The Positive Integer Solutions of Equation

4/(*x*+*y*+*z*)=1/*x*+1/*y*+1/*z*

Haofeng Zhang

Abstract: This paper proves that equation 4/(*x*+*y*+*z*)=1/*x*+1/*y*+1/*z* has no positive integer solutions using the method of solving third order equation.

1. Proving Method

The equation we talk about is

$$\frac{4}{x+y+z}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \tag{1-1}$$

which is

$$(x+y+z)(xy+xz+yz)=4xyz \tag{1-2}$$

has no positive integer solutions, in which x, y, z are positive integers. There are three cases for equation (1-1) when

Case I: $x=y=z$;

Case II: $x=y\neq z$;

Case III: $x\neq y\neq z$.

Here $x\neq y\neq z$ means $x\neq y, x\neq z, y\neq z$.

1.1 Case I

From equation (1-2) when $x=y=z$ we have

$$3x\times 3x^2=4x^3\Rightarrow 9=4$$

that is impossible which means there are no positive integers for x, y, z that meet equation (1-1) when $x=y=z$.

1.2 Case II

From equation (1-2) when $x=y\neq z$ we have

$$\begin{aligned}&(2x+z)(x^2+xz+xz)=4xyz\Rightarrow\\&(2x+z)(x+2z)=4xz\Rightarrow\\&2x^2+2z^2+5xz=4xz\end{aligned},$$

where

$$2x^2+2z^2+xz=0$$

that is impossible since the left side is greater than 0. So equation (1-2) has no positive integer solutions when $x = y \neq z$ and so it is with $x = z \neq y, y = z \neq x$.

1.3 Case III

From equation (1-2) when $x \neq y \neq z$, we let $z > x > y$ and

$$\begin{cases} y = x - f \\ z = x + e \end{cases},$$

we get

$$(x + x - f + x + e)[x(x - f) + x(x + e) + (x - f)(x + e)] = 4x(x - f)(x + e),$$

and

$$5x^3 + 5(e - f)x^2 + \left(2(e - f)^2 + ef\right)x + (f - e)ef = 0 \tag{1-3}$$

where

$$x = \frac{(e - f)ef}{5x^2 + 5(e - f)x + 2(e - f)^2 + ef} = \frac{(e - f)}{\dfrac{5x^2 + 5(e - f)x + 2(e - f)^2}{ef} + 1}, \tag{1-4}$$

if $(e - f) = 0$ then $x = 0$ that is impossible; If $(e - f) < 0$ then

$$5x^2 < 5(f - e)x \Rightarrow x < f - e$$

that is impossible since $y = x - f > 0$ and $x > f$, so we have the conclusion of $(e - f) > 0$.

The real root expression for $ax^3 + bx^2 + cx + d = 0$ is

$$\begin{cases} p = \dfrac{c}{a} - \dfrac{b^2}{3a^2} \\ q = \dfrac{d}{a} + \dfrac{2b^3}{27a^3} - \dfrac{bc}{3a^2} \\ w = \sqrt[3]{-\dfrac{q}{2} + \sqrt{\left(\dfrac{q}{2}\right)^2 + \left(\dfrac{p}{3}\right)^3}} + \sqrt[3]{-\dfrac{q}{2} - \sqrt{\left(\dfrac{q}{2}\right)^2 + \left(\dfrac{p}{3}\right)^3}} \\ x = w - \dfrac{b}{3a} \end{cases}.$$

For (1-3) we have

$$\begin{cases} p = \dfrac{2(f-e)^2 + ef}{5} - \dfrac{25(e-f)^2}{75} = \dfrac{(e-f)^2 + 3ef}{15} \\ q = \dfrac{(f-e)ef}{5} + \dfrac{2(e-f)^3}{27} - \dfrac{(e-f)\left(2(e-f)^2 + ef\right)}{15} = \left[\dfrac{-8(e-f)^2 - 36ef}{135}\right](e-f) \end{cases},$$

and

$$\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = \sqrt{\left[\frac{8(e-f)^2 + 36ef}{270}\right]^2 (e-f)^2 + \left(\frac{(e-f)^2 + 3ef}{45}\right)^3}$$

$$= \sqrt{\frac{64(e-f)^6 + 16 \times 36ef(e-f)^4 + 36^2 e^2 f^2 (e-f)^2}{270^2} + \left(\frac{(e-f)^2 + 3ef}{45}\right)^3}$$

$$= \sqrt{\frac{64(e-f)^6 + 16 \times 36ef(e-f)^4 + 36^2 e^2 f^2 (e-f)^2}{(15 \times 18)^2} + \frac{(e-f)^6 + 9ef(e-f)^4 + 27e^2 f^2 (e-f)^2 + 27e^3 f^3}{(15 \times 3)^3}}$$

$$= \sqrt{\frac{64 \times 45(e-f)^6 + 16 \times 36 \times 45ef(e-f)^4 + 36^2 \times 45e^2 f^2 (e-f)^2}{(15 \times 3)^2 \times 36 \times 45} + \frac{36(e-f)^6 + 9 \times 36ef(e-f)^4 + 27 \times 36e^2 f^2 (e-f)^2 + 27 \times 36e^3 f^3}{(15 \times 3)^3 \times 36}}$$

