Dark Energy and the Bohm-Poisson-Schrödinger Equation

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Abstract

We revisit the solutions to the *nonlinear* Bohm-Poisson (BP) equation with relevant cosmological applications. In particular, we obtain an exact analytical expression for the observed vacuum energy density and explain the origins of its *repulsive* gravitational nature. Further results are provided which include two possible extensions of the Bohm-Poisson equation to the full relativistic regime; how Bohm's quantum potential in four-dimensions could be re-interpreted as a gravitational potential in five-dimensions, and which explains why the presence of dark energy/dark matter in our $4D$ spacetime can only be inferred *indirectly*, but not be detected/observed directly. Solutions to the novel Bohm-Poisson-Schrödinger equation are provided encoding the repulsive nature of dark energy (repulsive gravity). We proceed with a discussion on Asymptotic safety, matter creation from the vacuum, and Finsler relativistic extensions of the Bohm-Poisson equation. Finally, we conclude with some comments about the Dirac-Eddington large numbers coincidences.

1 Dark Energy and the Bohm-Poisson-Schrödinger Equation

In physical cosmology and astronomy, dark energy is an unknown form of energy which is hypothesized to permeate all of space, tending to accelerate the expansion of the universe [1] . Assuming that the standard model of cosmology is correct, the best current measurements indicate that dark energy contributes 68.3 percent of the total energy in the present-day observable universe. The mass-energy of dark matter and ordinary (baryonic) matter contribute 26.8 and 4.9 percent respectively, and other components such as neutrinos and photons contribute a very small amount. The density of dark energy much less than the density of ordinary matter or dark matter within galaxies. However,

it dominates the mass-energy of the universe because it is uniform across space [1]. Two proposed forms for dark energy are the cosmological constant, [2] representing a constant energy density filling space homogeneously, and scalar fields such as quintessence or moduli, dynamic quantities whose energy density can vary in time and space.

The nature of dark energy is more hypothetical than that of dark matter, and many things about the nature of dark energy remain matters of speculation [1]. Dark energy is thought to be very homogeneous, not very dense and is not known to interact through any of the fundamental forces other than gravity. In the models based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, it can be shown that a strong constant negative pressure in all the universe causes an acceleration in universe expansion if the universe is already expanding, or a deceleration in universe contraction if the universe is already contracting. This accelerating expansion effect is sometimes labeled "gravitational repulsion". In this work we shall *depart* from the FLRW model based on an isotropic and homogeneous universe to the case where the density of dark energy/dark matter (dark-fluid) permeating all of space can *vary* in space and time.

A major outstanding problem is that quantum field theories predict a huge cosmological constant, more than 100 orders of magnitude too large. This would need to be almost, but not exactly, cancelled by an equally large term of the opposite sign. Some supersymmetric theories require a cosmological constant that is exactly zero, which does not help because supersymmetry must be broken. Nonetheless, the cosmological constant is the most economical solution to the problem of cosmic acceleration. Thus, the current standard model of cosmology, the Lambda-CDM (cold dark matter) model, includes the cosmological constant as an essential feature [1].

The evidence for dark energy is heavily dependent on the theory of general relativity. Therefore, it is conceivable that a modification to general relativity also eliminates the need for dark energy. There are very many such theories, and research is ongoing [3], [4]. The measurement of the speed of gravity with the gravitational wave event GW170817 ruled out many modified gravity theories as alternative explanation to dark energy [1].

In quintessence models of dark energy, the observed acceleration of the scale factor is caused by the potential energy of a dynamical field, referred to as quintessence field. Quintessence differs from the cosmological constant in that it can vary in space and time. In order for it not to clump and form structure like matter, the field must be very light so that it has a large Compton wavelength. This class of theories attempts to come up with an all-encompassing theory of both dark matter and dark energy as a single phenomenon that modifies the laws of gravity at various scales. This could for example treat dark energy and dark matter as different facets of the same unknown substance, a "dark fluid" [5], or postulate that cold dark matter decays into dark energy.

The Schrödinger-Newton equation has had a long history since the 1950's [6], [7]. It is the name given to the system coupling the Schrödinger equation to the Poisson equation. In the case of a single particle, this coupling is effected as follows: for the potential energy term in the Schrödinger equation take the gravitational potential energy determined by the Poisson equation from a matter density proportional to the probability density obtained from the wave-function. For a single particle of mass m the coupled system of equations leads to the nonlinear and nonlocal Newton-Schrödinger integro-differential equation

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \;=\; -\;\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t)\;+\;V(\vec{r},t)\;\Psi(\vec{r},t)\;-\;\left(\;Gm^2\;\int\;\frac{|\Psi(\vec{r}',t)|^2}{|\vec{r}-\vec{r}'|}\;d^3r'\right)\;\Psi(\vec{r},t) \tag{1}$$

Bohm's quantum potential $V_Q = -\frac{\hbar^2}{2m}(\nabla^2\sqrt{\rho}/\sqrt{\rho})$ was shown to be *proportional* to the Weyl scalar spatial curvature produced by an ensemble density of paths associated with one, and only one particle, as shown in [8]. The constant of proportionality is $-\frac{\hbar^2}{2m}$. It can be generalized to the relativistic case. This geometrization process of quantum mechanics allowed to derive the Schroedinger, Klein-Gordon [8] and Dirac equations [9]. Most recently, a related geometrization of quantum mechanics was proposed [10] that describes the time evolution of particles as geodesic lines in a curved space, whose curvature is induced by the quantum potential. This formulation allows therefore the incorporation of all quantum effects into the geometry of space-time, as it is the case for gravitation in the general relativity.

