Euler's Formula for $\zeta(2n)$

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Abstract

In this article we derive a formula for zeta(2) and zeta(2n).

Introduction

In this paper we derive from scratch

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^{2p}} = (-1)^{p-1} \frac{2^{2p-1}}{(2p!)} B_{2p} \pi^{2p} \tag{2}$$

where B_{2p} are the Bernoulli numbers. Both are attributed to Euler [5]. Our treatment is close to that of Eymard [5] and Knopp [6].

The justification for this article is that these texts seem rather unfocused: both authors develop other material sporadically as they prove this result. We wish here to isolate the result and just develop the mathematics necessary for an understanding of this formula. There are many treatments of these result [1]. Here we wish to motivate known, easiest proofs.

Taylor series for sin

At some point someone determined that there is a relationship between nth order derivatives and coefficients of polynomials. This can be anticipated by the easiest observation; if $f(x) = ax^2 + bx + c$, the coefficient of x^0

is given by the zero order derivative evaluated at $x = 0$: $f(0) = c$. As we take ever increasing derivatives the constant of the derivative becomes a new coefficient. So, $f'(x) = 2ax + b$ and $f'(0) = b$. When we repeat this pattern, we notice that a factorial is building by way of the formula $(cx^n)' = cnx^{n-1}$. Factorials need to be divided out. Here it is for the quadratic: $f''(x) = 2a$ gives

$$\frac{f^{(2)}(0)}{2!} = a.$$

In general, for a function $f(x)$ with derivatives

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This is termed the Taylor (actually Maclaurin) series expansion of $f(x)$ about the point 0. It is a Maclaurin expansion when the point used, the center is 0.

The power of these power series (an infinite series with x^n) is that they allow for approximations to an arbitrary precision. The transcendental functions in particular are in need of such. What after all can we say about sin(1.2387) and the like? We only have exact evaluations possible for this trigonometric function when the argument is a fraction with π: $\pi/2$, $\pi/3$, etc.. If we have a power series for sin we can evaluate any x value.

We know the derivative of sin is cos and taking nth derivatives is not difficult; the functions just cycle around:

$$\sin' = \cos; \cos' = -\sin; (-\sin)' = -\cos; \text{and } (-\cos)' = \sin.$$

As $\pm \sin(0) = 0$, $\cos(0) = 1$, and $-\cos(0) = -1$, we can easily generate a Maclaurin series for sin:

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{3}$$

The odd $2n + 1$ follows from the even terms, thanks to $\pm \sin(0) = 0$, vanishing.

Properties of polynomials

Power series are like an infinite polynomial and polynomials have coefficients that are related to their roots – what x values make them 0. So, for example, expanding $(x-a)(x-b)(x-c)$ gives

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \tag{4}$$

We can sense that in general the constant will be the sum of the roots taken all at a time, hence one term, and the coefficient of x will be the sum of the roots taken (or multiplied) $n-1$ at a time. We are obtaining sums that remind us of the goal of determining the sum in (1). In comparing this sum with the ones in (4) and the powers of x in (3), we have a puzzle.

Puzzle of (1)

We'd like to get the polynomial of $\sin x$ to have a x term and a 1 constant. If this were true then, using (4) as a model,

$$\frac{x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc}{abc}$$

gives a coefficient of x equal to $1/c + 1/b + 1/a$, a sum of the reciprocals of the roots. The roots of sin are $\pm n\pi$.

First

$$\sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

gives

$$\sin x = x(1 - x^2/3! + x^4/5! - \dots)$$

which gives

$$\frac{\sin x}{x} = (1 - x^2/3! + x^4/5! - \dots).$$

Letting $y = x^2$, we get a infinite polynomial which we set to 0:

$$0 = 1 - y/3! + \dots.$$

This has a constant of 1, so the sum of the roots is $1/3! = 1/6$ and the roots are given by the squares of $\sin x$'s roots (just using $y = x^2$). Thus

$$\frac{1}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2}$$

and this implies (1).

Puzzle of (2)

We start from the observation that there are three ways (at least) to generate a power series: using a division, like $1/(x-1)$; using Taylor series expansions, as above; and using a Fourier expansion. As (2) involves all

$$\zeta(2p) = \sum_{k=1}^{\infty} \frac{1}{k^{2p}}$$

and as all power series expansions have both a sequence of coefficients and powers of z,

$$\sum_{k=1}^{\infty} a_k x^k,$$

if we can use two different means of obtaining a power series expansion for a given same function, we can hope to equate the coefficients and arrive at (2). We would need one set of coefficients to be $\zeta(2p)$. We can use a division to generate that situation.