$$= \sqrt{\frac{2916(e-f)^6 + 26244ef(e-f)^4 + 59292e^2 f^2 (e-f)^2 + 27 \times 36e^3 f^3}{(15 \times 3)^2 \times 36 \times 45}}$$

$$= \sqrt{\frac{36(e-f)^6 + 324ef(e-f)^4 + 732e^2 f^2 (e-f)^2 + 12e^3 f^3}{(15 \times 3)^2 \times 4 \times 5}}$$

$$= \sqrt{\frac{3(e-f)^6 + 27ef(e-f)^4 + 61e^2 f^2 (e-f)^2 + e^3 f^3}{(15)^2 \times 5 \times 3}}$$

$$= \frac{\sqrt{3(e-f)^6 + 27ef(e-f)^4 + 61e^2 f^2 (e-f)^2 + e^3 f^3}}{15},$$

so we get

$$w=\sqrt[3]{(e-f)\left[\frac{4(e-f)^2+18ef+9\sqrt{3(e-f)^4+27ef(e-f)^2+61e^2f^2+\frac{e^3f^3}{(e-f)^2}}}{135}\right]}$$

$$+\sqrt[3]{(e-f)\left[\frac{4(e-f)^2+18ef-9\sqrt{3(e-f)^4+27ef(e-f)^2+61e^2f^2+\frac{e^3f^3}{(e-f)^2}}}{135}\right]},$$

and

$$w=\frac{(e-f)}{3}\sqrt[3]{\frac{4+\frac{18ef}{(e-f)^2}+9\sqrt{3+\frac{27ef}{(e-f)^2}+\frac{61e^2f^2}{(e-f)^4}+\frac{e^3f^3}{(e-f)^6}}}{5}}$$

$$+\frac{(e-f)}{3}\sqrt[3]{\frac{4+\frac{18ef}{(e-f)^2}-9\sqrt{3+\frac{27ef}{(e-f)^2}+\frac{61e^2f^2}{(e-f)^4}+\frac{e^3f^3}{(e-f)^6}}}{5}},$$

so we have

$$x=\frac{(e-f)}{3}\left[\sqrt[3]{\frac{4+\frac{18ef}{(e-f)^2}+9\sqrt{3+\frac{27ef}{(e-f)^2}+\frac{61e^2f^2}{(e-f)^4}+\frac{e^3f^3}{(e-f)^6}}}{5}}+\sqrt[3]{\frac{4+\frac{18ef}{(e-f)^2}-9\sqrt{3+\frac{27ef}{(e-f)^2}+\frac{61e^2f^2}{(e-f)^4}+\frac{e^3f^3}{(e-f)^6}}}{5}}-1\right], \qquad (1\text{-}5)$$

if $\gcd\left(x,\frac{(e-f)}{3}\right)=\frac{(e-f)}{3}$ then from (1-4) we have

$$x=\frac{(e-f)}{\frac{5x^2+5(e-f)x+2(e-f)^2}{ef}+1}=\left(\frac{e-f}{3}\right)\frac{3}{\left(\frac{5x^2+5(e-f)x+2(e-f)^2}{ef}+1\right)},$$

and

$$\frac{x}{\left(\frac{e-f}{3}\right)}=\frac{3}{\left(\frac{5x^2+5(e-f)x+2(e-f)^2}{ef}+1\right)}$$

so

$$\frac{5x^2+5(e-f)x+2(e-f)^2}{ef}=2,$$

$$x=\left(\frac{e-f}{3}\right),$$

we get

$$\frac{5\left(\frac{e-f}{3}\right)^2+5(e-f)\left(\frac{e-f}{3}\right)+2(e-f)^2}{ef}=2,$$

and

$$\frac{ef}{(e-f)^2}=\frac{19}{9},$$

from (1-5) we have

$$\sqrt[3]{\frac{4+\frac{18ef}{(e-f)^2}+9\sqrt{3+\frac{27ef}{(e-f)^2}+\frac{61e^2f^2}{(e-f)^4}+\frac{e^3f^3}{(e-f)^6}}}{5}}+$$

$$,$$

$$\sqrt[3]{\frac{4+\frac{18ef}{(e-f)^2}-9\sqrt{3+\frac{27ef}{(e-f)^2}+\frac{61e^2f^2}{(e-f)^4}+\frac{e^3f^3}{(e-f)^6}}}{5}}=2$$

but

$$\sqrt[3]{\frac{4+\frac{18\times19}{9}+9\sqrt{3+\frac{27\times19}{9}+\frac{61\times19^2}{9^2}+\frac{19^3}{9^2}}}{5}}+$$

$$,$$

$$\sqrt[3]{\frac{4+\frac{18\times19}{9}-9\sqrt{3+\frac{27\times19}{9}+\frac{61\times19^2}{9^2}+\frac{19^3}{9^2}}}{5}}\neq 2$$

if $\gcd(x,e-f)=e-f$ then from (1-4) we have

$$\frac{x}{e-f}=\frac{1}{\frac{5x^2+5(e-f)x+2(e-f)^2}{ef}+1}<1,$$

that is impossible, so we have the conclusion of there are no positive integer solutions for equation (1-1).

2. Conclusion

The purpose of this paper is to illustrate using third order equation's root expression can solve some indeterminate equations.