The Signum Function of the Second Derivative and its application to the determination of relative extremes of fractional functions (SF2D).

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Dedicated to Nityangi

ABSTRACT

Usually, the complexity of a fractional function increases significantly in its second derivative, so the calculation of the second derivative can be tedious and difficult to simplify and evaluate its value at a point, especially if the abscise isn't an integer.

However, to determine whether a point at which cancels the first derivative of a function is a relative extremum (maximum or minimum) of it, is not necessary to know the value of the second derivative at the point but only its sign.

Motivated by these facts, we define a signum function for the second derivative of fractional functions in the domain of the roots of the first derivative of the function.

The method can dramatically simplify the search for maximum and minimum points in fractional functions and can be implemented by means of a simple algorithm.

Signum Function of the Second Derivative (SF2D)

Let

$$\boxed{f(x) = \frac{u(x)}{v(x)}} \qquad (1)$$

Applying the rule for the derivative of a quotient in an unspecified point,

$$\boxed{f' = \frac{u'v - uv'}{v^2}} \qquad (2)$$

Let $x_0 \in \operatorname{Dom} f$ such that f is differentiable at x_0 and $f'(x_0) = 0$.

Then,

$$f'(x_0)=0 \Leftrightarrow \left.\frac{u'v-uv'}{v^2}\right|_{x_0}=0 \Rightarrow u'v-uv'\Big|_{x_0}=0.$$

And

$$f''(x_0)=\left.\frac{(u''v+\cancel{u'v'}-\cancel{u'v'}-uv'')v^2-2v'\overbrace{(u'v-uv')}^{0}}{v^4}\right|_{x_0}=\left.\frac{(u''v-uv'')}{v^2}\right|_{x_0}.$$

But $v^2>0$ as, by hypothesis, f is differentiable at x_0 and so must be $v\neq 0$ at $x_0\in \operatorname{Dom} f$; therefore, the sign of f'' doesn't depend on v^2 and so we can ignore v^2. Therefore, the sign of f'' at the root x_0 of f' is given by the simplest function

$$\boxed{\operatorname{sgn} f''_{x_0}=\operatorname{sgn}\,[u''v-uv'']_{x_0}} \qquad ③$$

Actually the sign function is not only a function but a class of functions, because if we define in a set of functions the relationship "take the same sign at x_0 that...," a *partition* is set in such set, classifying the functions in three *classes of equivalence* according to its sign at that point: "+", "-", "0"". Therefore, as the sign for the second derivative at the points where it exists and annul the first derivative we can adopt ③ or any other with the same sign at the same point. Taking into account this considerations, frequently, is possible to use as signum function a simplest function that ③.

Let's see through some examples how to implement the algorithm and its advantages over the standard method.

Exemple 1

Find all the *relative extremum*, *minimums* and *maximums*, in the following function:

$$f(x)=\frac{(2x-5)^2}{x^2+9}.$$

Proceeding in the **conventional way**, we have:

$$f'(x)=\frac{2(2x-5)(5x+18)}{(x^2+9)^2}=0\Rightarrow\begin{cases}x=\frac{5}{2}\\ x=-\frac{18}{5}\end{cases}$$

$$f''(x)=\frac{2(20x^3+33x^2-540x-99)}{(x^2+9)^3}.$$

Getting f'' has been a so ungrateful work as unnecessary; but we are still waiting for the worst: the evaluation of f'' at the roots of f'. This task would can be really deterrent, even using a calculator.

Applying the SF2D ::

$$\boxed{f'=0 \Leftrightarrow u'v-uv'=0} \quad \boxed{u=(2x-5)^2} \Rightarrow \boxed{u'=4(2x-5)} \Rightarrow \boxed{u''=8}$$

$$\boxed{\operatorname{sgn} f''=\operatorname{sgn}(u''v-uv'')} \quad \boxed{v=x^2+9} \Rightarrow \boxed{v'=2x} \Rightarrow \boxed{v''=2}$$

