Exploring Both the Shared and the Incompatible Characteristics of the X-Direction Galilean and Lorentz Transformations in a Common Framework

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Abstract

The x-direction Galilean and Lorentz space-time transformations are both effectively two-dimensional matrix transformations, so a simple four-parameter general framework of which both are special cases is easily devised. Moreover, passing from the general space-time transformation to its velocity counterpart uniquely singles out one of those four parameters as the general transformation's intrinsic x-direction constant velocity. This allows the "principle of relativity" to be extended to such general transformations; it applies when the transformation's inversion is accomplished by reversing the sign of its intrinsic velocity. Both the Galilean and Lorentz transformations abide by the "principle of relativity", and the Galilean transformation in addition refrains from altering the time coordinate. The Michelson-Morley null result, however, motivates the Lorentz transformation to refrain from changing the speed of light, which is readily shown to be outright incompatible with transformation-invariant time. The Lorentz transformation's pairing of invariant light speed with the "relativity principle" is closely allied to its preservation of the Minkowski quadratic form.

Introduction

The x-direction constant-velocity Galilean and Lorentz space-time transformations have a great deal in common: both are homogeneously linear transformations of the space-time coordinates (t, x, y, z) which are nontrivial *only* for the (t, x) pair, and therefore *both are of the general form*,

$$(t',\, x',\, y',\, z') = \left(\gamma_0 \left(t - \left(v_0/c^2\right) x\right),\, \gamma(x - vt),\, y,\, z\right), \tag{1a}$$

where γ_0 and γ are dimensionless parameters which are independent of the value of (t, x, y, z), while v_0 and v are parameters that have the dimension of velocity and are likewise independent of the value of (t, x, y, z). Note that *the homogeneously linear character* of the four-parameter general x-direction transformation of space-time given by Eq. (1a) *ensures coincidence of the space-time coordinate origins*, namely,

$$(t = 0,\, x = 0,\, y = 0,\, z = 0) \text{ transforms to } (t' = 0,\, x' = 0,\, y' = 0,\, z' = 0). \tag{1b}$$

The transformation of *velocity* which *corresponds* to the four-parameter general x-direction homogeneously linear transformation of *space-time* given by Eq. (1a) is,

$$(dx'/dt',\, dy'/dt',\, dz'/dt') = \frac{(dx'/dt,\, dy'/dt,\, dz'/dt)}{dt'/dt} = \frac{(\gamma((dx/dt) - v),\, dy/dt,\, dz/dt)}{\gamma_0 \left(1 - \left(v_0/c^2\right)(dx/dt)\right)}. \tag{2a}$$

Eq. (2a) shows that the transformation of *velocity* is in general a *rational* transformation rather than a *linear* one. In order to *ensure* that the rational *velocity* transformation given by Eq. (2a) *is well-defined*, we impose the following *two restrictions*,

$$\gamma_0 \neq 0, \tag{2b}$$

and,

$$|dx/dt| < \left(c^2/|v_0|\right). \tag{2c}$$

We now take note of *a key property* of the Eq. (2a) *transformation of velocity* (which of course *corresponds* to the Eq. (1a) *transformation of space-time*), namely,

$$(dx/dt,\, dy/dt,\, dz/dt) = (v, 0, 0) \text{ implies that } (dx'/dt',\, dy'/dt',\, dz'/dt') = (0, 0, 0). \tag{2d}$$

The result given by Eq. (2d) shows that the four-parameter general *x-direction transformation* described by Eq. (1a) or (2a) *expressly compensates for the x-direction constant velocity* $(v, 0, 0)$. Therefore, as long as,

$$|v| < \left(c^2/|v_0|\right), \tag{2e}$$

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in accord with the Eq. (2c) restriction, $(v, 0, 0)$ can be *identified* as the x-direction constant-velocity "boost" which is *intrinsic* to the Eq. (1a) or (2a) four-parameter general *x-direction transformation*.

