + \beta^2}}{1+m^2} - \frac{1-m}{\sqrt{1+m^2}} r, \frac{m\sqrt{\alpha^2 + \beta^2 m^2} + \sqrt{\alpha^2 m^2 + \beta^2}}{1+m^2} - \frac{1+m}{\sqrt{1+m^2}} r \right)$$

$$\begin{aligned} x_1 &= \frac{\beta \left( \sqrt{\frac{\alpha^2}{\beta^2} + m^2} - m\sqrt{\frac{\alpha^2}{\beta^2} m^2 + 1} \right)}{1+m^2} - \frac{1-m}{\sqrt{1+m^2}} r \\ y_1 &= \frac{\beta \left( m\sqrt{\frac{\alpha^2}{\beta^2} + m^2} + \sqrt{\frac{\alpha^2}{\beta^2} m^2 + 1} \right)}{1+m^2} - \frac{1+m}{\sqrt{1+m^2}} r \end{aligned} \tag{B.17}$$

or, as a function of the eccentricity of the ellipse:

$$\begin{aligned} x_1 &= \frac{\frac{r}{\varepsilon} \left( \sqrt{\frac{1}{1-\varepsilon^2} + m^2} - m\sqrt{\frac{m^2}{1-\varepsilon^2} + 1} \right) - r(1-m)\sqrt{1+m^2}}{1+m^2} \\ y_1 &= \frac{\frac{r}{\varepsilon} \left( m\sqrt{\frac{1}{1-\varepsilon^2} + m^2} + \sqrt{\frac{m^2}{1-\varepsilon^2} + 1} \right) - r(1+m)\sqrt{1+m^2}}{1+m^2} \end{aligned} \tag{B.18}$$

### Appendix C. Analysis of the solubility of the sextic (2.54).

As is well known, [3] and [4], by means of a *Tschirnhausen transformation* the general sextic,

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \quad (\text{C.01})$$

can be reduced to the form,

$$z^6 + z^2 + a_2z + a_1 = 0 \quad (\text{C.02})$$

Thus, via *Tschirnhausen transformation*, the sextic (2.54) can be reduced to the form,

$$z \left[ z^5 + z - \left( \frac{2r}{b} \right)^2 - \left( \frac{2r}{c} \right)^2 \right] = 0; \quad b, c, r \in \mathbb{R}^+; \quad 2r < b < c \quad (\text{C.03})$$

All equations of degree  $\leq 4$  are algebraically solvable over the complex number field  $\mathbb{C}$ , but from the work of *Ruffini*, *Abel* and *Galois* it is now well-known that all quintic equations do not [2].

In [1] are stated the *necessary and sufficient conditions* for the quintic factor of (C.03), a *Bring-Jerrard form* or *principal quintic* for the general quintic, be solvable by radicals: the existence of three *rational* numbers,  $\epsilon = \pm 1$ ,  $p > 0$  and  $q \neq 0$ , such that, applied to our specific values,

$$\frac{5q^4(3 - 4\epsilon p)}{p^2 + 1} = 1, \quad \frac{4q^5(11\epsilon + 2p)}{p^2 + 1} = \left( \frac{2r}{b} \right)^2 + \left( \frac{2r}{c} \right)^2 \quad (\text{C.04})$$

from where,

$$\frac{4q(11\epsilon + 2p)}{5(3 - 4\epsilon p)} = \left( \frac{2r}{b} \right)^2 + \left( \frac{2r}{c} \right)^2 \quad (\text{C.05})$$

that can be written as,

$$\frac{4q(11\epsilon + 2p)}{5(3 - 4\epsilon p)} = 4 \left( \frac{a}{bc} \right)^2 r^2 \quad (\text{C.06})$$

or,

$$\frac{4q(11\epsilon + 2p)}{5(3 - 4\epsilon p)} = \left( \frac{2r}{h} \right)^2 \quad (\text{C.07})$$

where  $r$  is the radius common to the three circumferences and  $h$  the height relative to the hypotenuse of  $\triangle ABC$ .

