KYUNGPOOK Math. J. 46(2006), 409-416

Smarandache Fresh and Clean Ideals of Smarandache BCI-algebras

Young Bae Jun
Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea
e-mail : ybjun@gsnu.ac.kr

Abstract. The notion of Smarandache fresh and clean ideals is introduced, examples are given, and related properties are investigated. Relations between Q-Smarandache fresh ideals and Q-Smarandache clean ideals are given. Extension properties for Q-Smarandache fresh ideals and Q-Smarandache clean ideals are established.

1. Introduction

Generally, in any human field, a *Smarandache Structure* on a set A means a weak structure $\mathbf{W}$ on A such that there exists a proper subset B of A which is embedded with a strong structure $\mathbf{S}$. In [4], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [3]. It will be very interesting to study the Smarandache structure in BCK/BCI-algebras. In [1], Y. B. Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigated some related properties. In this paper we deal with Smarandache ideal structures in Smarandache BCI-algebras. We introduce the notion of Smarandache fresh ideals and Smarandache clean ideals in Smarandache BCI-algebras, and investigate its useful properties. We give relations between Q-Smarandache fresh ideals and Q-Smarandache clean ideals. We also establish extension properties for Q-Smarandache fresh ideals and Q-Smarandache clean ideals.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the

Received March 24, 2005.
2000 Mathematics Subject Classification: 06F35, 03G25.
Key words and phrases: Smarandache BCI-algebra, Smarandache (fresh, clean) ideal.
This paper was supported by Korea Research Foundation Grant (KRF-2003-005-C00013).

following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)\ (x * x = 0)$,

(a4) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* We can define a partial order '$\leq$' on X by $x \leq y$ if and only if $x * y = 0$. Every BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.

(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

(b3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.

(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.

A *Smarandache BCI-algebra* [1] is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

(s1) $0 \in Q$ and $|Q| \geq 2$,

(s2) Q is a BCK-algebra under the operation of X.

3. Smarandache fresh ideals

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 3.1. A nonempty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, *Q-Smarandache ideal* of X) [1] if it satisfies:

(c1) $0 \in I$,

(c2) $(\forall x \in Q)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a *Smarandache ideal* of X.

Proposition 3.2. *If Q satisfies $Q * X \subset Q$, then every Q-Smarandache ideal I of X satisfies the following implication.*

(1) $$(\forall x, y \in I)\,(\forall z \in Q)\,((z * y) * x = 0 \Rightarrow z \in I).$$

Proof. Assume that $Q * X \subset Q$ and let I be a Q-Smarandache ideal of X. Suppose that $(z * y) * x = 0$ for all $x, y \in I$ and $z \in Q$. Then $z * y \in Q$ by assumption, and $(z * y) * x \in I$, and so $z * y \in I$ by (c2). Since $y \in I$, it follows from (c2) again that $z \in I$. This completes the proof. □

Problem 3.3. If a nonempty subset I of X satisfies the condition (1), then is I a Q-Smarandache ideal of X?

Definition 3.4. A nonempty subset I of X is called a *Smarandache fresh ideal* of X related to Q (or briefly, *Q-Smarandache fresh ideal* of X) if it satisfies the condition (c1) and

(c3) $(\forall x, y, z \in Q)\ ((x * y) * z \in I,\ y * z \in I \ \Rightarrow\ x * z \in I)$.

Example 3.5. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Table 3.1

Then $(X; *, 0)$ is a Smarandache BCI-algebra. Note that $Q = \{0, 1, 2, 3, 4\}$ is a BCK-algebra which is properly contained in X. It is easily checked that subsets $I_1 = \{0, 1, 3\}$ and $I_2 = \{0, 1, 2, 3\}$ are Q-Smarandache fresh ideals of X.

Theorem 3.6. Let Q_1 and Q_2 be BCK-algebras which are properly contained in X such that $Q_1 \subset Q_2$. Then every Q_2-Smarandache fresh ideal is a Q_1-Smarandache fresh ideal.

Proof. Straightforward. □

The following example shows that the converse of Theorem 3.6 is not true in general.