The simplest transformation of velocity: the Galilean case

The *simplest* conceivable transformation of velocity $(dx/dt,\ dy/dt,\ dz/dt)$ which possesses *the essential intrinsic* $(v, 0, 0)$ *x-direction constant-velocity compensation property* noted in Eq. (2d) is of course the straightforward *velocity-difference transformation*,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = (dx/dt,\ dy/dt,\ dz/dt) - (v, 0, 0) = (((dx/dt) - v),\ dy/dt,\ dz/dt). \tag{3a}$$

Galileo *motivated* the Eq. (3a) *velocity-difference* transformation by considering a boat moving at the constant velocity $(v, 0, 0)$ relative to a wharf, and *also* considering a person aboard the boat moving at velocity $(dx/dt,\ dy/dt,\ dz/dt)$ *relative to that wharf*. The person aboard the boat would be moving at velocity $(dx'/dt',\ dy'/dt',\ dz'/dt') = (dx/dt,\ dy/dt,\ dz/dt) - (v, 0, 0)$ *relative to the floor (and the other fixed structure) of the boat*.

Comparison of Eq. (3a) with the four-parameter *general* x-direction transformation of *velocity* given by Eq. (2a) reveals *the three fixed parameter values* $v_0 = 0$, $\gamma_0 = 1$ *and* $\gamma = 1$. *Insertion* of these parameter values into the Eq. (1a) *general* x-direction homogeneously linear transformation of *space-time* yields,

$$(t',\ x',\ y',\ z') = (t,\ (x - vt),\ y,\ z), \tag{3b}$$

which is *readily recognized* as the x-direction constant-velocity *Galilean space-time transformation. A salient feature* of the Eq. (3b) Galilean space-time transformation is $t' = t$, namely *the universality of time*, which when *imposed* on the Eq. (1a) four-parameter *general* space-time transformation, *compels the two fixed parameter values* $v_0 = 0$ *and* $\gamma_0 = 1$. *Another salient feature* of the Eq. (3b) Galilean space-time transformation is that its *inversion*, namely,

$$(t,\ x,\ y,\ z) = (t',\ (x' + vt'),\ y',\ z'), \tag{3c}$$

can be accomplished *by reversing the sign of its intrinsic constant velocity* $(v, 0, 0)$, namely $(v, 0, 0) \to (-v, 0, 0)$, which is referred to as "the principle of relativity of constant-velocity motion"—*swapping the boat and wharf of Galileo's example* is of course accomplished by $(v, 0, 0) \to (-v, 0, 0)$.

Combining the *imposition* of this "principle of relativity" on the Eq. (1a) four-parameter *general* space-time transformation *with the above-mentioned imposition on it of the* $t' = t$ *constraint of the universality of time* compels the fixed parameter value $\gamma = 1$ *in addition to the two fixed parameter values* $v_0 = 0$ *and* $\gamma_0 = 1$. Thus the Eq. (1a) four-parameter *general* space-time transformation is *compelled* to become the *Galilean* space-time transformation if the $t' = t$ constraint of the universality of time *and* the "principle of relativity" are *both* imposed on it. *Alternatively*, we have of course noted above that the *simplest* conceivable transformation of *velocity*, which is the Eq. (3a) *velocity-difference* transformation, is *as well* the *Galilean* transformation of velocity.

Galileo's velocity difference, or an immutable speed of light?

Although there is much empirical evidence in favor of the validity of the Galilean velocity-difference transformation of Eq. (3a) *from feasible ordinary-precision velocity measurements* which test the like of Galileo's expectations of boat-passenger velocity relative to both the boat and a wharf, *the ultra-high-precision null results of Michelson-Morley type experiments support a speed of light which physically correct velocity transformations are incapable of changing* [1]. Unfortunately, the Galilean velocity-difference transformations of Eq. (3a) *are in fact all too capable of changing the speed of light*.

For example, if,

$$(dx/dt,\ dy/dt,\ dz/dt) = (0, c, 0),$$

which, inter alia, ensures that,

$$\left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\right)^{\frac{1}{2}} = c,$$

we obtain from Eq. (3a) that,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = (0, c, 0) - (v, 0, 0) = (-v, c, 0),$$

which has the upshot,

$$\left((dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2\right)^{\frac{1}{2}} = \left(v^2 + c^2\right)^{\frac{1}{2}}.$$

Thus the Galilean velocity-difference transformations of Eq. (3a) *are capable of increasing the speed of light to an arbitrary extent instead of being incapable of changing that speed*!