$$\Downarrow$$

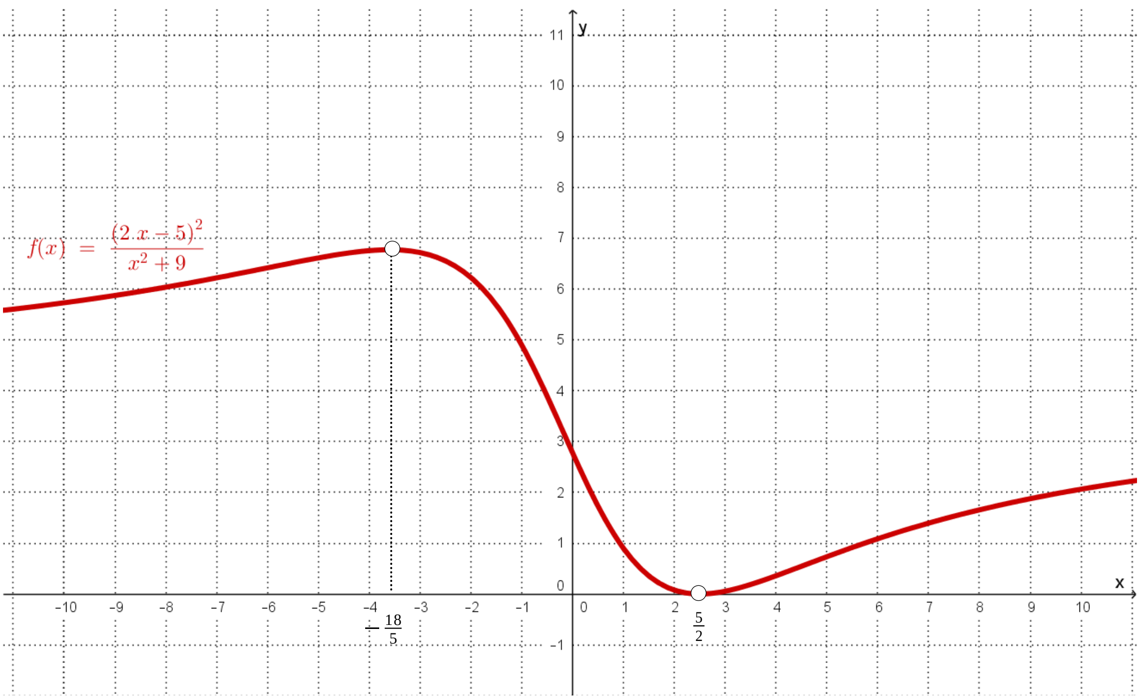
$$\boxed{f'=0 \Leftrightarrow (2x-5)(5x+18)=0 \Leftrightarrow \begin{cases} x=\frac{5}{2} \\ x=-\frac{18}{5} \end{cases}}$$

$$\boxed{\operatorname{sgn} f''=\operatorname{sgn}\left[8(x^2+9)-2(2x-5)^2\right]=\operatorname{sgn}\left[2(20x+11)\right]}$$

The **canonical form of the SF2D** (the simplest *representative sign function of the class*) is $\operatorname{sgn}[20x+11]$, as the factor 2 can be neglected since does not affect the sign.

Now, the determination of the sign of f'' at the roots of f' is immediate:

$$\boxed{\operatorname{sgn} f''_{x_0}=\operatorname{sgn}[20x+11]_{x_0}} \Rightarrow \begin{cases} \operatorname{sgn} f''\left(\frac{5}{2}\right)>0 \Rightarrow f''\left(\frac{5}{2}\right)>0 \Rightarrow \boxed{\text{min of } f \text{ at } x=\frac{5}{2}} \\ \operatorname{sgn} f''\left(-\frac{18}{5}\right)<0 \Rightarrow f''\left(-\frac{18}{5}\right)<0 \Rightarrow \boxed{\text{max of } f \text{ at } x=-\frac{18}{5}} \end{cases}$$



Exemple 2

Find all the *relative extremum*, *minimums* and *maximums*, in the following function:

$$f(x) = \frac{3}{\ln(x^4 - 3x^2 + 3)}.$$

Conventional way ::

$$f'(x) = \frac{-6x(2x^2-3)}{(x^4-3x^2+3)\left[\ln(x^4-3x+3)\right]^2} = 0 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = +\frac{\sqrt{6}}{2} \\ x_3 = -\frac{\sqrt{6}}{2} \end{cases}$$

$$f''(x) = 6\cdot\frac{(2x^6-3x^4-9x^2+9)\ln(x^4-3x+3)+4x^2(2x^2-3)^2}{(x^4-3x^2+3)^2\left[\ln(x^4-3x+3)\right]^3}$$

As can be seen, even being a simple fractional function, its second derivative can present monstrous appearance. In this case, although it is not necessary to use f'' to determine the sign of f'', since

$$\operatorname{sgn}\ f''(x) = \operatorname{sgn}\left[\frac{(2x^6-3x^4-9x^2+9)\ln(x^4-3x+3)+4x^2(2x^2-3)^2}{\ln(x^4-3x+3)}\right],$$

the calculation is still complicated.