Based on these results we proposed in [11] the following nonlinear quantum-like Bohm-Poisson equation for *static* solutions $\rho = \rho(\vec{r})$

$$\nabla^2 V_Q \;=\; 4\pi Gm\rho \;\Rightarrow\; -\;\frac{\hbar^2}{2m}\;\nabla^2\;(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}})\;=\;4\pi Gm\rho \tag{2}$$

such that one could *replace* the nonlinear Newton-Schrödinger equation for the above non-linear quantum-like Bohm-Poisson equation (2) where the fundamental quantity is *no* longer the wave-function Ψ (complex-valued in general) but the real-valued probability density $\rho = \Psi^*\Psi$. The logic behind eq-(2) is based on the idea that the laws of physics should themselves determine the distribution of matter. This is going one step further from General Relativity where a given distribution of matter determines the geometry. Eq-(2) is based on Bohm's quantum potential

$$V_Q \;\equiv -\frac{\hbar^2}{2m}\;\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{3}$$

If, in addition to the Bohm-Poisson equation (2), one were to *add* the Schrödinger equation for the complex-valued wave-function $\Psi \equiv \sqrt{\rho}e^{iS/\hbar}$, one can obtain consistent solutions, which *avoids* having an overdetermined system of equations, when the external potential is itself a function of ρ as we shall show at the end of this section. Hence in this scenario we will have the Bohm-Poisson-Schrödinger equation instead of the Newton-Schrödinger equation (1).

If one wishes to introduce a temporal evolution to ρ via a Linblad-like equation, for instance, this would lead to an overdetermined system of differential equations for $\rho(\vec{r},t)$. Replacing ∇^2 in eqs-(1,2) for the D'Alambertian operator $\Box = \nabla_\mu\nabla^\mu, \mu = 0,1,2,3$ has the caveat that in QFT $\rho(x^\mu) = \rho(\vec{r},t)$ *no* longer has the interpretation of a probability density but is now related to the particle number current. Despite this caveat we will propose an eq-(10) below involving the D'Alambertian $\Box$ operator and a proper mass density (mass per proper four-volume).

For the time being we shall just focus on static solutions $\rho(\vec{r})$. The de Sitter space metric associated with an exponentially expanding universe can be written in static and spherical coordinates in the form $g_{tt}(r) = -(1 - \frac{\Lambda}{3}r^2); g_{rr}(r) = -(g_{tt})^{-1}, \cdots$, and given in terms of the cosmological constant $\Lambda = (3/R_H^2)$, where R_H is the Hubble radius. Hence, there is no inconsistency in focusing for now on static and spherically symmetric solutions $\rho_m(r)$ for the probability density in order to simplify matters.

Since almost 95 percent of the energy/mass content of the Universe is comprised of dark energy/dark matter, we may envisioned the Universe's dark energy/dark matter (dark-fluid) density distribution as being proportional to a QM probability density obeying the Bohm-Poisson (BP) equation (2), in the same vain that one can view an electron orbiting the Hydrogen nucleus as an "electron probability cloud" surrounding the nucleus, permeating all of space, and whose mass density distribution is $\rho = m_e \Psi^* \Psi$, where $\Psi(\vec{r})$ are the stationary wave-function solutions to the Schrödinger equation, and m_e is the electron's mass.

The density $\rho_m = m\rho$ of dark energy/dark matter (dark-fluid) permeating all of space is postulated to be a solution to the nonlinear quantum-like Bohm-Poisson (BP) equation. It is straightforward to verify that a spherically symmetric solution to eq-(2) in a $3D$ spatially $flat$ background [1] is given by

$$\rho_m(r) \;=\; \frac{A}{r^4}, \;\; A \;=\; -\;\frac{\hbar^2}{2\pi G m^2} \;<\; 0 \tag{4}$$

At first glance, since $\rho_m(r) \leq 0$ one would be inclined to dismiss such solution as being unphysical. Nevertheless, we can bypass this problem by noticing that in the Einstein field equations with a cosmological constant, in the spherically symmetric case,

$$R_{tt} \;-\; \frac{1}{2}\, g_{tt}\, R \;+\; \Lambda\, g_{tt} \;=\; 8\pi G\, g_{tt}\, \rho_m(r) \tag{5a}$$

$$R_{rr} \;-\; \frac{1}{2}\, g_{rr}\, R \;+\; \Lambda\, g_{rr} \;=\; 8\pi G\, g_{rr}\, p_r(r) \tag{5b}$$

one can *absorb* the cosmological constant into a redefinition of $\rho_m(r)$, and the radial pressure $p_r(r)$, by rewriting

$$\rho_m \;=\; \tilde{\rho}_m \;+\; \frac{\Lambda}{8\pi G} \;=\; \tilde{\rho}_m \;+\; \rho_o, \;\; p_r \;=\; \tilde{p}_r \;+\; \frac{\Lambda}{8\pi G} \;=\; \tilde{p}_r \;+\; \rho_o \tag{6}$$

leading then to field equations rewritten in terms of $\tilde{\rho}_m(r)$ and $\tilde{p}_r(r)$

$$R_{tt} \;-\; \frac{1}{2}\, g_{tt}\, R \;=\; 8\pi G\, g_{tt}\, \tilde{\rho}_m(r) \tag{7a}$$

$$R_{rr} \;-\; \frac{1}{2}\, g_{rr}\, R \;=\; 8\pi G\, g_{rr}\, \tilde{p}_r(r) \tag{7b}$$