Theorem 1.

$$\sum_{n=1}^{\infty} \frac{x^2}{x^2-n^2} = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{x^{2p}}{n^{2p}} = \sum_{p=1}^{\infty} \zeta(2p) x^{2p} \tag{5}$$

Proof. First,

$$\frac{x^2}{x^2-n^2}\frac{1/n^2}{1/n^2} = \frac{x^2}{n^2}\left(\frac{1}{x^2/n^2-1}\right).$$

Letting $m = x^2/n^2$, we have

$$\frac{m}{m-1}\frac{1/m}{1/m} = \frac{1}{1-1/m}.$$

This last is the formula for a geometric series:

$$\frac{x^2}{x^2-n^2} = \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k}}.$$

Substituting,

$$\sum_{n=1}^{\infty} \frac{x^2}{x^2-n^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k}}$$

and transposing summation gives (5). $\square$

Next is the puzzle of finding a function that is amenable to two methods of power series generation one of which gives (5). This is not as far fetched as it seems. Consider that

$$\frac{x}{x+n} + \frac{x}{x-n} = \frac{2x^2}{x^2-n^2}$$

and

$$x\int \cos[(x+n)t]\ \mathrm{dt} = \frac{x}{x+n}\sin[(x+n)t]$$
$$x\int \cos[(x-n)t]\ \mathrm{dt} = \frac{x}{x-n}\sin[(x-n)t].$$

Using a product to sum trigonometric identity and given that the integration limits when computing Fourier coefficients have an upper limit of π, there is some hope that say the $\cos(nt)$ Fourier expansion of $\cos xt$ might yield the desired function with power series (5).

Theorem 2. *The* $\cos(nt)$ *Fourier expansion of* $\cos(xt)$ *yields*

$$\pi x \cot \pi x = 1 - 2\sum_{k=1}^{\infty} \frac{x^2}{x^2-n^2}.$$

Proof. By definition of a Fourier series (and half series), the Fourier half series expansion of $\cos(xt)$ using $\cos(nt)$ is

$$\cos(xt) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(xt)\cos(nt)\ \mathrm{dt}. \tag{6}$$

Using a trigonometric identity (product to sum) and cos is an even function, this is

$$\begin{aligned}
&\frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos[(x+n)t] + \cos[(x-n)t]\right)\\
&= \frac{1}{\pi}\left(\frac{\sin[(x+n)\pi]}{x+n} + \frac{\sin[(x-n)\pi]}{x-n}\right)\\
&= \frac{\sin(\pi x)\cos(nx)}{\pi}\left(\frac{1}{x+n} + \frac{1}{x-n}\right)\\
&= (-1)^n \frac{2x\sin(\pi x)}{\pi}\frac{1}{x^2-n^2}.
\end{aligned}$$

So we have

$$\cos(xt) = \frac{\sin(\pi x)}{\pi x} + \frac{2x \sin \pi x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^n \cos(nt)}{x^2 - n^2} \tag{7}$$

Setting $t = \pi$, (7) becomes

$$\pi x \cot \pi x = 1 + 2 \sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2}.$$

□

Corollary 1.

$$\pi x \cot \pi x = 1 - 2 \sum_{k=1}^{\infty} \frac{x^2}{x^2 - n^2} = 1 - 2 \sum_{p=1}^{\infty} \zeta(2p) x^{2p}.$$

Proof. This follows from Theorem 1. □

Now if we just have another way to obtain a power series expansion for $z \cot z$, we could equate the coefficients for each. Why not use Taylor? We have already an expansion for sin and the one for cos can be derived easily. We could divide the two series. This is what Larson in his calculus text [7] does to arrive at the beginnings of a power series expansion for tan. He doesn't follow through and divide 1 by this tan result: $1/\tan = \cot$.

The catch with this approach is that we are not getting a closed form of the coefficients whereby we could compute them at will. Of course the even more natural approach is to take derivatives of $z \cot z$, per Taylor's theorem, and develop the series this wise. The catch is repeated differentiation of cot (and tan) is not nearly so neat and nice as taking such for sin and cos. It is fast a mess.