Applying the SF2D ::

$$f(x) = \frac{u}{v} \Rightarrow \begin{cases} \boxed{u=3} \Rightarrow \boxed{u'=0} \Rightarrow \boxed{u''=0} \\ \boxed{v=\ln(x^4-3x^2+3)} \Rightarrow \boxed{v' = \frac{4x^3-6x}{x^4-3x^2+3}} \Rightarrow \boxed{v'' = -2\cdot\frac{2x^6-3x^4-9x^2+9}{(x^4-3x^2+3)^2}} \end{cases}$$

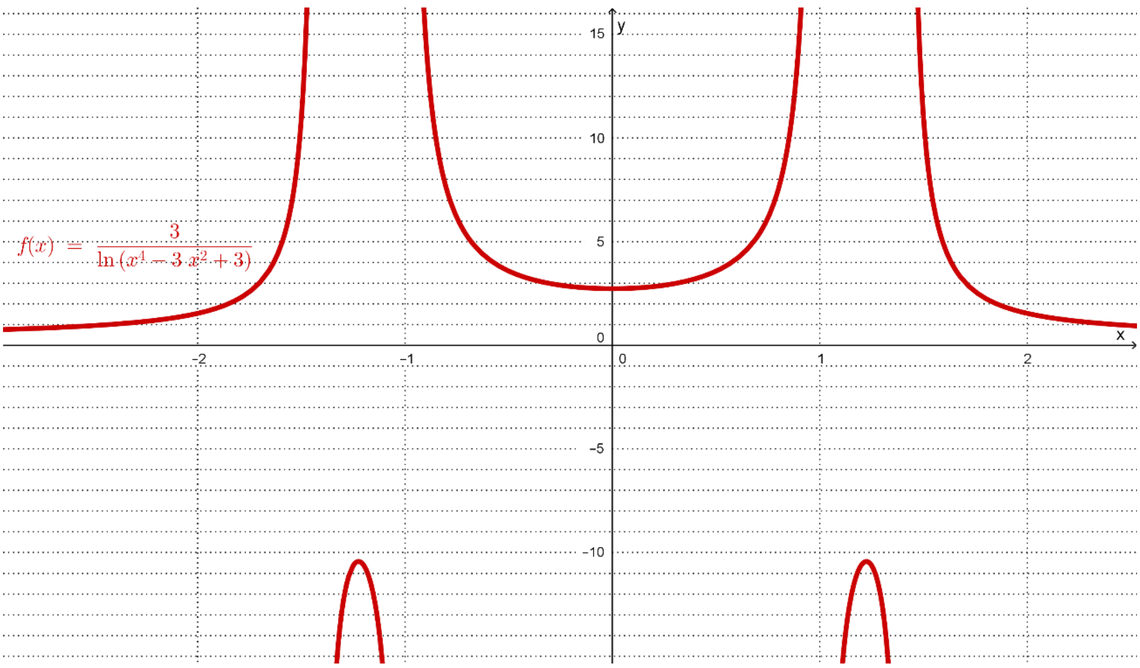
$$f'=0 \Leftrightarrow u'v-uv'=0 \Leftrightarrow 3\cdot\frac{4x^3-6x}{x^4-3x^2+3} = 0 \Leftrightarrow x(2x^2-3)=0 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = +\frac{\sqrt{6}}{2} \\ x_3 = -\frac{\sqrt{6}}{2} \end{cases}$$

$$\operatorname{sgn}\left[u''v-uv''\right]_{x_0} = \operatorname{sgn}\left[-6\cdot\frac{2x^6-3x^4-9x^2+9}{(x^4-3x^2+3)^2}\right]_{x_0} = \operatorname{sgn}\left[-2x^6+3x^4+9x^2-9\right]_{x_0}$$