Hence, focusing instead on the $shifted$ density $\tilde{\rho}_m(r) \equiv \rho_m(r) - \rho_0$ to be the actual solution of the Bohm-Poisson equation we are interested in

$$-\;\frac{\hbar^2}{2m}\, \nabla^2 \left(\frac{\nabla^2 \sqrt{\tilde{\rho}_m}}{\sqrt{\tilde{\rho}_m}}\right) \;=\; 4\pi G m \tilde{\rho}_m, \;\; \nabla^2 f(r) \equiv r^{-2}\partial_r(r^2 \partial_r f(r)). \tag{8}$$

[1] For the time being we shall not discuss solutions in *curved* backgrounds

one has now that

$$\tilde{\rho}_m \;=\; A/r^4 \;=\; \rho_m(r) \;-\; \rho_o \;\leq 0, \quad \Rightarrow\; \rho_m(r) \;=\; \frac{A}{r^4} \;+\; \rho_0, \;\; A \;=\; -\; \frac{\hbar^2}{2\pi G m^2} \qquad (9)$$

and consequently, when $\Lambda > 0$, one can now concentrate on the domain of values of r where $\rho_m(r) > 0$. And, in doing so, it will permit us to show that the value of ρ_0 can be made to coincide exactly with the (extremely small) observed vacuum energy density, by simply introducing an ultraviolet length scale l that is *very close* to the Planck scale, and infrared length scale L equal to Hubble scale R_H.

A covariant (relativistic) extension of the BP equation, for signature $(-,+,+,+)$, may be defined in terms of the D'Alambertian operator, and a proper mass density $\sigma(\vec{r},t)$ of physical dimensions $(length)^{-5}$, such that $m = \int \sigma(\vec{r},t)\, \sqrt{|g|}\; d^4x$, as follows

$$-\;\Box\;\left(\frac{\Box\sqrt{\sigma(\vec{r},t)}}{\sqrt{\sigma(\vec{r},t)}}\right) \;=\; 4\pi G m\; \sigma(\vec{r},t), \;\; \Box \equiv \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\; g^{\mu\nu}\; \partial_\nu), \;\; \hbar = c = 1 \qquad (10)$$

The relativistic case will be the subject of future investigation. Focusing for now on the static spherically symmetric solutions (8) of the BP equation, let us choose the ultraviolet scale l to coincide with the node of $\rho_m(r)$ such that from eq-(9) one has

$$\rho_m(r=l) \;=\; -\; \frac{\hbar^2}{2\pi G m^2}\; \frac{1}{l^4} \;+\; \rho_o \;=\; 0 \;\Rightarrow\; \rho_o \;=\; \frac{\hbar^2}{2\pi G m^2}\; \frac{1}{l^4} \qquad (11)$$

The domain of physical values of r must be $r \geq l$ in order to ensure a positive-definite density $\rho_m(r) \geq 0$. One could include all the values of r from 0 to ∞. The density diverges at $r = 0$, while the integral $\int_0^\infty \rho(r) 4\pi r^2 dr = \infty - \infty$. The $+\infty$ contribution stems from the region $r \geq l$, while the $-\infty$ contribution stems from the region $r < l$. Therefore one needs to introduce a suitable and judicious regularization involving an ultraviolet and infrared scale.

In natural units of $\hbar = c = 1$, after introducing the ultraviolet scale l and infrared scale $L = R_H$ in the normalization condition (otherwise the mass would diverge) it yields the integral

$$m \;=\; \int_l^{R_H} \rho(r)\, 4\pi r^2\, dr \;=\; \int_l^{R_H} (\frac{A}{r^4} + \rho_0)\, 4\pi r^2\, dr \;=\; \int_l^{R_H} \left(-\frac{1}{2\pi G m^2}\; \frac{1}{r^4} \;+\; \rho_0\right)\, 4\pi r^2\, dr \qquad (12)$$

In conventional QM, the plane wave solutions $\Psi = e^{i\vec{k}.\vec{r}}$ are *not* square integrable. Nevertheless we bypass this problem after introducing an infrared cutoff by putting the free particle in a box of finite volume. Similarly, we follow this regularization procedure in eq-(12). Upon performing the integral in eq-(12), after plugging in the value of ρ_0 derived from eq-(11), with the provision that when $R_H >> l$ the dominant contribution to the integral stems solely from ρ_o, one ends up with the following relationship

$$\frac{4\pi R_H^3}{3}\,\rho_o \;=\; \frac{4\pi R_H^3}{3}\,\frac{1}{2\pi G m^2 l^4} \;=\; m \;\;\Rightarrow\; m^3 \;=\; \frac{2}{3}\,\frac{R_H^3}{G l^4} \tag{13}$$