Can we look it up in a reference book? The following line occurs in Spiegel [8]:

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!}.$$

Multiplying this by x gives

$$x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \cdots - \frac{2^{2n} B_n x^{2n}}{(2n)!}$$

and substituting πx for x gives

$$\pi x \cot \pi x = 1 - \frac{(\pi x)^2}{3} - \frac{(\pi x)^4}{45} - \frac{2(\pi x)^6}{945} - \cdots - \frac{2^{2n} B_n (\pi x)^{2n}}{(2n)!} - \ldots$$

Equating

$$\pi x \cot \pi x = 1 - \sum_{k=1}^{\infty} \frac{2^{2n} B_n (\pi x)^{2n}}{(2n)!} = 1 - 2 \sum_{p=1}^{\infty} \zeta(2p) x^{2p}$$

The same reference book gives $B_1 = 1/6$, so what is $\zeta(2)$? Well

$$\frac{4(1/6)(\pi)^2}{2!} = 2\zeta(2)$$

implies $\zeta(2) = \pi^2/6$.

Bernoulli numbers

Typically calculus textbooks do not include power series expansions for $\tan x$ and $\cot x$. This seems to disallow a tabular and systematic understanding of the subject matter. Instead it favors uninformed understandings of mathematics that just drops natural questions leaving the student thinking that mathematics consists of a sequence of problems without any particular rhyme or reason. Of course the writers of calculus textbooks have a good reason not to include such series; they are difficult. But why can't this be stated? In this section we will derive the series for cot.

1. Let
$$\frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2 z^2}{2!} + \frac{B_3 z^3}{3!} + \ldots \tag{8}$$
   (a) Show that the numbers B_n, called the Bernoulli numbers, satisfy the recursion formula $(B + 1)^n = B^n$ where B^k is formally replaced by B_k after expanding.
   (b) Using (a) or otherwise determine $B_1, \ldots, B_6$.

Answer: For (a) multiply RHS by 1/LHS of (8) and get 1. We can then infer coefficient properties. To wit:

$$\frac{e^z - 1}{z}\left(1 + B_1 z + \frac{B_2}{2!} z^2 + \frac{B_3 z^3}{3!} + \ldots\right) = 1.$$

Now

$$e^z - 1 = \left(\frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^3}{3!} + \dots\right) - 1$$
$$= z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

and divided by z this is

$$= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots.$$

So

$$\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)\left(\frac{B_0}{0!} + \frac{B_1}{1!}z + \frac{B_2}{2!}z^2 + \frac{B_3 z^3}{3!} + \dots\right) = 1$$

and this is really (z powers)(z powers) $=$ 1+ 0 times z powers which implies that the coefficients of the z powers on the right hand side are 0, except for the first one. Speaking of which, equating constants

$$B_0 = 1.$$

Equating coefficients of z,

$$\frac{1}{2!}\frac{B_0}{0!} + \frac{1}{1!}\frac{B_1}{1!} = 0.$$

Equating coefficients of z^2,

$$\frac{1}{3!}\frac{B_0}{0!} + \frac{1}{2!}\frac{B_1}{1!} + \frac{1}{1!}\frac{B_2}{2!} = 0. \tag{9}$$

We see something close to

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

emerging. If we multiply (9) by 3!, we get

$$\binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 = 0.$$

Now this is a row of Pascal's triangle, absent the last $\binom{3}{3}B_3$. Temporarily confusing super with subscripts, we have $(B+1)^3$ is too much by $\binom{3}{3}(=1)B_3$, so $(B+1)^3 - B^3 = 0$ and $(B+1)^3 = B^3$. In general, $(B+1)^n = B^n$.

For (b), we solve the recursive equations one after the other starting with $(B+1)^2 = B^2$: $B^2 + 2B + 1 = B^2$ implies $2B + 1 = 0$ or $B_1 = -1/2$. Next, $(B+1)^3 = B^3$ implies $3B^2 + 3B + 1 = 0$ and substituting $B = -1/2$, this gives $3B^2 - 3/2 + 1 = 0$ and $B_2 = 1/6$. And so on for B_3: $(B+1)^4 = B^4$ implies $4B^3 + 6(1/6) + 4(-1/2) + 1 = 0$ for $B_3 = 0$. And for B_4: $(B+1)^5 = B^5$ implies $B_4 = -1/30$. $B_5 = 0$ and $B_6 = 1/42$.

2. (a) Prove that
$$\frac{z}{e^z - 1} = \frac{z}{2}\left(\coth\frac{z}{2} - 1\right). \tag{10}$$
(b) Use problem 163 and part (a) to show that $B_{2k+1} = 0$, if $k = 1, 2, 3, \ldots$.

