Developing a Phenomenon for Analytic Number Theory

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Abstract

A phenomenon is described for analytic number theory. The purpose is to coordinate number theory and to give it a specific goal of modeling the phenomenon.

Introduction

Number theory is a strange branch of mathematics. On the one hand it is the most concrete branch. What could be more concrete than the natural numbers. On the other hand, a glance at a book like [1] shows that it is not an easy subject. Indeed, studying such a book one senses a lack of a coordinating concept that embraces much of the theory. This idea can be best brought about by comparison with calculus. There the integral calculus has one central concept that guides the evolution of the theory: the area under a general continuous curve. One can unpack the ideas in an orderly way and build the subject up. What is the equivalent for number theory. Perhaps it is the study of natural numbers. But what about the natural numbers in particular? The prime numbers. But what about the prime numbers in particular? There are so many things one can say about them that it is hard to see an over arching concept. The distribution of the prime numbers captures a lot of the effort of number theory and within this rests the Riemann $\zeta(s)$ function and the Riemann hypotheses about its zeros. The Riemann hypotheses is the number one problem in abstract mathematics and efforts to resolve it consumes a lot of energy in number theory. But in anyway is the function given by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}$$

somehow affiliated with a concept as simple as the area under a curve or the slope of the derivative of a tangent line? Is there some phenomenon that can be ascribed to this function that makes its importance more natural?

This lack of a coordinating phenomenon for number theory is a genuine problem. In this article a phenomenon or behavior is evolved. The goal can be stated. We wish to describe a phenomenon that coordinates much of the subject matter of Riemann's famous article of 1859 [2], that motivates it, and helps in its understanding. We need a picture in our minds. If our picture works, many topics in [1] will have a coordinating mechanism and number theory in general will have an answer to an important question: what is number theory about?

0	$[\underline{0}1]_2$
1	$[0\underline{1}]_2[\underline{0}, \underline{1}, 2]_3$
2	$[\underline{0}, 1]_2^1[0, 1, \underline{2}]_3$
3	$[0, \underline{1}]_2^1[\underline{0}, 1, 2]_3^1$
4	$[\underline{0}, 1]_2^2[0, \underline{1}, 2]_3^1$
5	$[0, \underline{1}]_2^2[0, 1, \underline{2}]_3^1[\underline{0}, 1, 2, 3, 4]_5^1$
6	$[\underline{0}, 1]_2^3[\underline{0}, 1, 2]_3^2[0, \underline{1}, 2, 3, 4]_5^1$
7	$[0, \underline{1}]_2^3[0, \underline{1}, 2]_3^2[0, 1, \underline{2}, 3, 4]_5^1[\underline{0}, 1, 2, 3, 4, 5, 6]_7^1$

Table 1: When non-zeros occur in each circle, given by bracketed numbers, a new circle is added. The subscript gives primes and the superscript gives powers of the prime. The underlined numbers gives the tick mark resting on the x-axis.

The phenomenon: all odometers

Circles turn. The quest is to make all circles that turn in lock step and generate numbers. The first circle has two tick marks: one labeled zero the other 1. If the 1 rests on the $x - axis$, another circle is generated. It has all the previous tick marks plus one. So $0, 1, 2$. If both circles have non-zero tick marks on the x-axis, another circle is added. Each circle has a number inside that indicates the number of its revolutions and a tick mark on the x-axis that indicates its current position. Using these two the number of *days* since the beginning of the phenomenon can be calculated redundantly on each circle.

Modulus base

This picture can be thought of as allowing a different number base. Instead of just a base 10 system, we now have base all primes within a moduli idea. The numbers in order are given in Table 1.

The prime number theorem

The prime number theorem has a picture. What is the ratio of the growth of new circles to the number of days? Can one predict when a new circle will be attached to our string of circles. What is the dependency relationship of all previous circles to new circles?

Similarity to $\zeta(s)$

The growth of this *caterpillar* or *tree* is complex. Its states of mind are constantly evolving, but also reflect periodic returns to previous states. Like an infinite series, its current state is dependent on all its segments, the equivalent of an infinite series terms. There are three elements in each segment: a power, a base, and a current position. This parallels $1/n^s$ where $s = a + bi$: three things. Also, we have a periodically occurring non-zero product of the tick marks of each circle. This is accompanied by a new circle being added that has a 0 value and thus defines the position of a root.

Relation to complex analysis

Complex analysis might be called circle analysis and the phenomenon we have been describing consists of a lot of circles. Complex analysis allows for the use of differentiation and integration, series summations and all kinds of applications. Is it the best way to model our caterpillar or tree or human being? Probably not.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] H. M. Edwards, *Riemann's Zeta Function*, Dover, 1974.