solving for m one gets

$$m \;=\; (\frac{2}{3Gl^4})^{1/3}\; R_H \tag{14}$$

One can verify that when the ultraviolet scale l is chosen to be *very* close to the Planck scale, and given by

$$l^4 \;=\; \frac{4}{3}\,L_P^4 \;\Rightarrow\; l \;=\; (\frac{4}{3})^{1/4}\,L_p \;=\; 1.0745\,L_p \tag{15}$$

then upon inserting the values for m and l obtained in eqs-(14,15) into the expression for ρ_o *derived* in eq-(11), after setting $L_p^2 = 2G$, [2] it gives in natural units of $\hbar = c = 1$

$$\rho_o \;=\; \frac{1}{2\pi G m^2}\,\frac{1}{l^4} \;=\; \frac{1}{2\pi G}\,(\frac{3\;G\;l^4}{2})^{2/3}\,\frac{1}{R_H^2\;l^4} \;=\; \frac{3}{8\pi G}\,\frac{L_p^4}{R_H^2\;L_p^4} \;=\; \frac{3}{8\pi G R_H^2} \tag{16}$$

which is precisely *equal* to the observed vacuum energy density $\rho = (\Lambda/8\pi G)$ associated with a cosmological constant $\Lambda = (3/R_H^2)$ and corresponding to a de Sitter expanding universe whose throat size is the Hubble radius R_H ($= c/H_o$, H_o is today's value of the Hubble parameter).

The physical reason behind the choice of the ultraviolet scale l in eq-(15) is based on re-interpreting ρ_o as the uniform energy (mass) density inside a black hole region of Schwarzschild radius $R = 2Gm$

$$\rho_{bh} \;=\; \frac{m}{(4\pi/3)R^3} \;=\; \frac{3}{8\pi G R^2}, \quad L_P^2 = 2G, \quad \hbar = c = 1 \tag{17}$$

In the regime $R = 2Gm >> l$, when the dominant contribution to the integral (12) stems from the ρ_o term, we may equate the expression for ρ_o in eq-(11) to ρ_{bh} in eq-(17) giving

$$\frac{1}{2\pi G m^2 l^4} \;=\; \frac{1}{2\pi l^4}\,\frac{(2G)^2}{GR^2} \;=\; \frac{1}{2\pi l^4}\,\frac{L_p^4}{GR^2} \;=\; \frac{3}{8\pi G R^2} \;\Rightarrow\; l \;=\; (\frac{4}{3})^{1/4}\,L_p, \;\; \hbar = c = 1 \tag{18}$$

and leading once again to the value of $l = 1.0745 L_p$ in eq-(15). Therefore, when $R = 2Gm >> l$, the value of l is always *very* close to the Planck scale, and *independent* of $R = 2Gm$, because the scale R has *decoupled* in eq-(18).

In this way, one can effectively view the observable universe as a "black-hole" whose Hubble radius R_H encloses a mass M_U given by $2GM_U = R_H$. From eq-(14) it follows that when $R = R_H$, the black hole density $\rho_{bh} = \rho_o = \rho_{obs}$ coincides with the observed

[2] Some authors absorb the factor of 2 inside the definition of L_p, we define the Planck scale such that the Compton wavelength coincides with the Schwarzschild radius

vacuum energy density. It is well known that inside the black hole horizon region the roles of t and r are exchanged due to the switch in the signature of the g_{tt}, g_{rr} metric components. Cosmological solutions based on this $t \leftrightarrow r$ exchange were provided by the Kantowski-Sachs metric. For references on Black-Hole Cosmology see [12].

To sum up our main result : By postulating that dark energy/dark matter is a dark-fluid permeating all of space, whose mass density is proportional to the probability density, and after finding a particular static spherically symmetric solution $\tilde{\rho}_m$ in eq-(9) to the Bohm-Poisson equation (8), while introducing an ultraviolet (very close to the Planck scale) and an infrared (Hubble) scale to regularize the mass, one can naturally obtain a value for the vacuum energy density ρ_o in eq-(16) which coincides *exactly* with the extremely small observed vacuum energy density. It is remarkable that the Bohm-Poisson equation chooses for us a lower scale to be basically equal to the Planck scale. It was not put it in by hand, but is a direct result of the solutions to the Bohm-Poisson equation. The only assumption made was to choose the Hubble scale R_H for the infrared cutoff, and which makes physical sense since R_H is the cosmological horizon. Is it a numerical coincidence or design ? Because Bohm's formulation of QM is by construction non-local, it is this non-locality which casts light into the crucial ultraviolet/infrared entanglement of the Planck/Hubble scales which was required in order to obtain the observed values of the vacuum energy density.

Furthermore, one can also explain the origins of its *repulsive* gravitational nature. The Bohm-Poisson's (BP) equation is invariant under $\tilde{\rho}_m \to -\tilde{\rho}_m$, and $G \to -G$. Consequently $-\tilde{\rho}_m \geq 0$ is a solution to a BP equation associated to a *negative* gravitational coupling $-G < 0$ which is tantamount to *repulsive* gravity. This is perhaps one of the most salient feature of these results, in particular because of the positive definite values $-\tilde{\rho}_m \geq 0$.