Answer For (a), we use a hyperbolic identity:

$$\text{RHS} = \frac{z}{2}\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} - 1\right) \tag{11}$$
$$= \frac{z}{2}\left(\frac{\cancel{e^{z/2}} + e^{-z/2} - \cancel{e^{z/2}} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right) \tag{12}$$
$$= \frac{z}{\cancel{2}}\left(\frac{\cancel{2}e^{-z/2}}{e^{z/2} - e^{-z/2}}\right)\frac{e^{z/2}}{e^{z/2}} \tag{13}$$
$$= \frac{z}{e^z - 1} = \text{LHS}. \tag{14}$$

For (b), we need to show $z/(e^z - 1) + z/2$ is an even function using (a). It is important to note that B_1 has an odd subscript with $k = 0$. We are to show all other odd subscripts are 0. We have (10) and

$$\frac{z}{e^z - 1} = \frac{B_0}{0!}z^0 + \frac{B_1}{1!}z^1 + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \ldots.$$

Using these two and knowing $B_1 = -1/2$,

$$\frac{z}{e^z - 1} + \frac{1}{2}z = \frac{z}{2}\coth\frac{z}{2} = \frac{B_0}{0!}z^0 + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \ldots.$$

Now there is no odd that we know is not zero. Let $f(z) = (z/2)\coth(z/2)$, we will show that $f(-z) = f(z)$ using a hyperbolic identity:

$$f(-z) = \left(-\frac{z}{2}\right)\frac{e^{-z/2} + e^{z/2}}{e^{-z/2} - e^{z/2}}$$
$$= \left(\frac{z}{2}\right)\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = f(z).$$

So $f(z)$ is an even function.

For power series such that $\sum a_n z^n = \sum a_n(-z)^n$, it is clear that $a_n z^n = a_n(-z)^n = 0$. This implies that $a_n(-1)^n = a_n$, which can only be true if n is even. This implies that $a_{2n+1} = 0$ for power series representations, $\sum a_n z^n$, of even functions.

3. Derive the series expansions:

   (a) $\coth z = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \cdots + \frac{B_{2n}(2z)^{2n}}{(2n)!z} + \ldots, (|z| < \pi)$

   (b) $\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \ldots (-1)^n \frac{B_{2n}(2z)^{2n}}{(2n)!z} + \ldots, (|z| < \pi)$

   (c) $\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots (-1)^{n-1} \frac{2(2^{2n}-1)B_{2n}(2z)^{2n-1}}{(2n)!} + \ldots, (|z| < \frac{\pi}{2})$

   (d) $\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \ldots (-1)^{n-1} \frac{2(2^{2n-1}-1)B_{2n}(2z)^{2n-1}}{(2n)!} + \ldots, (|z| < \pi)$

Answer For (a), we know

$$\frac{z}{e^z - 1} = \frac{z}{2}\left(\coth\frac{z}{2} - 1\right)$$

and this implies

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\coth\frac{z}{2}.$$

The z term cancels, for

$$1 + \cancel{B_1 z} + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots + \cancel{\frac{z}{2}} = \frac{z}{2}\coth\frac{z}{2}.$$

Substitute $2z$ for z:

$$1 + \frac{B_2}{2!}(2z)^2 + \frac{B_3}{3!}(2z)^3 + \cdots + \frac{B_n}{n!}(2z)^n + \cdots = z\coth(z).$$

Dividing both by z and knowing $B_{2k+1} = 0$, we get

$$\frac{1}{z} + \frac{B_2}{2!}4z + \frac{B_4}{4!}16z^3 + \cdots + \frac{B_{2n}}{(2n)!z}(2z)^{2n} + \cdots = z\coth(z).$$

As $B_2 = 1/6$ and $B_4 = -1/30$, we get a confirmation of the terms in (a).

For (b), we use the hyperbolic identity $\coth(zi) = \cot(z)$ and just substitute into the above series for coth:

$$\coth zi = \sum_{n=0}^{\infty} \frac{B_{2n}(2iz)^n}{(2n)!z} = \sum_{n=0}^{\infty}(-1)^n \frac{B_{2n}(2z)^n}{(2n)!z} = \cot(z).$$

We note $i^0 = 1$; $i^2 = -1 = (-1)^1$; $i^4 = 1 = (-1)^2$; and $i^6 = -1 = (-1)^3$ shows the accuracy of the $(-1)^n$ factor in the above.

For (c), we first establish the trigonometric identity $\tan z = \cot z - 2\cot 2z$. Here's the sequence of steps:

$$\tan z/2 - \cot z/2 = \frac{\sin z/2}{\cos z/2} - \frac{\cos z/2}{\sin z/2} \tag{15}$$

$$= \frac{-(\cos^2 z/2 - \sin^2 z/2)}{(1/2)(2\sin z/2 \cos z/2} \tag{16}$$

$$= \frac{-\cos 2z}{(1/2)\sin 2z} = -2\cot z \tag{17}$$

and substituting $2z$ for z gives

$$\tan z = \cot z - 2\cot 2z.$$

As we have the series for cot, we can find the series for tan.