Example 3.7. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	4	4	5	1	0

Table 3.2

Then $(X; *, 0)$ is a Smarandache BCI-algebra. Note that $Q_1 = \{0, 1\}$ and $Q_2 = \{0, 1, 2, 3\}$ are BCK-algebras which are properly contained in X. It is easily checked that a subset $I = \{0, 2, 3\}$ is a Q_1-Smarandache fresh ideal of X, but it is not a Q_2-Smarandache fresh ideal of X since $(1 * 2) * 3 = 0 * 3 = 0 \in I$ and $2 * 3 = 2 \in I$ but $1 * 3 = 1 \notin I$.

Proposition 3.8. *If I is a Q-Smarandache fresh ideal of X, then*

(i) $(\forall x, y \in Q)\ ((x * y) * y \in I \Rightarrow x * y \in I)$.

(ii) $(\forall x, y, z \in Q)\ ((x * y) * z \in I \Rightarrow (x * z) * (y * z) \in I)$.

Proof. (i). Assume that $(x * y) * y \in I$ for all $x, y \in Q$. Since $y * y = 0 \in I$ by (a3) and (c1), it follows from (c3) that $x * y \in I$ which is the desired result.

(ii). Suppose that $(x * y) * z \in I$ for all $x, y, z \in Q$. Since

$$(((x * z) * (y * z)) * z) * ((x * y) * z) = 0 \in I,$$

we have $((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \in I$. Applying (i) and (b2), we conclude that $(x * z) * (y * z) = (x * (y * z)) * z \in I$. □

Theorem 3.9. *Every Q-Smarandache fresh ideal which is contained in Q is a Q-Smarandache ideal.*

Proof. Let I be a Q-Smarandache fresh ideal of X which is contained in Q and let $x \in Q$ and $y \in I$ be such that $x * y \in I$. Then $(x * y) * 0 = x * y \in I$ and $y * 0 = y \in I$. Since $x \in Q$ and $y \in I \subset Q$, it follows from (c3) and (b1) that $x = x * 0 \in I$ so that I is a Q-Smarandache ideal of X. □

The following example shows that the converse of Theorem 3.9 is not true in general.

Example 3.10. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	1	5
2	2	1	0	1	2	5
3	3	1	1	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Table 3.3

Then $(X;*,0)$ is a Smarandache BCI-algebra. Note that $Q=\{0,1,2,3,4\}$ is a BCK-algebra which is properly contained in X. It is easily checked that a subset $I=\{0,4\}$ is a Q-Smarandache ideal of X which is not a Q-Smarandache fresh ideal of X since $(2*1)*3=0\in I$ and $1*3=0\in I$, but $2*3=1\notin I$.

We provide conditions for a Q-Smarandache ideal to be a Q-Smarandache fresh ideal.

Theorem 3.11. *If I is a Q-Smarandache ideal of X such that*

$$(\forall x,y,z\in Q)\,((x*y)*z\in I \;\Rightarrow\; (x*z)*(y*z)\in I), \tag{2}$$

then I is a Q-Smarandache fresh ideal of X.

Proof. Assume that $(x*y)*z\in I$ and $y*z\in I$ for all $x,y,z\in Q$. Then $(x*z)*(y*z)\in I$ by (2), and so $x*z\in I$ by (c2). Therefore I is a Q-Smarandache fresh ideal of X. □

Corollary 3.12. *If I is a Q-Smarandache ideal of X that satisfies the following implication*:

$$(\forall x,y\in Q)\,((x*y)*y\in I \;\Rightarrow\; x*y\in I), \tag{3}$$

then I is a Q-Smarandache fresh ideal of X.