We should remark that we found solutions to the BP equation in $flat$ spatial $3D$ backgrounds. The operator ∇^2 is metric-dependent, thus solutions of the BP equation in *curved* backgrounds will differ from those found above. In general, one must have a *coupled* system of equations involving the relativistic analog of the BP equation *plus* Einstein's field equations. Matter affects the geometry (metric), and the latter metric determines the form of the $\nabla^2, \Box$ operators, which in turn will affect the solutions for ρ_m, and which in turn will have an affect on the metric, and so forth, $\cdots$ In eqs-(33, 34) we show how this *coupled* system of equations can be lumped into a single equation.

For consistency checks, given $\rho_m = \rho_o + A/r^4$, and a radial pressure $p = -\rho_m$, one can find solutions to the Einstein field equations with a cosmological constant if one introduces a cutoff ϵ to avoid the singularity at $r = 0$, such that $M_U\ \epsilon \sim 1$, and $\rho_m \equiv 0$ in the region $r < \epsilon$. The temporal and radial metric components solutions to the field equations are then given by

$$g_{tt} = -(\ 1\ -\ \frac{\Lambda}{3}\ r^2\ +\ \frac{C_1}{r}\ +\ \frac{C_2}{r^2}\ ),\quad g_{rr} = -(g_{tt})^{-1},\quad \Lambda = 8\pi G\rho_o \qquad (19a)$$

where

$$C_1 \;=\; 8\pi G(\frac{\rho_o\epsilon^3}{3} - \frac{A}{\epsilon}) \;>\; 0,\;\; C_2 \;=\; 8\pi GA \;<\; 0,\;\; A \;=\; -\;\frac{1}{2\pi Gm^2},\;\; \hbar = c = 1 \;\; (19b)$$

when the mass parameter is $m \sim M_U = (R_H/2G) \Rightarrow \epsilon \sim L_p(L_p/R_H) \sim 10^{-61}L_p$, one can verify that the terms defining the coefficients C_1, C_2 are infinitesimals of orders $\epsilon, \epsilon^2, \epsilon^3$. Therefore, one recovers the de Sitter metric as expected without having to go to the asymptotic region $r \to \infty$.

The Bohm-Poisson-Schrödinger equation

To finalize this section, let us include the Schrödinger equation in addition to the Bohm-Poisson equation and see what is the potential V which reproduces the positive definite probability density ρ which is related to the matter density $\tilde{\rho}_m$ solutions of the Bohm-Poisson equation (8). As shown by David Bohm, the Schrödinger equation for the complex valued wave function $\Psi \equiv \sqrt{\rho(\vec{x},t)}\; e^{iS(\vec{x},t)/\hbar}$ is equivalent to the coupled pair of equations

$$-\;\frac{\partial S}{\partial t} \;=\; \frac{p^2}{2m} \;+\; V_Q \;+\; V \;=\; \frac{(\nabla S)^2}{2m} \;-\; \frac{\hbar^2}{2m}\;\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \;+\; V \qquad (20a)$$

$$\frac{\partial \rho}{\partial t} \;+\; \nabla(\rho\frac{\nabla S}{m}) \;=\; 0 \qquad (20b)$$

The first equation is the Hamilton-Jacobi equation corresponding to an external potential V *and* including Bohm's quantum potential V_Q; the second equation is the continuity equation. The momentum (not to be confused with pressure) is $\vec{p} = \nabla S$. Inserting the spherically symmetric static solutions $|\tilde{\rho}_m|/m = \rho = |A|/mr^4$ of the Bohm-Poisson equation into the above eqs-(20a, 20b) leads to a coupled system of differential equations which determine the potential $V(r)$ and the action (phase) $S(r,t) = S(r) - Et$. After some lengthy but straightforward algebra, the solutions to the Bohm-Poisson-Schrödinger equation in the spherically symmetric case are

$$\rho(r) \;=\; \frac{|\tilde{\rho}_m|}{m} \;=\; \frac{|A|}{mr^4} \;=\; \frac{1}{2\pi Gm^3r^4},\;\; V(r) \;=\; E \;-\; (E - V_o)\;(\frac{r}{r_o})^4 \;+\; \frac{\hbar^2}{mr^2}$$

$$V_Q \;=\; -\;\frac{\hbar^2}{2m}\;\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \;=\; -\;\frac{\hbar^2}{mr^2}$$

$$S(r,t) \;=\; S(r) \;-\; E\;t,\;\; S(r) \;=\; \sqrt{2m(E-V_o)}\;\frac{r^3}{3r_o} \;+\; S_o,\;\; E > V_o \qquad (21a)$$

One can verify that the spherically symmetric wave function $\Psi(r) \equiv \sqrt{\rho(r)}\; e^{iS(r,t)/\hbar}$ built from the expressions in eq-(21a) is a solution of the Schrödinger equation

$$i\hbar\;(\frac{\partial\Psi(r,t)}{\partial t}) \;=\; -\;\frac{\hbar^2}{2m}\;\frac{1}{r^2}\;\partial_r(r^2\partial_r\Psi(r,t)) \;+\; V(r)\;\Psi(r,t) \qquad (21b)$$

As expected, the external potential $V(r)$ cannot be arbitrary but is itself determined in terms of ρ. It can be rewritten as

$$V(\rho(r)) \;=\; c_o \;+\; \frac{c_1}{\rho(r)} \;+\; c_2\,\sqrt{\rho(r)} \tag{22}$$

where c_o, c_1, c_2 are numerical coefficients given in terms of E, V_o, r_o. Such potential leads to a *repulsive* force $F = -\nabla V > 0$, for $r > 0$. Therefore, the Bohm-Poisson-Schrödinger equation admits solutions encoding the *repulsive* nature of dark energy (repulsive gravity).