In [4] are given explicit formulas to get the roots for this irreducible particular form of quintic when it is algebraically solvable:

$$z_j = \sum_{k=1}^4 e^{jk\left(\frac{2\pi i}{5}\right)} u_k, \quad j = [0, 4] \cap \mathbb{Z} \quad (\text{C.08})$$

with,

$$u_1 \equiv \sqrt[5]{\frac{v_1^2 v_3}{\Delta^2}}, u_2 \equiv \sqrt[5]{\frac{v_3^2 v_4}{\Delta^2}}, u_3 \equiv \sqrt[5]{\frac{v_2^2 v_1}{\Delta^2}}, u_4 \equiv \sqrt[5]{\frac{v_4^2 v_2}{\Delta^2}}, \Delta \equiv p^2 + 1 \quad (\text{C.09})$$

$$v_1 \equiv \sqrt{\Delta} + \sqrt{\Delta - \epsilon\sqrt{\Delta}}, v_2 \equiv -\sqrt{\Delta} - \sqrt{\Delta + \epsilon\sqrt{\Delta}}, v_3 \equiv -\sqrt{\Delta} + \sqrt{\Delta + \epsilon\sqrt{\Delta}}, v_4 \equiv \sqrt{\Delta} - \sqrt{\Delta - \epsilon\sqrt{\Delta}} \quad (\text{C.10})$$

Furthermore, any *irreducible* equation with rational coefficients is solvable by radicals if and only if its *Galois group*  $G$  is a subgroup of one of the two transitive subgroups of  $S_5$ :  $J = S_2 \# S_3$  and  $K = S_3 \# Z_2$ , denoting the *wreath product* of two groups by  $\#$ , [5].

At first glance, the lack of the first-degree term in (2.54) might seem to facilitate the analysis of solvability of the equation, but is just the opposite because the *resolvent equation* of (C.02) is of the form [1],

$$z^{15} - 6a_2^2 z^{13} - (42a_1 + 3)a_1^3 z^{12} + \dots + (144a_1 a_2^4 - 32a_2^4 - 3)a_1^{15} a_2^2 = 0 \quad (\text{C.11})$$

and (C.02) is solvable by radicals and its *Galois group* is a subgroup of  $J$  iff (C.11) has a rational root, but it is straightforward that this happens if the constant term vanishes because then  $z = 0$ , i.e., if

$$a_1 = \frac{32a_2^4 + 3}{144a_2^2} \quad (\text{C.12})$$

and taking, for instance,  $a_2 = 1/2$ , we find that the *Galois group* of the irreducible polynomial  $36z^6 + 36z^2 + 18z + 5$  is a subgroup of  $J$  and solvable. But in our case,  $a_1 = 0$ , and so (C.02) is *reducible*, therefore this theorem is not applicable.

Moreover, the second condition of (C.04) involves a relationship between  $b$ ,  $c$  and  $r$ , which is the final goal of the problem, so this condition neither is implementable.

But, according to *Dummit's analysis* [3], solvable irreducible quintic equations are rare, hence, it is reasonable to expect that solvable irreducible sextics are also rare.

Through the exhaustive analysis carried out we have seen that the insurmountable barrier to solving the asymmetric case is the sextic in  $\varepsilon$  (2.54), the only way to eliminate the parameter  $\varepsilon$ .



**Appendix D. Warnings about the polynomial 6<sup>th</sup> degree equation (2.54).**

One of the roots of the sextic (2.54), expressed in its most compact form,

$$\sum_{q=0}^6 c_q \varepsilon^q = 0 \quad \left\{ \begin{array}{l} c_0 \equiv \left( \frac{2r^2}{bc} \right)^2 \\ c_1 \equiv 0 \\ c_2 \equiv - \left( \frac{2ar}{bc} \right)^2 \\ c_3 \equiv \frac{4r(a^2 - 2r^2)}{abc} \\ c_4 \equiv 4r^2 \left[ \left( \frac{a}{bc} \right)^2 + \frac{1}{a^2} \right] - 1 \\ c_5 \equiv - \frac{4ar}{bc} \\ c_6 \equiv 1 \end{array} \right. \quad (D.01)$$

is the eccentricity  $\varepsilon$  of the ellipse, in terms of  $a$ ,  $b$ ,  $c$  and  $r$ , but this equation could be unsolvable by radicals.