Universal time, or an immutable speed of light?

It isn't *only* the Galilean velocity-difference transformations of Eq. (3a) which are capable of changing the speed of light; *all* Eq. (1a) space-time transformations that *constrain t' to equal t*, namely *that enforce the universality of time* permit Eq. (2a) *velocity* transformations which are capable of changing the speed of light. We have noted in the discussion below Eq. (3b) that Eq. (1a) space-time transformations that constrain t' to equal t have the two fixed parameter values $v_0 = 0$ and $\gamma_0 = 1$. It therefore follows that their corresponding Eq. (2a) *velocity* transformations are given by,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = (\gamma((dx/dt) - v),\ dy/dt,\ dz/dt). \tag{4a}$$

If in Eq. (4a) $\gamma \neq 0$, we proceed exactly as in the immediately preceding section, namely we select,

$$(dx/dt,\ dy/dt,\ dz/dt) = (0, c, 0), \tag{4b}$$

which, inter alia, of course ensures that,

$$\left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\right)^{\frac{1}{2}} = c. \tag{4c}$$

In consequence of Eq. (4b), Eq. (4a) yields,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = (-\gamma v,\ c,\ 0), \tag{4d}$$

which has the upshot,

$$\left((dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2\right)^{\frac{1}{2}} = \left((\gamma v)^2 + c^2\right)^{\frac{1}{2}}. \tag{4e}$$

Thus the velocity transformation of Eq. (4a) is clearly capable of changing the speed of light if $\gamma \neq 0$.

However, if in Eq. (4a) $\gamma = 0$, we *instead* select,

$$(dx/dt,\ dy/dt,\ dz/dt) = (c, 0, 0), \tag{4f}$$

which, inter alia, of course ensures that,

$$\left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)\right)^{\frac{1}{2}} = c. \tag{4g}$$

In consequence of Eq. (4f), Eq. (4a) with $\gamma = 0$ yields,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = (0, 0, 0), \tag{4h}$$

which has the upshot,

$$\left((dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2\right)^{\frac{1}{2}} = 0. \tag{4i}$$

Thus if $\gamma = 0$ in Eq. (4a), that velocity transformation is capable of changing speed c of light to speed zero.

The principle of relativity plus an immutable speed of light

The ultra-high-precision Michelson-Morley null results drive us to seek values of the three Eq. (1a) and (2a) transformation parameters v_0, γ_0 and γ *which don't change the speed of light*. At *the same time*, Occam's razor and conservatism in theoretical physics impel us *to salvage as many attributes of the Galilean transformation as feasible*. The immediately preceding section has shown us that in the context of the Eq. (1a) and (2a) general transformations we *cannot* salvage the Galilean universality of time, $t' = t$. The *remaining* salient attribute of the Galilean transformation is the "principle of relativity", namely that the *inversion* of the transformation can be accomplished by reversing the sign of its *intrinsic* x-direction constant velocity $(v, 0, 0)$. This is clearly *a basic symmetry requirement*, a profoundly *self-evident* one whose *violation* seems all but *impossible to justify by adducing physical arguments*, so its pursuit now becomes our top priority. The *first* step to undertake is the *inversion* of the Eq. (1a) four-parameter *general* x-direction homogeneously linear space-time transformation, with the result,

$$(t,\, x,\, y,\, z) = \left(\frac{\left(t' + (\gamma_0/\gamma)\left(v_0/c^2\right) x'\right)}{\left(\gamma_0 \left(1 - (vv_0/c^2)\right)\right)},\, \frac{\left(x' + (\gamma/\gamma_0)\, vt'\right)}{\left(\gamma \left(1 - (vv_0/c^2)\right)\right)},\, y',\, z' \right). \tag{5a}$$