Using the general term for cot with $\cot z - 2\cot 2z$ gives that the general term for tan

$$\begin{aligned}
&= \frac{(-1)^n B_{2n}(2z)^{2n}}{(2n)!z} - 2\frac{(-1)^n B_{2n}(2 \cdot 2z)^{2n}}{(2n)!(2z)} \\
&= \frac{(-1)^n B_{2n}(2z)^{2n}}{(2n)!z} - \cancel{2}\frac{(-1)^n B_{2n}2^{2n}(2z)^{2n}}{(2n)!(\cancel{2}z)} \\
&= \frac{(-1)^n B_{2n}(2z)^{2n}(1-2^{2n})}{(2n)!z} \\
&= \frac{(-1)^n B_{2n}(2z)^{2n}(-1)(1-2^{2n})}{(2n)!z(-1)} \\
&= \frac{(-1)^{n-1} B_{2n}(2z)^{2n}(2^{2n}-1)}{(2n)!z} \\
&= \frac{(-1)^{n-1} B_{2n}(2z)^{2n}2(2^{2n}-1)}{(2n)!2z} \\
&= \frac{(-1)^{n-1}2(2^{2n}-1)B_{2n}(2z)^{2n-1}}{(2n)!}.
\end{aligned}$$

For (d), we establish the trigonometric identity $\csc z = \cot z + \tan z/2$. This follows from a standard trigonometric identity [**?**]:

$$\begin{aligned}
\frac{1-\cos z}{\sin z} &= \tan\frac{z}{2} \\
\csc z - \cot z &= \tan\frac{z}{2}.
\end{aligned}$$

This identity follows, in turn, from half angle identities for sin and cos.

Using the general terms for cot and tan with $\csc z = \cot z + \tan z/2$, we can deduce the general term for $\csc z$

$$
\begin{aligned}
&= \frac{(-1)^n B_{2n}(2z)^{2n}}{(2n)!z} + \frac{(-1)^{n-1}2(2^{2n}-1)B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{(-1)^n B_{2n}2^{2n}z^{2n-1}}{(2n)!} + \frac{(-1)^{n-1}2(2^{2n}-1)B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{[(-1)^n 2^{2n} + (-1)^{n-1}2(2^{2n}-1)]B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{[(-1)^{n-1}(-1)2^{2n} + (-1)^{n-1}2(2^{2n}-1)]B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{(-1)^{n-1}[2(2^{2n}-1) - 2^{2n}]B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{(-1)^{n-1}[2(2^{2n}-1) - 2 \cdot 2^{2n-1}]B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{(-1)^{n-1}2[(2^{2n}-1) - 2^{2n-1}]B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{(-1)^{n-1}2[(2 \cdot 2^{2n-1}-1) - 2^{2n-1}]B_{2n}z^{2n-1}}{(2n)!} \\
&= \frac{(-1)^{n-1}2(2^{2n-1}-1)B_{2n}z^{2n-1}}{(2n)!}.
\end{aligned}
$$

References

[1] T. M. Apostol, Another Elementary Proof of Euler's Formula for $\zeta(2n)$. *Amer. Math. Monthly* **80** (1973) 425–31.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[3] R. Blitzer, *Algebra and Trigonometry*, 4th ed., Upper Saddle, NJ, 2010.

[4] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[5] G. Chrystal, *Algebra: An Elementary Textbook*, 7th ed., vol. 1, American Mathematical Society, Providence, RI, 1964.

[6] P. Eymard and J.-P. Lafon, *The Number π*, American Mathematical Society, Providence, RI, 2004.

[7] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[8] K. Knopp, *Theory and Application of Infinite Series*, Dover, 1990.

[9] B. Larson and B. H. Edwards, *Calculus*, 9th ed., Brooks/Cole, Belmont, CA, 2010.

[10] M. R. Spiegel, *Complex Variables: With an Introduction to Conformal Mapping and Its Applications*, McGraw Hill, New York, 1964.

[11] M. R. Spiegel and J. Liu, *Mathematical Handbook of Formulas and Tables*, 2nd ed., McGraw-Hill, New York, 1999.

[12] D. Zwillinger, *CRC Standard Mathematical Tables and Formulae*, 31st ed., Chapman and Hall, Boca Raton, FL, 2003.