As a “warning to sailors”, it can be useful to reproduce the following chain of thoughts: we know that (D.01) has to have, at least, one real solution, and taking into account that not real roots appear only in pairs of conjugate complex, necessarily, the number of real roots have to be even, and by the *uniqueness* of the solution discussed before, we'd can think (wrongly) that the unique real solution of this equation has to have multiplicity 2, 4 or 6, and, therefore, the derivative of this polynomial should preserve the same real root, allowing the substitution of the before equation by the following:

$$\sum_{k=1}^6 k c_k \varepsilon^{k-1} = 0 \quad (D.02)$$

But  $c_1 = 0$ , therefore, (D.02) can be written as,

$$\sum_{k=2}^6 k c_k \varepsilon^{k-1} = 0 \quad (F.03)$$

At this point, we would think that the lack of the first grade term in  $\varepsilon$  in the equation (D.01) should be the key for the door to the solution, since it allow to get down one grade more by elimination of the common factor  $\varepsilon$  in (D.03) (it has been proved yet that the solution  $\varepsilon = 0$  it isn't admissible), getting finally the following quartic:

$$\sum_{k=2}^6 k c_k \varepsilon^{k-2} = \sum_{p=0}^4 (p+2) c_{p+2} \varepsilon^p = 0 \quad (D.04)$$

Unfortunately, the hypothesis that the equation (F.01) has one unique manifold real root can't be proved, hence, it isn't warranted to reach the solution trough this path. Indeed, for the numerical example presented in 2.11, the six roots of (2.54) are all real and different.

## Appendix E.

### Exploring another ways.

The transformation by *inversion*, the most widely-used method in Sangaku challenges, applied to the ellipse, transform its canonical equation into a quartic equation which is much more complicated.

For instance, the curves on the right picture arise from the mapping

$$(x, y) \rightarrow \frac{(x, -y)}{x^2 + y^2}$$

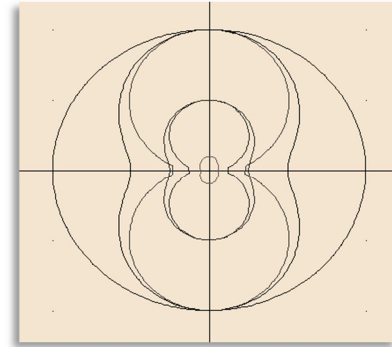
by *inversion on the unit circle*, that transforms the ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

into the *quartic*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = (x^2 + y^2)^2.$$

It is neither possible to simplify the problem through affine transformations, since an *affinity* that transform the ellipse into a circumference, it also will transform, at the same time, the circumferences into ellipses.



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# Interview with Tony Rothman

Sujatha Ramdorai



Tony Rothman

**Sujatha Ramdorai:** *Tell us a little about yourself.*

**Tony Rothman:** I'm a theoretical physicist who has specialised in general relativity and cosmology. Most of my research has concerned the very early universe and black holes. I am just finishing a six-year appointment in the physics department at Princeton University and will be teaching at The College of New Jersey in the fall. I've also done a fair amount of writing for the general public (sometimes I hesitate to call it "popular"). Quite recently I published my tenth book, *Firebird*, which is a novel set in a fusion-research laboratory. I think it is unusual in that it isn't science fiction, but an attempt to base a novel on real science. To the best of my abilities the science is totally accurate and, unfortunately, the politics too. I've also just drafted a play about the famous sixteenth-century Cardano–Tartaglia feud over the cubic equation. It's been fun, but difficult. One question has been how "accurate" to make it. Much of what is written about it in the semi-popular literature is nothing more than fairy tales. Several recent books have Tartaglia causing Cardano's arrest for heresy — 13 years after Tartaglia died! At any rate, it's something that I needed to get done.