$$\operatorname{sgn} f''(0) = \operatorname{sgn}\left[-2x^6+3x^4+9x^2-9\right]_{x=0} = \operatorname{sgn}\left[-9\right] < 0 \Rightarrow \boxed{\text{max at } x=0}$$

$$\operatorname{sgn} f''\left(\pm\tfrac{\sqrt{6}}{2}\right) = \operatorname{sgn}\left[x^4\left(-2x^2+3+\frac{9}{x^2}\right)-9\right]_{x=\pm\frac{\sqrt{6}}{2}} = \operatorname{sgn}\left[\frac{6^2}{2^4}\left(-2\cdot\frac{6}{2^2}+3+9\cdot\frac{2^2}{6}\right)-9\right]$$

$$= \operatorname{sgn}\left[\frac{9}{4}(-3+3+6)-9\right] = \operatorname{sgn}\left[9\left(\frac{3}{2}-1\right)\right] > 0 \Rightarrow \boxed{\text{min at } x=\pm\frac{\sqrt{6}}{2}}$$



Exemple 3 [1] ____

Find all the *local extremum*, *minimums* and *maximums*, in the following function:

$$f(x) = \frac{16}{x(4-x^2)}.$$

Conventional way ::

$$f'(x) = \frac{16(3x^2-4)}{x^2(x^2-4)^2} = 0 \Rightarrow \begin{cases} x_1 = +\frac{2}{\sqrt{3}} \\ x_2 = -\frac{2}{\sqrt{3}} \end{cases}$$

$$f''(x) = 64\cdot\frac{3x^4-6x^2+8}{x^3(4-x^2)^3}$$

$$\operatorname{sgn} f''\left(+\tfrac{2}{\sqrt{3}}\right) = \operatorname{sgn}\left[\frac{3x^4 - 6x^2 + 8}{x^3(4-x^2)}\right]_{x=+\frac{2}{\sqrt{3}}} .$$

$$\operatorname{sgn} f''\left(-\tfrac{2}{\sqrt{3}}\right) = \operatorname{sgn}\left[\frac{3x^4 - 6x^2 + 8}{x^3(4-x^2)}\right]_{x=-\frac{2}{\sqrt{3}}} .$$

whose calculation is complicated.

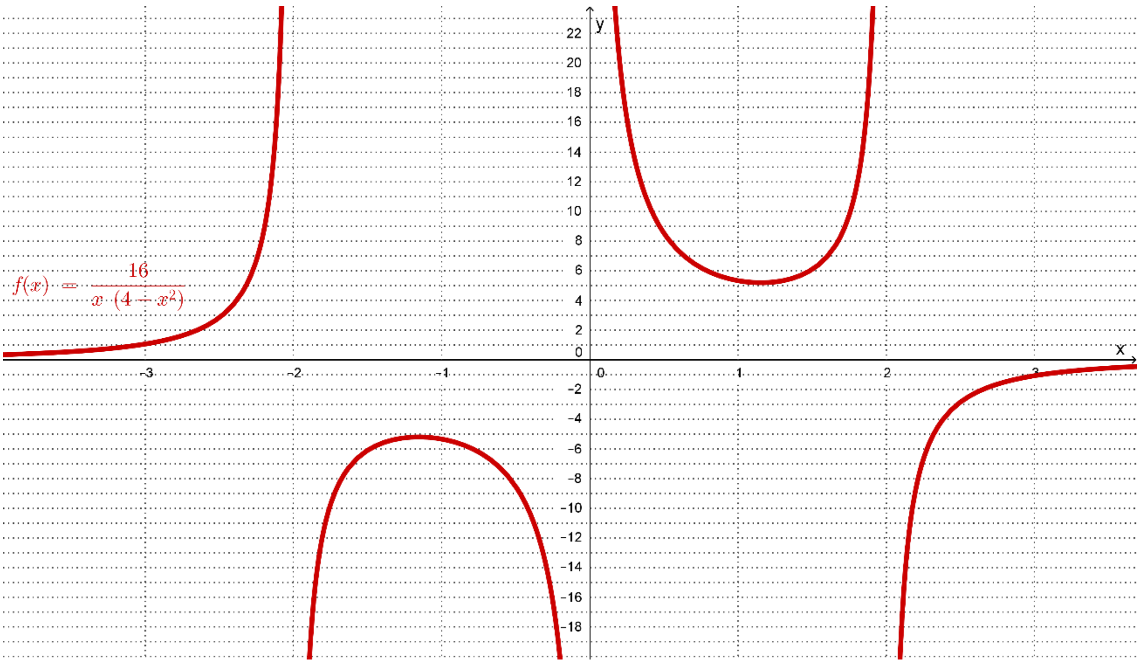
Applying the SF2D ::

