

Question 423: On Pi

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abstract

This note presents some series for π^2 .

Introduction. The numbers π and π^2 .

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad , \text{Leibniz-Gregory} \quad (1)$$

$$\pi^2 = 6 \sum_{n=1}^{\infty} \frac{1}{n^2} \quad , \text{Euler} \quad (2)$$

Some series for π^2 .

1. For $a = \sqrt{3 - \frac{1}{\sqrt{3 - \frac{1}{\sqrt{3 - \dots}}}}}$ we have

$$\frac{\pi^2}{54} = \frac{1}{2} \left(\ln \left(1 + \frac{1}{a} \right) \right)^2 + \sum_{n=1}^{\infty} \frac{(a^2 + a^3)^{-n}}{n^2 \binom{3n}{n}} \quad (3)$$

Remark 1: $a = \frac{1}{2} \left(-4 + 4i\sqrt{3} \right)^{1/3} + 2 \left(-4 + 4i\sqrt{3} \right)^{-1/3}$.

2. For $a = -\sqrt[3]{1 + 3\sqrt{1 + 3\sqrt{1 + \dots}}}$ we have

$$\frac{\pi^2}{54} = \frac{1}{2} \left(\ln \left(1 + \frac{1}{a} \right) \right)^2 + \sum_{n=1}^{\infty} \frac{(a^2 + a^3)^{-n}}{n^2 \binom{3n}{n}} \quad (4)$$

Remark 2:

$$a = -\frac{1}{4}(-4+4i\sqrt{3})^{1/3} - (-4+4i\sqrt{3})^{-1/3} + \frac{i\sqrt{3}}{2} \left(\frac{1}{2}(-4+4i\sqrt{3})^{1/3} - 2(-4+4i\sqrt{3})^{-1/3} \right)$$

3. For $a = \sqrt{6+3\sqrt{3} - \frac{2+\sqrt{3}}{\sqrt{6+3\sqrt{3} - \frac{2+\sqrt{3}}{\sqrt{6+3\sqrt{3} - \dots}}}}}$ we have

$$\frac{\pi^2}{216} = \frac{1}{2} \left(\ln \left(1 + \frac{1}{a} \right) \right)^2 + \sum_{n=1}^{\infty} \frac{(a^2 + a^3)^{-n}}{n^2 \binom{3n}{n}} \quad (5)$$

Remark 3:

$$a = \frac{1}{2}(-8+28i-4\sqrt{3}+16i\sqrt{3})^{1/3} + 2(2+\sqrt{3})(-8+28i-4\sqrt{3}+16i\sqrt{3})^{-1/3}$$

4. For $a = -\sqrt[3]{2+\sqrt{3}+3(2+\sqrt{3})} \sqrt[3]{2+\sqrt{3}+3(2+\sqrt{3})} \sqrt[3]{2+\sqrt{3}+3(2+\sqrt{3})} \sqrt[3]{2+\sqrt{3}+\dots}$ we have

$$\frac{\pi^2}{216} = \frac{1}{2} \left(\ln \left(1 + \frac{1}{a} \right) \right)^2 + \sum_{n=1}^{\infty} \frac{(a^2 + a^3)^{-n}}{n^2 \binom{3n}{n}} \quad (6)$$

Remark 4:

$$a = -\frac{1}{4}(-8+28i-4\sqrt{3}+16i\sqrt{3})^{1/3} - (2+\sqrt{3})(-8+28i-4\sqrt{3}+16i\sqrt{3})^{-1/3} + \frac{i\sqrt{3}}{2} \left(\frac{1}{2}(-8+28i-4\sqrt{3}+16i\sqrt{3})^{1/3} - 2(2+\sqrt{3})(-8+28i-4\sqrt{3}+16i\sqrt{3})^{-1/3} \right) \quad (7)$$

References

1. Berndt, B.C.: Ramanujan's Notebooks, Part IV. New York, Springer-Verlag, 1994.
2. Gourdon, X. and Sebah, P. : Collection of Series for π .
<http://numbers.computation.free.fr/Constants/Pi/piSeries.html>.