**SR:** *What appealed to you in the Sangaku story and Hidetoshi's work that you decided to collaborate on the book?*

**TR:** While in high school my favourite math subject was certainly geometry. I think mathematicians have either algebraic imaginations or geometric imaginations. I don't consider myself a terribly creative mathematician — like most physicists I use mathematics to solve problems — but my own imagination is certainly geometric. I suspect this is one reason I fell in love with relativity, which is a very geometric subject. My contact with Sangaku came about during a very specific space-time event. One day, as I recall, in winter 1989–1990, I stopped by Freeman Dyson's office at the Institute for Advanced Study in Princeton. We were probably just planning to have lunch. As soon as Freeman raised his hand to say hello, he said, "Take a look at this," and handed me the Sangaku-problem book Hidetoshi had just published with Dan Pedoe, who had been Freeman's math teacher long ago in England. I had absolutely no idea of what "Sangaku" or "temple geometry" meant. As I leafed through the book with dropped jaw, Freeman stood there laughing. I found the problems visually striking, quite different from anything I had seen in school — they even looked Japanese. But the main impression was how damn difficult they were. For all my love of geometry, I quickly realised to my embarrassment that I hadn't the faintest idea of how to solve most of them. The fact that they were found in temples and had evidently been largely created by farmers and peasants was an additional embarrassment, not to mention extremely intriguing. I bought a copy of the book for myself, worked on some of the problems and eventually contacted Hidetoshi about a possible *Scientific American* article, which I wrote with his assistance. The article sat at the magazine for three years before it was published — even though I had been an editor there. When it finally appeared, it proved fairly influential — I think it was the first major piece in the West about Sangaku — and it helped make temple geometry part of the world heritage of mathematics.

So I am happy about that. I hadn't initially intended to write an entire book about temple geometry. I prefer to write fiction, actually, but it has become almost impossible to get fiction published. And so, around 2005, when all my fiction projects had collapsed, I contacted Princeton University Press about doing a book on Sangaku and the editor there, Vickie Kearn, quickly agreed (literally within about twenty minutes, although it was actually the second time I approached them; a previous editor had expected me to pay for it!) I did all this without even telling Hidetoshi, but he was quite glad to hear the news.

**SR:** *One of the things that strikes the reader — and which you comment yourself — is the aesthetic side of the Sangaku. What are your thoughts on this.*

**TR:** As I said, I found the problems quite beautiful — miniature Japanese works of art. Like most scientists, I suspect, I am drawn to “clean” artwork — I am always struck by how many of my colleagues have collections of African art, and I do as well. Japanese art certainly fits the bill. In fact, I think the Japanese are incapable of creating anything ugly. I was also struck by the asymmetry in many of the problems, compared to the Greek-inspired problems we all face in high school. From Daisetz Suzuki's books on Zen, it seems that asymmetry is a characteristic feature of Japanese art. So I suspect that the whole aesthetic of Sangaku evolved from the Japanese artistic aesthetic. I'm certain that many of the problems evolved from everyday situations and objects — like fans, which are really sectors of circles, and origami designs. The tablets themselves are also beautiful, brightly coloured, and in one case, surrounded by a striking dragon frame.

**SR:** *Can you tell us more about the whole collaborative process?*

**TR:** It wasn't easy. To this day, Hidetoshi and I have never met. I don't speak any Japanese and his English, although it's improved over the years, is far from his native language. The whole thing was done by email. Over the two years we worked on it, I'd guess we exchanged about a thousand emails. Luckily, mathematical terminology is limited and, usually, well defined, but sometimes we would exchange ten emails just to clarify one sentence. Hidetoshi is the expert on Sangaku; my role was basically editorial. He would send me the raw material and I would check it for errors and rewrite the problem statements into

respectable English. The intro chapters I wrote pretty much from scratch. Also, I wanted the book to appeal to non-mathematicians, so I tried to avoid technical terms when possible, even when they might have made things clearer to geometers. We had some organisational issues as well. Hidetoshi wanted to organise the book by tablet, but this resulted in very easy problems being placed side by side with nearly impossible problems, and I felt that this would discourage a lot of readers, not to mention make presentation of solutions extremely difficult. So I reordered everything, placing easy problems first and harder problems later. Most of the solutions were either traditional or Hidetoshi's, but I also contributed a few and did all the line drawings, mainly because Hidetoshi's drawing software wasn't compatible with anything I had. The whole thing ended up being a gigantic jigsaw puzzle. From a design perspective, it was certainly the most complicated book I've worked on. The Princeton University Press art director, Dimitri Karetnikov, was very helpful in this regard.