2 Asymptotic Safety and Covariant Extensions of the BP Equation

In the previous section we studied solutions to the BP equation involving a large mass m of the order of $M_U = (R_H/2G)$ and which followed directly from eqs-(13, 14). In this section we shall be focusing on a particle with a very small mass (large Compton wavelength) of the order of $m \sim (1/R_H)$, and on the effects of the Renormalization Group. This scalar particle might be related to quintessence [5].

The Renormalization Group (RG) improvement of Einstein's equations is based on the possibility that Quantum Einstein Gravity might be non-perturbatively renormalizable and asymptotically safe due to the presence of interacting ultraviolet fixed points [19]. In this program one has k (energy) dependent modifications to the Newtonian coupling $G(k)$, the cosmological constant $\Lambda(k)$ and energy-dependent spacetime metrics $g_{ij,(k)}(x)$.

In $D = 4$ there is a nontrivial interacting (non-Gaussian) ultraviolet fixed point $G_* = G(k)k^2 \neq 0$. The fixed point G_* by definition is *dimensionless* and the running gravitational coupling has the form [20], [19]

$$G(k) \;=\; G_N\,\frac{1}{1 + [G_N k^2/G_*]} \tag{23a}$$

The scale dependence of $\Lambda(k)$ in the de Sitter case was found to be [20]

$$\Lambda(k) \;=\; \Lambda_0 \;+\; \frac{b\,G(k)}{4}\,k^4, \quad \Lambda_0 > 0 \tag{23b}$$

where b is positive numerical constant.

In $D = 4$, the dimensionless gravitational coupling has a nontrivial fixed point $G = G(k)k^2 \to G_*$ in the $k \to \infty$ limit, and the dimensionless variable $\Lambda = \Lambda(k)k^{-2}$ has also a nontrivial ultraviolet fixed point $\Lambda_* \neq 0$ [20]. The infrared limits are $\Lambda(k \to 0) = \Lambda_0 > 0,\ G(k \to 0) = G_N$. Whereas the ultraviolet limit is $\Lambda(k = \infty) = \infty; G(k = \infty) = 0$.

Let us choose now an actual positive-definite solution $\hat{\rho}_m \equiv -\tilde{\rho}_m = |A|/r^4 \geq 0$; $|A| = \hbar^2/2\pi G m^2$, of the BP equation associated to *repulsive* gravity $-G < 0$, as explained earlier

$$-\ \frac{\hbar^2}{2m}\ \nabla^2\ (\frac{\nabla^2\sqrt{-\tilde{\rho}_m}}{\sqrt{-\tilde{\rho}_m}})\ =\ 4\pi\ (-G)\ m\ (-\tilde{\rho}_m) \tag{24}$$

The mass density solution of (24) to focus on (in $\hbar = c = 1$ units) is

$$\hat{\rho}_m(r)\ =\ -\ \tilde{\rho}_m(r)\ =\ \frac{1}{2\pi G m^2 r^4}\ \geq 0 \tag{25}$$

If one selects $m = (1/R_H)$ to coincide with the Compton mass of a particle corresponding to the Hubble scale R_H, then at the Hubble scale $r = R_H$ one has

$$\hat{\rho}_m(r = R_H)\ =\ \frac{1}{2\pi G m^2 R_H^4}\ =\ \frac{1}{2\pi G\ (R_H)^{-2}\ R_H^4}\ \sim\ (L_P R_H)^{-2} \sim 10^{-122} M_{Planck}^4 \tag{26a}$$

and which agrees with the observed vacuum energy density. It is well known (to the experts) that such extremely small value is of the same order of magnitude as $(m_{neutrino})^4$.

The problem arises when one evaluates $\hat{\rho}_m(r)$ at L_p, given $m = 1/R_H$. One gets a huge value

$$\hat{\rho}_m(r = L_p)\ \sim\ \frac{1}{G m^2 L_p^4}\ =\ (\frac{R_H}{L_p})^2\ L_p^{-4}\ \sim\ 10^{122}\ M_p^4 \tag{26b}$$