For the Eq. (5a) *inversion* of Eq. (1a) to be accomplished *by the sign reversal* $v \to -v$, it is *necessary* that,

$$\gamma_0 = \gamma = \pm\left(1 - \left(vv_0/c^2\right)\right)^{-\frac{1}{2}}, \text{ and } \textit{also} \text{ that } v_0 \textit{ be an odd function of } v. \tag{5b}$$

The $\pm$ sign ambiguity which appears in Eq. (5b) for γ and γ_0 *must be resolved in favor of* $\pm = +$ in order for Eqs. (1a) and (5a) to reduce to *the identity space-time transformation* as $v \to 0$. Note that the Eq. (5b) upshot of the "principle of relativity" *is fully compatible with the Galilean transformation*: with $\pm = +$, the parameter choice $v_0 = 0$, which makes v_0 a (trivial) *odd function of* v, causes Eq. (5b) *to in addition yield the two parameter values* $\gamma_0 = \gamma = 1$*, which completes the entire parameter-value description of the Galilean transformation*.

Having obtained the Eq. (5b) *pair* of parameter-value consequences of the "principle of relativity" from the Eq. (1a) *space-time* transformation, we now turn to the Eq. (2a) *velocity* transformation to obtain the *third* parameter value from the Michelson-Morley null-result requirement that the Eq. (2a) velocity transformation *must not change the speed of light*. That third parameter value is quickly obtained by inserting into Eq. (2a) *two distinct advantageous velocity values whose magnitudes equal* c, specifically,

$$(dx/dt,\; dy/dt,\; dz/dt) = (\pm c,\; 0,\; 0). \tag{6a}$$

We also *simplify* Eq. (2a) by inserting into it the Eq. (5b) result that $\gamma_0 = \gamma$. The *two outputs* of this *simplified* Eq. (2a) *which arise from the two* Eq. (6a) *inputs* are,

$$(dx'/dt',\; dy'/dt',\; dz'/dt') = \left(\pm c \left(\frac{1 \mp (v/c)}{1 \mp (v_0/c)} \right),\; 0,\; 0 \right). \tag{6b}$$

Since Eq. (2a), simplified *in accord with* Eq. (5b), *must not change the speed of light*, we conclude from Eqs. (6a) and (6b) that,

$$(1 \mp (v_0/c))^2 = (1 \mp (v/c))^2, \tag{6c}$$

which is a set of *two quadratic equations for the parameter* v_0, whose *unique shared root* is,

$$v_0 = v. \tag{6d}$$

This result is *consistent* with the Eq. (5b) *requirement that* v_0 *be an odd function of* v. Insertion of Eq. (6d) into Eq. (5b) with $\pm = +$ yields the familiar Lorentz-transformation parameter values,

$$\gamma_0 = \gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}. \tag{6e}$$

We also note that insertion of Eq. (6d) into the *restriction* given by Eq. (2e) yields the familiar Lorentz-transformation "speed limit" $|v| < c$, which is obviously *necessary* in view of Eq. (6e).

Insertion of the Eq. (6d) and (6e) parameter values into Eq. (1a) reveals the familiar $(v, 0, 0)$-"boost" Lorentz transformation of space-time, namely,

$$(t',\, x',\, y',\, z') = \left(\gamma\left(t - \left(v/c^2\right) x\right),\, \gamma(x - vt),\, y,\, z\right), \tag{6f}$$

where,

$$\gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}. \tag{6g}$$

An important characteristic of the $(v, 0, 0)$-"boost" Lorentz transformation of space-time of Eqs. (6f) and (6g) is that *it preserves the Minkowski quadratic form* $(ct)^2 - x^2 - y^2 - z^2$, namely,

$$\begin{aligned} (ct')^2 - (x')^2 - (y')^2 - (z')^2 &= \gamma^2\left[(ct - (v/c)x)^2 - (x - vt)^2\right] - y^2 - z^2 = \\ &\gamma^2\left[(ct)^2\left(1 - (v/c)^2\right) - x^2\left(1 - (v/c)^2\right)\right] - y^2 - z^2 = \\ &\gamma^2\left[(ct)^2\gamma^{-2} - x^2\gamma^{-2}\right] - y^2 - z^2 = (ct)^2 - x^2 - y^2 - z^2. \end{aligned} \tag{6h}$$