**SR:** *Has working on this project led you to explore other areas of “Japanese Science” or other forms of Japanese knowledge or culture, especially from that period or earlier?*

**TR:** I am not a scholar of Japanese culture — and don't speak Japanese — so I haven't plunged far into related areas, but I am intrigued by certain aspects of Japanese mathematical history, which seem to me not well understood. For instance, the feudal Japanese mathematicians didn't know calculus — at least what we regard as calculus — and we don't know anything about how they handled differentiation. Yet some Sangaku problems seem to require differential calculus for their solution. It is a mystery to me how the Japanese solved them. Also, there is a whole field known as “Rangaku”, literally “Dutch Learning”, which concerns foreign knowledge that seeped into Japan during the period of national isolation through the Dutch trading post on Deshima island. There seems to be quite a debate among scholars about just what the Japanese knew of foreign science and when they knew it, but the Wikipedia article on Rangaku, for example, isn't very satisfactory. I would be interested in learning more about Japanese knowledge of foreign developments in mathematics. Finally, my *Scientific American* article and the book seem to have given people the impression that everybody in Edo-period Japan was creating Sangaku. It is difficult to estimate the number of original tablets, but even if there were

50,000, that would only be about 150 a year over three centuries, and that is surely an upper limit. Since many of the problems on different Sangaku are duplicates, I suspect that we have most of the problems that were created — only several thousand. So it may not have been such a widespread cultural phenomenon. One or two other Sangaku investigators have written to me about such matters and perhaps they should all get together and try to sort them out.

**SR:** *The book is well publicised in the West. What have the reactions been to what one might call a “Coffee Table” book on mathematics?*

**TR:** It was my intent from the start to create exactly the first coffee-table math book; I even called it that and I guess it has been received as such, although I don't know of too many people who bought it just to look at the pictures. (Given the available resources, Princeton University Press did a good job. The book would have been even more beautiful if Abrams had published it, but it would have cost \$100.) Most of the feedback I've gotten has been from mathematics faculty members at various universities who are planning a trip to Japan and would like to see a Sangaku in the flesh. I pass them on to Hidetoshi, who knows where they all are. (I should say that most of the original Sangaku have been removed from the temples and are either in museums or in temple storehouses, where you need prior permission to view them.) One interesting outcome is that an artist in Santa Fe, Jean Constant, has based a whole series of his and his students' works on Sangaku. They're quite striking. A furniture maker has also created a “sangaku” line. I'm always glad when science or math inspires artists, even if their creations are metaphorical. Nevertheless, in this case most of my mail has come from mathematics people.

**SR:** *Tell us a little about your experience in this whole transcultural, mathematical journey that straddles two civilisations.*

**TR:** I think I've already given some idea about that. In general I believe in culture shock — it keeps you on your toes. Certainly, to interact with someone from a different culture whom you've never met — especially by email — takes a lot of patience. The whole situation is a minefield for misunderstanding, and sometimes I think Hidetoshi and I blew each other up. When I taught in Korea a few years ago and would hang out with the students, the long silences made me uncomfortable,

until they told me that silence was admired in their culture. Luckily, mathematics itself is universal. The problems were basically Euclidean geometry problems, and although the Japanese often attacked them with methods that wouldn't have occurred to me, I was nevertheless able to understand what they were doing. I do feel that I wasn't the ideal person to collaborate with on the book. A mathematician fluent in Japanese and versed in Japanese history would have been a better choice. My only qualification was that I stepped up to the plate.

**SR:** *When did you actually first see a Sangaku in reality?*

**TR:** Believe it or not, I never have. I've travelled widely around the world, and have lived for many years abroad, but for some reason have never been to Japan. As I just said, I wasn't the ideal person to do this book. I hope someday to get there and then Hidetoshi will show me some.

**SR:** *Did you notice anything different in the way the problems were posed and answered, compared to your own training in the West?*

**TR:** Sure. The repeated, intricate use of the Pythagorean theorem was really ingenious, if at times cumbersome. It's amazing how much you can do with just the Pythagorean theorem. One eye-opener was the Japanese way of dealing with ellipses, which is quite different from ours. The Japanese mathematicians viewed an ellipse as a slice through a right circular cylinder, not as a conic section. A circle inscribed in the ellipse was the projection of a sphere in the cylinder onto the slice. They could then use the Pythagorean theorem to connect the various important lengths involved. This “3-D” approach allows you to solve some really difficult problems, which I never would have been able to do using the usual equation for an ellipse. In fact, you don't even need the usual equation for an ellipse.