We will see how the Asymptotic Safety scenario comes to our rescue by realizing that a Renormalization Group flow of G and m^2 solves the problem. The key idea, based on dimensional grounds, is simply to postulate that the flow of $m^2(k)$ has the *same* functional form as the flow of $\Lambda(k)$ in eq-(23b)

$$m^2(k)\ =\ m_o^2\ +\ \frac{b\ G(k)}{4}\ k^4,\ \ m_o^2 > 0 \tag{27}$$

The only thing remaining is to related the scale r in eq-(17) with the energy (momentum) scale k. The authors [20] expressed k as the inverse of $d(r)$ where $d(r)$ was a proper distance derived from the Schwarzschild metric. If one opts for the simplest choice $k = 1/r$, eq-(25) can be rewritten as

$$\hat{\rho}_m(r)\ =\ \frac{1}{2\pi}\ \frac{1}{G(k)\ m^2(k)\ r^4}\ =\ \frac{1}{2\pi}\ \frac{k^4}{G(k) m^2(k)} \tag{28}$$

note that strictly speaking eq-(28) is *not* a solution to the BP equation, because if it were one must have that $G(k)m^2(k) = constant$, for all values of k, which is *not* the case. Similarly, the renormalization-group-improved black hole solutions of [20] are *not* solutions to the Einstein vacuum field equations [17]. Nevertheless, from eqs-(23,27) one learns that

$$lim_{k\to 0}\ (G(k)\ m^2(k))\ =\ G_N\ m_o^2 \tag{29a}$$

$$lim_{k\to \infty}\ (G(k)\ m^2(k))\ =\ \frac{b}{4}\ (G_*)^2 \tag{29b}$$

whereas at the Planck scale $k = M_p$

$$lim_{k\to M_p}\ (G(k)\ m^2(k))\ \sim\ \frac{b}{4}\ (G_*)^2 \tag{30}$$

Consequently, eqs-(28,30) lead to

$$\hat{\rho}_m(r = L_p)\ \sim\ \frac{1}{2\pi\ G(k = M_p)\ m^2(k = M_p)\ L_p^4}\ \sim\ \frac{2}{b\pi}\ (G_*)^{-2}\ L_p^{-4}\ \sim\ M_p^4 \tag{31}$$

which is the expected result for the vacuum energy density at the Planck scale.

To sum up, the renormalization group machinery (Asymptotic Safety) can be implemented such that eq-(26a) furnishes the observed vacuum energy density at the Hubble scale, while eq-(31) is the expected vacuum energy density at Planck scale. Naturally, one needs to generalize the BP equation to the fully relativistic regime as described by eq-(10). The key question is what is the "particle" represented by the mass m in the BP equation (21) ? i.e. a mass that experiences a renormalization group flow (27) similar to the flow experienced by Λ (23b). Is this a scalar particle related to quintessence ? [5].

We emphasized earlier the key role that $-G < 0$ plays in all of this and which *stems* directly from the invariance of the BP equation under $\rho \to -\rho; G \to -G$. Our solutions for $\hat{\rho}_m(r) \geq 0$ correspond to $-G < 0$, thus the "particle" in question exerts a *repulsive* gravitational force which mimics "dark energy". The RG flow behavior of G displayed in eq-(23a) shows that G grows as k decreases. Meaning that G *increases* with distance, so that the magnitude of the repulsive force exemplified by $-G < 0$ becomes larger, and larger, as the universe expands. This is what is observed. Next we shall provide a different view of our findings so far.

Matter Creation from the Vacuum

The second interpretation of the solution (25) to the BP equation (24) is that involving matter creation from the vacuum, as advocated by Hoyle long ago. Imagine one pumps matter out of the vacuum in lumps/units of Planck masses. Let us assume that the Universe expands in such a way that matter is being replenish from the vacuum so that the mass at any moment is linearly proportional to the size of the Universe. As the mass of the universe grows the vacuum energy density decreases since the vacuum is being depleted. In this scenario, at the Hubble scale R_H, one has $M_U \sim R_H$.

This result is also compatible with Mach's principle. By equating GmM_U/R_H to the rest mass m of a particle one arrives at $GM_U = R_H$, which once again is very close to the Schwarzschild radius $2GM_U$. Hence, one arrives at the scaling relation

$$\frac{M_p}{L_P}\ =\ \frac{M_U}{R_H} \tag{32}$$

which we interpreted long ago [18] as equating the proper forces (after re-introducing c) $M_U c^2/R_H = M_p c^2/L_p$ and leading to some sort of maximal/minimal acceleration duality.

Inserting the values of M_P, M_U, and $r = L_P$, into the solution (25) of the BP equation gives

$$\hat{\rho}(r = L_p; m = M_p) \;=\; \frac{1}{2\pi G M_P^2\; L_p^4} \;\sim\; L_p^{-4} = M_p^4, \quad L_p^2 = 2G \tag{33a}$$

which is compatible with the large density at the Planck scale, and

$$\hat{\rho}(r = L_p; m = M_U) \;=\; \frac{1}{2\pi G M_U^2\; L_p^4} \;\sim\; \frac{G}{2\pi R_H^2\; L_p^4} \;\sim\; (L_p\; R_H)^{-2} \tag{33b}$$

which agrees with the observed vacuum energy density at the Hubble scale and obtained above in eq-(11). Before concluding we add some important remarks.

Does Dark Energy Resides in the Bulk of a 5D spacetime ?