$$f(x) = \frac{u}{v} \Rightarrow \begin{cases} \boxed{u = 16} \Rightarrow \boxed{u' = 0} \Rightarrow \boxed{u'' = 0} \\ \boxed{v = 4x - x^3} \Rightarrow \boxed{v' = 4 - 3x^2} \Rightarrow \boxed{v'' = -6x} \end{cases}$$

$$f' = 0 \Leftrightarrow u'v - uv' = 0 \Leftrightarrow 4 - 3x^2 = 0 \Rightarrow \begin{cases} x_1 = +\frac{2}{\sqrt{3}} \\ x_2 = -\frac{2}{\sqrt{3}} \end{cases}$$

$$\boxed{\operatorname{sgn} f'' = \operatorname{sgn}[u''v - uv'']} = \operatorname{sgn}[x] \Rightarrow \begin{cases} \operatorname{sgn} f''(x_1) = + \Rightarrow \boxed{\text{min at } x = +\frac{2}{\sqrt{3}}} \\ \operatorname{sgn} f''(x_2) = - \Rightarrow \boxed{\text{max at } x = -\frac{2}{\sqrt{3}}} \end{cases}$$

Therefore, in this case the signum of f'' is the same as that of the roots of f', so it is not necessary to perform any computations to know whether there is a maximum or a minimum at each critical points.



Example 4 [2]

A generator of E volts is connected to an inductor of L henrys, a resistor of R ohms, and a second resistor of x ohms. Heat is dissipated from the second resistor, the power ***P*** being given

$$P = \frac{E^2 x}{(2\pi L)^2 + (x+R)^2}$$

(a) Find the resistance value x_0 which makes the power as large as possible. Justify with the second derivative test.

(b) Find the *maximum* power which can be achieved by adjustment of the resistance x.

Conventional way ::

a)

$$\frac{dP}{dx} = E^2 \cdot \frac{(2\pi L)^2 + (x+R)^2 - 2x(x+R)}{\left[(2\pi L)^2 + (x+R)^2\right]^2} = -E^2 \cdot \frac{x^2 - (2\pi L)^2 - R^2}{\left[(2\pi L)^2 + (x+R)^2\right]^2}.$$

Therefore, if $E \neq 0$,

$$\frac{dP}{dx} = 0 \Leftrightarrow x^2 - (2\pi L)^2 - R^2 = 0,$$

whose only solution in the context of the problem is

$$\boxed{x_0 = \sqrt{(2\pi L)^2 + R^2}}$$

$$\begin{aligned}\frac{d^2 y}{dx^2} &= -E^2 \cdot \frac{2x\left[(2\pi L)^2 + (x+R)^2\right]^2 - 4(x+R)\left[x^2 - (2\pi L)^2 - R^2\right]\left[(2\pi L)^2 + (x+R)^2\right]}{\left[(2\pi L)^2 + (x+R)^2\right]^4} \\ &= -E^2 \cdot \frac{2x\left[(2\pi L)^2 + (x+R)^2\right] - 4(x+R)\left[x^2 - (2\pi L)^2 - R^2\right]}{\left[(2\pi L)^2 + (x+R)^2\right]^3} \\ &= \frac{2E^2\left\{x^3 - 3\left[(2\pi L)^2 + R^2\right]x - 2R\left[(2\pi L)^2 + R^2\right]\right\}}{\left[(2\pi L)^2 + (x+R)^2\right]^3}.\end{aligned}$$

The second derivative of P not only involves a considerable time of calculation and simplification, with a high risk of committing some error, but the evaluation of it for the root x_0 of the first derivative is little less than dissuasive.

$$P''\left(\sqrt{(2\pi L)^2+R^2}\right)=2E^2\cdot\frac{\left[\sqrt{(2\pi L)^2+R^2}\right]^3-3\left[(2\pi L)^2+R^2\right]\sqrt{(2\pi L)^2+R^2}-2R\left[(2\pi L)^2+R^2\right]}{\left[(2\pi L)^2+\left(\sqrt{(2\pi L)^2+R^2}+R\right)^2\right]^3}.$$

The sign of this expression is the same as

$$\left[\sqrt{(2\pi L)^2+R^2}\right]^3-3\left[(2\pi L)^2+R^2\right]\sqrt{(2\pi L)^2+R^2}-2R\left[(2\pi L)^2+R^2\right]$$

since $2E^2$ and the denominator are positives.

