THE EULER – MASCHERONI CONSTANT AND ITS APPLICATION IN PHYSICAL RESEARCH

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ABSTRACT: *The paper covers applicability areas of the Euler – Mascheroni constant in General Theory of Relativity and Quantum Theory.*

1: FUNDAMENTAL MATHEMATICAL CONSTANTS

In mathematics, natural sciences and technology, the term „constant“ represents a certain known fixed number or invariable quantity. It can be said, in general, that fundamental mathematical and physical constants are numbers defining the very essence of our Universe. If any of the constants just a little changed, the life in the Universe, in the form we know, would not evolve. As for pure mathematical constants, they are always dimensionless and their impact can be identified not just in mathematics but also in physics and related disciplines.

The most fundamental mathematical constants are Ludolph's number π (widely applied in physics and technology), Euler's number *e* (also applied in physics and technology), golden section constant φ (used in technology, biology, architecture and art) and finally the Euler – Mascheroni constant γ, (up to now applied mainly in mathematics).

This constant is defined as the limiting difference between the harmonic series and the natural logarithm (eq. 1) and used predominantly in the theory of numbers.

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \ln n\right) \qquad (1)$$

Its value is approximately γ = 0.57721 56649 01532 86060 65120 90082 40243 10421 59335.

Little is known about the properties of the number itself. It is not known if gamma is algebraic or transcendental. It is not even known if gamma is irrational like other important mathematical constants such as π and *e*.

There is no need to introduce Leonard Euler in more detail. He is generally considered as the most prolific and important mathematician in the history of mankind. His achievements promptly found their application in physical research. The first steps leading to deducing the Euler – Mascheroni constant did Euler already in the year 1734.

Lorenzo Mascheroni is less known. He was born in Bergamo in 1750. Officially functioning as a priest, he dedicated his whole life to mathematics and poetry. In 1786 he became professor of geometry at Pavia University. In his *Adnotationes ad calculum integrale Euleri* (1790) he published a calculation of what is now known as the Euler–Mascheroni constant, γ and calculated its value. Mascheroni died untimely in 1801.

2: PHYSICAL SIGNIFICANCE OF THE EULER – MASCHERONI CONSTANT

Photosphere represents the lowest orbit around a Schwarzschild black hole. This area is located at the distance equalling 3/2 of its gravitational radius. It can be said that photosphere is the most significant region of black holes in the Universe. For Lorentz contraction factor β on the photosphere it must hold

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 0.57735 \tag{2}$$

The value in eq.2 is suspiciously close to that of the Euler – Mascheroni constant γ. Anyway, it is an interesting relationship of GTR and mathematics (theory of numbers). For the photosphere it holds β = γ, i.e. Lorentz contraction on the black hole photosphere is almost equal to the Euler – Mascheroni constant. It should be bear in mind that Lorentz factor has a constant value for each black hole photosphere.

The following application is also of GTR field. Davis et all. In their paper published in 1971 [1] calculated that a body falling into a rapidly rotating Kerr black hole can lose up to 0.44 of it mass-energy due to emission of gravitational waves. In a limiting case:

$$E_{\mathrm{emit}} = \left(0.44\, m c^2\right) e^{-2l} \tag{3}$$

The spectrum of the outgoing radiation is the superposition of a series of overlapping peaks, each peak corresponding to a certain multipole order *l*. At the Kerr black hole horizon, *l* approaching zero.

More precise calculations documented that the amount of emitted energy must lie closely over 0.42 m [2] (Cheng, 2010, p 169). Even more precisely value of maximum mass-energy loss due to gravitational waves emission – 0.423 *m* – is mentioned in the book [3] (Ullmann 1986, p 207).

It means that the minimum amount of matter which is able to pass through the Kerr black hole horizon is 0.577 *m*, which is equal to the value of Euler –Mascheroni constant (dividing both quantities by the mass *m*, dimensionless values are obtained).

There is a further interesting area of Euler – Mascheroni constant functioning. In 1968, G. Veneziano [4] published an important paper expressing the scattering amplitude $A_{k(n)}$ for strong interactions via Euler gamma-function Γ:

$$A_{k(n)} = \frac{\Gamma\left(-1 + \frac{1}{2}(k_1 + k_2)^2\right)\Gamma\left(-1 + \frac{1}{2}(k_2 + k_3)^2\right)}{\Gamma\left(-2 + \frac{1}{2}\left((k_1 + k_2)^2 + (k_2 + k_3)^2\right)\right)} \tag{4}$$

where $k(n)$ is the vector (such as a four vector) referring to the momentum of the *n*th- particle, Γ is the gamma function.

It is worth realizing that the scattering amplitude was obtained from data accumulated by accelerators of elementary particles constructed during the second half of the 20th century while Euler gamma-function was invented in the 18th century.

What should be taken into account is that the Euler – Mascheroni constant participates significantly in the derivation of Euler gamma-function and has an indirect impact on quantum theory, at least in the description of strong interactions. For all areas with $x > 0$, gamma-function is defined as follows:

$$\Gamma_{(x)} = \int_0^\infty t^{x-1} e^{-t}\, dt \tag{5}$$

where Euler gamma function may be defined through the Euler – Mascheroni constant.

A lesser known derivation by Weierstrass is as follows:

$$\Gamma_{(X)} = \frac{e^{-x\gamma}}{x} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right)^{-1} e^{\frac{x}{n}} \qquad (6)$$

It is obvious that the constant γ participates indirectly in the description of strong interactions in quantum theory.

3: CONCLUSIONS

The present paper manifests the significance of the Euler – Mascheroni constant in physical research.

Directly, this constant is of importance also in theory of weak interactions [5]. Its significance for quantum theory is obvious; its role in GTR deserves a rationalization, which should not, however, be complicated. In both above mentioned GTR-related cases potential gravitational field comes into play. Potential energy is proportional to 1/r. This energy is, of course, quantized and summing n levels (i.e. integrating the relation 1/r) leads to a relation composed of ln *n* plus an integration constant. In limiting cases (such as photosphere of Schwarzschild black holes or a body falling into rotating Kerr black holes) this integration constant equals just the Euler – Mascheroni constant. This is a gist of the issue; the rest follows from the definition of the Euler – Mascheroni constant. Since electric and magnetic field are potential ones, in limiting – relativistic cases Euler – Mascheroni constant must appear too.

We can conclude the paper by a statement declaring that there is no a pure mathematical constant without its involvement in physical reality. It is in accordance with a known and ancient truth that mathematics is the best tool for exploring and understanding the real World. In particular, the Euler – Mascheroni constant plays a considerable role in description of all four fundamental physical interactions.

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