Proof. Let $x,y,z\in Q$ be such that $(x*y)*z\in I$. Since

$$(((x*z)*(y*z))*z)*((x*y)*z)=0\in I,$$

it follows from (b2) and (c2) that $((x*(y*z))*z)*z=((x*z)*(y*z))*z\in I$ so from (b2) and (3) that $(x*z)*(y*z)=(x*(y*z))*z\in I$. Hence I is a Q-Smarandache fresh ideal of X by Theorem 3.11. □

Proposition 3.13. *If I is a Q-Smarandache fresh ideal of X which is contained in Q, then*

$$(\forall x,y\in Q)\,(\forall z\in I)\,(((x*y)*y)*z\in I \;\Rightarrow\; x*y\in I). \tag{4}$$

Proof. Assume that $((x*y)*y)*z \in I$ for all $x, y \in Q$ and $z \in I$. If I is a Q-Smarandache fresh ideal of X which is contained in Q, then I is a Q-Smarandache ideal of X (see Theorem 3.9). Using (c2), we know that $(x*y)*y \in I$ and so $x*y \in I$ by Proposition 3.8(i). This completes the proof. □

Theorem 3.14 (Extension Property)**.** *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache fresh ideal of X, then so is J.*

Proof. Let $x, y, z \in Q$ be such that $(x*y)*z \in J$. Using (a3) and (b2), we have

$$((x*((x*y)*z))*y)*z = ((x*y)*z)*((x*y)*z) = 0 \in I.$$

Since I is a Q-Smarandache fresh ideal of X, it follows from Proposition 3.8(ii) and (b2) that

$$((x*z)*(y*z))*((x*y)*z) = ((x*((x*y)*z))*z)*(y*z) \in I \subset J$$

so from (c2) that $(x*z)*(y*z) \in J$. This proves that J is a Q-Smarandache fresh ideal of X. □

4. Smarandache clean ideals

Definition 4.1. A nonempty subset I of X is called a *Smarandache clean ideal* of X related to Q (or briefly, *Q-Smarandache clean ideal* of X) if it satisfies the condition (c1) and

(c4) $(\forall x, y \in Q)\ (\forall z \in I)\ ((x*(y*x))*z \in I \Rightarrow x \in I)$.

Example 4.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	5
2	2	1	0	1	0	5
3	3	3	3	0	0	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Table 4.1

Then $(X; *, 0)$ is a Smarandache BCI-algebra. Note that $Q = \{0, 1, 2, 3, 4\}$ is a BCK-algebra which is properly contained in X. It is easily checked that a subset $I = \{0, 1, 2, 3\}$ is a Q-Smarandache clean ideal of X.

Theorem 4.3. *Every Q-Smarandache clean ideal is a Q-Smarandache ideal.*

Proof. Let I be a Q-Smarandache clean ideal of X and let $x \in Q$ and $z \in I$ be such that $x*z \in I$. Taking $y = x$ in (c4), we have

$$(x*(x*x))*z = (x*0)*z = x*z \in I$$

and so $x \in I$ by (c4). Hence I is a Q-Smarandache ideal of X. □

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra described in Example 3.5. It is easily checked that a subset $I = \{0, 2\}$ is a Q-Smarandache ideal of X, but it is not a Q-Smarandache clean ideal of X since $(1 * (3 * 1)) * 2 = (1 * 1) * 2 = 0 * 2 = 0 \in I$ and $1 \notin I$.

Note that if Q is implicative, that is, Q satisfies the condition:

$$(\forall x, y \in Q)\,(x = x * (y * x)),$$

then every Q-Smarandache ideal is a Q-Smarandache clean ideal.

Theorem 4.5. *Every Q-Smarandache clean ideal is a Q-Smarandache fresh ideal.*

Proof. Let I be a Q-Smarandache clean ideal of X. Assume that $(x * y) * z \in I$ and $y * z \in I$ for all $x, y, z \in Q$. Note that

$$(((x * z) * z) * (y * z)) * ((x * y) * z) = (((x * z) * z) * (y * z)) * ((x * z) * y) = 0 \in I.$$

Since I is a Q-Smarandache ideal, it follows from (b4) and (c2) that

$$(x * z) * (x * (x * z)) = (x * (x * (x * z))) * z = (x * z) * z \in I$$

so from (b1) that $((x * z) * (x * (x * z))) * 0 = (x * z) * (x * (x * z)) \in I$. Thus $x * z \in I$ by (c4). This completes the proof. □

The converse of Theorem 4.5 is not true in general as seen in the following example.