… and after this hard work, we have not yet gotten to know what sign takes the second derivative.

Applying the SF2D ::

a)

$$P(x)=\frac{u}{v}\Rightarrow\begin{cases}\boxed{u=E^2x}\Rightarrow\boxed{u'=E^2}\Rightarrow\boxed{u''=0}\\ \boxed{v=(2\pi L)^2+(x+R)^2}\Rightarrow\boxed{v'=2(x+R)}\Rightarrow\boxed{v''=2}\end{cases}$$

$$P'=0\Leftrightarrow u'v-uv'=0\Leftrightarrow E^2\left[(2\pi L)^2+(x+R)^2\right]-E^2x\cdot 2(x+R)\Leftrightarrow x_0=\pm\sqrt{(2\pi L)^2+R^2}$$

$$\boxed{\operatorname{sgn} P''=\operatorname{sgn}\left[u''v-uv''\right]}=\operatorname{sgn}\left[-2E^2x\right]=-\operatorname{sgn}[x]\Rightarrow\operatorname{sgn} P''\left(\sqrt{(2\pi L)^2+R^2}\right)=\boxed{-}$$

Therefore, P reaches a *maximum* at $x_0=\sqrt{(2\pi L)^2+R^2}$.

b)

The value of the maximum of P is

$$P\left(\sqrt{(2\pi L)^2+R^2}\right)=\frac{E^2\sqrt{(2\pi L)^2+R^2}}{(2\pi L)^2+\left[\sqrt{(2\pi L)^2+R^2}+R\right]^2}.$$

Example 5

Determine the relative *maximum* and *minimum* of the function $f(x)=\dfrac{3+x-2x^2}{3+2x^2}$.

Applying the SF2D ::

$$f(x)=\frac{u}{v}\Rightarrow\begin{cases}\boxed{u=3+x-2x^2}\Rightarrow\boxed{u'=1-4x}\Rightarrow\boxed{u''=-4}\\ \boxed{v=3+2x^2}\Rightarrow\boxed{v'=4x}\Rightarrow\boxed{v''=4}\end{cases}$$

$$f'=0\Leftrightarrow u'v-uv'=0\Leftrightarrow(1-4x)\left(3+2x^2\right)-\left(3+x-2x^2\right)\cdot 4x=0$$

$$\Leftrightarrow 2x^2+24x-3=0\Rightarrow\begin{cases}x_1=-\frac{5\sqrt{6}}{2}-6\\ x_2=+\frac{5\sqrt{6}}{2}-6\end{cases}$$

$$\boxed{\operatorname{sgn}\, f''=\operatorname{sgn}\left[u''v-uv''\right]}=-\operatorname{sgn}\left[x+6\right]=-\operatorname{sgn}\left[\mp\tfrac{5\sqrt{6}}{2}\right]\Rightarrow\begin{cases}\operatorname{sgn}\, f''(x_1)=\boxed{+}\\ \operatorname{sgn}\, f''(x_2)=\boxed{-}\end{cases}$$

Hence, $f(x)$ has a *minimum* at $x=-\frac{5\sqrt{6}}{2}-6$ of value $f\left(-\frac{5\sqrt{6}}{2}-6\right)=-\frac{5\sqrt{6}}{12}$ and a *maximum* at $x=\frac{5\sqrt{6}}{2}-6$ of value $f\left(\frac{5\sqrt{6}}{2}-6\right)=\frac{5\sqrt{6}}{12}$.

References

[**1**] Demidovich (Editor). Problems in Mathematical Analysis.
MIR Publishers – Moscow. Problem 835.

[**2**] Jerrold Marsden, Alan Weinstein. Calculus I. 2nd Edition.
Chapter 3: Graphing and Maximum-Minimum Problems.
Exercises for Section 3.3: Page 162. Exercise 40.