I did find some of the problems ill-posed. Hidetoshi tells me this is a feature of traditional Japanese mathematics. For instance, take problem 7.12 in the book, which is an unsolved problem in which you are asked to find the radius of three identical circles, two of which are inscribed in an ellipse, which is itself inscribed in a right triangle along with the third circle. It wasn't at all clear to me at first that there even was an analytic solution. Recently Jesu Alvarez Lobo from Spain has sent me his solution. I haven't worked through it,

but it seems to be a tour-de-force, over thirty pages (arXiv:1110.1299). He does find an analytic, closed solution for an isosceles right triangle, but for the general case he can only get an implicit solution. Some of the traditional solutions are from another galaxy. We are often taught to draw auxiliary lines in high school geometry courses to solve problems. In at least one Sangaku problem (6.3), the author of the solution introduces an entire auxiliary circle, which at first seems to have nothing to do with the problem whatsoever. I doubt I'd ever have thought of that solution. And again, it wasn't entirely obvious to me that there even was one. Lobo showed that the solution exists only for a particular base angle of the isosceles triangle involved in the problem. Also, many of the solutions, especially by Yoshida Tameyuki, assume rather sophisticated lemmas or other steps, which aren't stated. I don't know whether Yoshida just assumed everyone knew them or what, but I doubt I'd ever present a proof with so many important details missing. One thing I learnt, is that to solve these intricate Sangaku problems, you have to make a *really* good drawing, not just a sketch. And in proving things about triangles you should never draw a 45-degree line.

**SR:** *There were algebraic, arithmetic as well as geometric problems that were posed, though geometry seems to have been the most popular. Some of the problems, as you observe in the book have appeared in other guises in other cultures. Why do you think Geometry was more popular?*

**TR:** Many problems, both algebraic and geometric, pop up in different cultures, just as most scientific discoveries are made multiple times. I don't know whether the duplication of the math problems was due to cross-fertilisation by "word of mouth" over the centuries, or whether they cropped up independently. Certainly every time I do something in physics, no matter how obscure it seems, somebody else always claims to have done it first! If geometry problems have been more popular, it must be because of the visual appeal, and in some sense geometry is easier than algebra. Even if you are algebraically challenged, as I often am, you can often solve problems geometrically, and many of the basic geometric theorems regarding angles and so forth are pretty self-evident, so you don't really have to prove them before solving a problem. In writing this play about Tartaglia and Cardano, I was looking at the Tartaglia–Cardano solution to the depressed cubic equation. They did this before modern

algebraic notation existed and so you might think it really hard. But to find the cubic formula is actually really easy if you think geometrically and remember that a cubic equation must give the volume of a cube. If you slice up the cube as those fellows did, the cubic formula falls out almost immediately. It's a good lesson. Nowadays we have algebraised geometry to such an extent that we've often make things more complicated than they really are.

**SR:** *"Sacred Math", "Temple Geometry".... In the West it would almost be anachronistic to juxtapose science and the spiritual in this manner, yet this appears natural in the East. Especially the thought of the problems and solutions being offerings to the divine! What are your thoughts on this....*

**TR:** The idea that mathematical tablets were presented as religious offerings is very appealing to me, but of course, the Buddhist idea of God, or the divine, is much more abstract than Western ideas. Some readers of our book have suggested that our use of the word "God" in the translation of some of the tablet inscriptions may not be accurate. Unfortunately, since I don't know Japanese, I am unable to say. Russians often speak of "lighting a candle to God", whether they are talking about creating a work of art or solving a scientific problem. That's more or less how I feel about it, although in my case "God" may be even vaguer than it is for the average Buddhist. I think, though, that Sangaku served several purposes. Since hanging tablets in temples and shrines was a long-established tradition in Japan before the advent of Sangaku, one can't rule out the possibility that many people were just doing it for "fun", or maybe with about as much religious fervour as most Americans celebrate Christmas. Some tablets were apparently created by classes at small schools, called *Jyuku*, and almost certainly these were hung in the temples as advertisements for the school. In any case, since the Sangaku almost always contain an answer, but rarely the solution, they were pretty clearly issued as challenges to other "worshippers".

**SR:** *Any lessons you have learnt or any other related thoughts you might want to share?*

**TR:** It's become clear to me that geometry education has declined considerably, at least in the US. To write *Sacred Mathematics*, I consulted a number of textbooks that are currently used in high schools, but they were totally inadequate to solve most Sangaku problems. To write

the chapter on inversion, I needed to go back to texts that were literally a century old, e.g. Clement Durrell's geometry. Inversion simply isn't taught anymore, except in some advanced college math courses. And it's not even that hard!

**SR:** *Thanks very much for taking the time for the interview.*



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