Example 4.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra described in Example 3.5. Then $I := \{0, 1, 3\}$ is a Q-Smarandache fresh ideal, but not a Q-Smarandache clean ideal since $(2 * (4 * 2)) * 3 = 0$.

We provide a condition for a Q-Smarandache fresh ideal to be a Q-Smarandache clean ideal.

Theorem 4.7. *If a Q-Smarandache fresh ideal I of X which is contained in Q satisfies the following implication:*

$$(\forall x, y \in Q)\,(y * (y * x) \in I \;\Rightarrow\; x * (x * y) \in I), \tag{5}$$

then I is a Q-Smarandache clean ideal of X.

Proof. Let $x, y \in Q$ and $z \in I$ be such that $(x * (y * x)) * z \in I$. If I is a Q-Smarandache fresh ideal which is contained in Q, then I is a Q-Smarandache ideal (see Theorem 3.9). Thus $x * (y * x) \in I$ by (c2). Since

$$((y * (y * x)) * (y * x)) * (x * (y * x)) = 0 \in I,$$

it follows from (c2) that $(y*(y*x))*(y*x) \in I$ so from Proposition 3.8(i) that $y*(y*x) \in I$. Hence, by (5), $x*(x*y) \in I$. On the other hand, note that

$$((x*y)*z)*(x*(y*x)) = 0 \in I.$$

Using (c2), we get $(x*y)*z \in I$, and so $x*y \in I$. Therefore $x \in I$. This completes the proof. □

Proposition 4.8. *Every Q-Smarandache clean ideal I of X satisfies the condition* (5).

Proof. Let I be a Q-Smarandache clean ideal of X. Then I is a Q-Smarandache ideal of X (see Theorem 4.3). Assume that $y*(y*x) \in I$ for all $x, y \in Q$. Since

$$((x*(x*y))*(y*(x*(x*y))))*(y*(y*x)) = 0 \in I,$$

it follows from (b1) and (c2) that

$$((x*(x*y))*(y*(x*(x*y))))*0 = (x*(x*y))*(y*(x*(x*y))) \in I$$

so from (c4) that $x*(x*y) \in I$. This completes the proof. □

Theorem 4.9 (Extension Property)**.** *Let I and J be Q-Smarandache ideals of X and $I \subset J \subset Q$. If I is a Q-Smarandache clean ideal of X, then so is J.*

Proof. If I is a Q-Smarandache clean ideal of X, then I is a Q-Smarandache fresh ideal of X (see Theorem 4.5), and so J is a Q-Smarandache fresh ideal of X by Theorem 3.14. Let $x, y \in Q$ be such that $y*(y*x) \in J$. Since $(y*(y*x))*(y*(y*x)) = 0 \in I$, it follows from (b2) and Proposition 3.8(ii) that

$$(y*(y*(y*x)))*((y*(y*(y*x)))*x) = (y*(y*(y*x)))*((y*x)*(y*(y*x))) \in I$$

so from (b4) and Proposition 4.8 that

$$x*(x*(y*x)) = x*(x*(y*(y*(y*x)))) \in I \subset J.$$

Since $((x*(x*y))*(x*(x*(y*x))))*(y*(y*x)) = 0 \in J$, we get $x*(x*y) \in J$ by (c2). Using Theorem 4.7, we conclude that J is a Q-Smarandache clean ideal of X. □

References

[1] Y. B. Jun, *Smarandache BCI-algebras*, Sci. Math. Jpn., **62(1)**(2005), 137-142.

[2] J. Meng and Y. B. Jun, *BCK*-algebras, Kyungmoonsa Co. Seoul, Korea (1994).

[3] R. Padilla, *Smarandache algebraic structures*, Bull. Pure Appl. Sci., Delhi, **17E(1)**(1998), 119-121; http://www.gallup.unm.edu/ smarandache/alg-s-tx.txt.

[4] W. B. Vasantha Kandasamy, Smarandache groupoids; http://www.gallup.unm.edu/ smarandache/Groupoids.pdf.