The Eq. (6d) and (6e) parameter values put into Eq. (2a) yield the Lorentz transformation of *velocity*,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = \frac{(\gamma((dx/dt) - v),\ dy/dt,\ dz/dt)}{\gamma\left(1 - (v/c^2)\,(dx/dt)\right)}. \tag{7a}$$

Combining the *restrictions* of Eqs. (2c) and (2e) with the Eq. (6d) value of the parameter v_0, namely $v_0 = v$, permits one to conclude that in Eq. (7a), $|dx/dt| \le c$.

Comparison of the Eq. (7a) Lorentz transformation of velocity with the simple Eq. (3a) Galilean transformation of velocity reveals that their failure to agree *only* encompasses effects that are of *at least second order in the ratios of the speeds involved to the speed of light*! It is little wonder, then, that the Galilean transformation *is so utterly adequate* for matters which involve boats and wharves, or even satellites and space vehicles. One notable exception to this state of affairs occurs when the mind-boggling accuracy which can be achieved by atomic clocks is brought into play, as is the case for GPS applications. The properties of the Galilean transformation are notably *intuitive*, whereas the rigid preservation of the speed of light by the Lorentz transformation—and the consequences of that preservation—are notably *counterintuitive*; these facts are of course reflections *of the complete absence over eons* of any experience by terrestrial creatures *of relative velocities whose magnitudes approach the speed of light*.

It is straightforward, if somewhat tedious, *to demonstrate conclusively* that the Eq. (7a) Lorentz transformation of velocity *doesn't change the speed of light*; the initial series of steps of that demonstration *very closely parallel* the steps of the Eq. (6h) demonstration that the Lorentz transformation of space-time preserves the Minkowski quadratic form $(ct)^2 - x^2 - y^2 - z^2$; those initial steps are as follows,

$$\begin{aligned}
&\left[c^2 - (dx'/dt')^2 - (dy'/dt')^2 - (dz'/dt')^2\right]\left(\gamma\left(1 - \left(v/c^2\right)(dx/dt)\right)\right)^2 = \\
&\gamma^2\left[(c - (v/c)(dx/dt))^2 - ((dx/dt) - v)^2\right] - (dy/dt)^2 - (dz/dt)^2 = \\
&\gamma^2\left[c^2\left(1 - (v/c)^2\right) - (dx/dt)^2\left(1 - (v/c)^2\right)\right] - (dy/dt)^2 - (dz/dt)^2 = \\
&\gamma^2\left[c^2\gamma^{-2} - (dx/dt)^2\gamma^{-2}\right] - (dy/dt)^2 - (dz/dt)^2 = c^2 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2.
\end{aligned} \tag{7b}$$

Dividing the Eq. (7b) result by the factor $\left(\gamma\left(1 - \left(v/c^2\right)(dx/dt)\right)\right)^2$ produces,

$$c^2 - (dx'/dt')^2 - (dy'/dt')^2 - (dz'/dt')^2 = \frac{\left[c^2 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2\right]}{\left(\gamma\left(1 - (v/c^2)\,(dx/dt)\right)\right)^2}. \tag{7c}$$

From Eq. (7c) it is apparent that,

$$\text{If } \left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\right)^{\frac{1}{2}} = c, \text{ then } \left((dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2\right)^{\frac{1}{2}} = c. \tag{7d}$$

Eq. (7d) shows conclusively that the Eq. (7a) Lorentz transformation of velocity *doesn't change the speed of light*. As a matter of fact, it is *as well* apparent from Eq. (7c) that,

$$\text{If } \left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\right)^{\frac{1}{2}} < c, \text{ then } \left((dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2\right)^{\frac{1}{2}} < c. \tag{7e}$$

Therefore the Eq. (7a) Lorentz transformation of velocity leaves any entity which is traveling at *less* than the speed of light *still* traveling at *less* than the speed of light.

References

[1] Michelson-Morley experiment–Wikipedia,
https://en.wikipedia.org/wiki/Michelson%E2%80%93Morley_experiment.