Evaluating Bohm's quantum potential (3) for $\rho(r)$ given by eq-(4) yields

$$V_Q \;=\; -\; \frac{\hbar^2}{mr^2} \tag{34}$$

and which is reminiscent of an effective gravitational potential in 4 spatial dimensions (a $5D$ spacetime). The "quantum" force $F_Q = -\partial_r V_Q$ corresponding to V_Q in eq-(34) scales as $F_Q \sim -r^{-3}$ which has the same behavior as the gravitational force between two masses m_1, m_2 in $5D$

$$F \;=\; -\; G_5\; \frac{m_1\; m_2}{r^3} \tag{35}$$

since the $5D$ gravitational constant G_5 has dimensions of $(length)^3$. Despite the possibility, we are not going to speculate at the moment as to whether or not the "quantum" force originating from Bohm's quantum potential is the "fifth" force. The main point is that one should consider the possibility that Bohm's quantum potential in 3 spatial dimensions ($4D$ spacetime) mimics classical gravity in 4 spatial dimensions ($5D$ spacetime), and for this reason one can only *indirectly* infer the gravitational effects of dark energy/dark matter in our $4D$ universe without directly detecting it because such dark energy/dark matter resides in $5D$, which is reminiscent of the brane-world scenarios.

Finsler-Relativistic Extension of the Bohm-Poisson equation

If one wishes to introduce a temporal dependence to $\hat{\rho}_m$ we should extend the BP equation to full the relativistic regime. It is interesting that a simple exchange of $r \leftrightarrow t$ as it occurs in the Schwarzschild metric leading to the Kantowski-Sachs metric, yields $\hat{\rho}_m(t) = |A|t^{-4}$, and similar findings are obtained for the values of the vacuum energy density, simply by exchanging $L_p \leftrightarrow ct_p; R_H \leftrightarrow ct_H$ in the equations. t_p, t_H are the Planck and Hubble times, respectively. Upon doing so it leads to a Big-Bang-like singularity at $t = 0, \hat{\rho}_m(t = 0) = \infty$. This combined with the repulsive gravitational feature of our model, implies naturally that an expansion would follow.

Besides eq-(10), another relativistic generalization of the BP equation can be constructed from the Lagrange-Finsler geometrical formulation of QM recently advocated by [10]. He described the time evolution of particles as geodesic lines in a curved space,

whose *curvature* is induced by the quantum potential. This formulation incorporates all quantum effects into the geometry of space-time, as it is the case for gravitation in the general relativity.

The explicit expression of the metric components in terms of the quantum potential V_Q were provided by [10]. This is not the first time where the metric components are expressed in terms of a potential. In General Relativity (GR) we learned that in the linearized weak gravity limit, and for slow moving bodies, the temporal metric component $g_{00} \sim \eta_{00} + 2V_N$ (c = 1), can be expressed in terms of the Newtonian potential V_N. Hence, given the explicit expressions [10] of the metric $g_{\mu\nu} = g_{\mu\nu}(V_Q)$ in terms of Bohm's quantum potential, one can write down the curvature tensors, and the Einstein tensor leading to the field equations

$$R_{\mu\nu}(V_Q) \; - \; \frac{1}{2} \; g_{\mu\nu}(V_Q) \; R(V_Q) \; + \; \Lambda \; g_{\mu\nu}(V_Q) \; = \; 8\pi G \; T_{\mu\nu} \tag{36}$$

where the stress energy tensor is the one associated with a dark energy/dark matter fluid permeating all of space and given in terms of ρ_m, p. In particular, the BP equation (2) can be generalized to

$$R_{00}(V_Q) \; - \; \frac{1}{2} \; g_{00}(V_Q) \; R(V_Q) \; + \; \Lambda \; g_{00}(V_Q) \; = \; 8\pi G \; T_{00} \tag{37}$$

where $T_{00} = g_{00}\rho_m$. Solutions to eq-(36) will be provided in future investigations as well as the study of these equations in higher dimensions.

3 Conclusions

To conclude we add some remarks pertaining the Dirac-Eddington large numbers coincidences. Nottale [16] found long ago a direct relationship between the fine structure constant α and the cosmological constant Λ. In $\hbar = c = 1$ units, $\alpha = e^2 = 1/137$, the expression is

$$\Lambda \simeq \frac{(L_p)^4}{(r_e)^6} \; = \; \frac{(m_e)^6 \; (L_p)^4}{(\alpha)^6} \; = \; 10^{-56} \; cm^{-2} \tag{38}$$

the classical electron radius r_e is defined in terms of the charge e, and electron mass m_e, as

$$\frac{e^2}{r_e} \; = \; \frac{\alpha}{r_e} \; = \; m_e \tag{39}$$

This important relation between Λ and α [16] warrants further investigation within the context of the Bohm-Poisson equation and the Dirac-Eddington large number coincidences.

We should mention that of the many articles surveyed in the literature pertaining the role of Bohm's quantum potential and cosmology, [13], [14], [15] we did not find any related

to the Bohm-Poisson equation proposed in this work.[3] The authors [14], for instance, have shown that replacing classical geodesics with quantal (Bohmian) trajectories gives rise to a quantum corrected Raychaudhuri equation (QRE). They derived the second order Friedmann equations from the QRE, and showed that this also contains a couple of quantum correction terms, the first of which can be interpreted as cosmological constant (and gives a correct estimate of its observed value), while the second as a radiation term in the early universe, which gets rid of the big-bang singularity and predicts an infinite age of our universe.

To finalize, we must say that the most attractive project is to find nontrivial solutions to the relativistic Bohm-Poisson equation (10). A careful inspection reveals that a separation of variables does not work. Solutions to eq-(36) are more difficult to find. We delegate this difficult task for future investigations as well as the study of these equations and solutions in higher dimensions.

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