On the integral: $I=\int_0^1 \coth\left(1+x^3\right)dx$

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abstract

This note presents some formulas related with the integral:

$$I=\int_0^1 \coth\left(1+x^3\right)dx$$

1. Some identities.

$$I=\int_0^1 \coth\left(1+x^3\right)dx=\int_0^1 \frac{1}{\tanh\left(1+x^3\right)}dx=\int_0^1 \frac{1+e^{-2-2x^3}}{1-e^{-2-2x^3}}dx \qquad (1)$$

$$I=\coth 2+\int_{\coth 2}^{\coth 1} \sqrt[3]{-1+\coth^{-1} x}\,dx=\coth 2+\int_{\coth 2}^{\coth 1} \sqrt[3]{-1+\tanh^{-1}\frac{1}{x}}\,dx \qquad (2)$$

$$I=\frac{1}{3}\int_0^1 \frac{x^{-2/3}}{\tanh(1+x)}dx=\frac{1}{3}\int_1^2 \frac{(x-1)^{-2/3}}{\tanh x}dx \qquad (3)$$

$$I=\frac{1}{1-e^{-4}}+\frac{1}{e^4-1}+\int_{\left(1-e^{-4}\right)^{-1}}^{\left(1-e^{-2}\right)^{-1}} \sqrt[3]{-1+\frac{1}{2}\ln\left(\frac{x}{x-1}\right)}\,dx+\int_{\left(e^4-1\right)^{-1}}^{\left(e^2-1\right)^{-1}} \sqrt[3]{-1+\frac{1}{2}\ln\left(1+\frac{1}{x}\right)}\,dx \qquad (4)$$

2. Some series.

$$I=\frac{\ln 2}{3}+\frac{\pi}{3\sqrt{3}}+\sum_{n=1}^{\infty}\frac{(-1)^n\, 2^{2n} B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{1}{3k+1} \qquad (5)$$

Remark 1: B_n , Bernoulli numbers,

$$B_n=\left\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\frac{691}{2730},\frac{7}{6},\ldots\right\} \qquad (6)$$

$$I=1+2\sum_{n=1}^{\infty} e^{-2n}\int_0^1 e^{-2nx^3}dx \qquad (7)$$

Remark 2:

$$\int_0^1 e^{-2nx^3}dx = \frac{3\cdot 2^{5/6}}{8n^{1/6}} e^{-n} M\left(\frac{1}{6},\frac{2}{3},2n\right) + \frac{2^{5/6}}{4n^{7/6}} e^{-n} M\left(\frac{7}{6},\frac{2}{3},2n\right) \quad ,n\geq 1$$

$M := M(\mu,\nu,z)$ Whittaker function

$$M(\mu,\nu,z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\nu} F\left(\frac{1}{2}+\nu-\mu;1+2\nu;z\right)$$

$F(a;b;z)$ is the hypergeometric function.

$$I = 1 + 2\sum_{n=1}^{\infty} e^{-2n}\, {}_3F_3\left(\frac{1}{3},\frac{2}{3},1;\frac{2}{3},1,\frac{4}{3};-2n\right) \tag{8}$$

Remark 3: ${}_3F_3$ is the hypergeometric function.

$$I = 1 + \frac{2}{3\sqrt[3]{2}}\Gamma\left(\frac{1}{3}\right)\sum_{n=1}^{\infty} e^{-2n}\sqrt[3]{\frac{1}{n}} - \frac{2}{3\sqrt[3]{2}}\sum_{n=1}^{\infty} e^{-2n}\sqrt[3]{\frac{1}{n}}\,\Gamma\left(\frac{1}{3},2n\right) \tag{9}$$

Remark 4: $\Gamma(x)$ is the gamma function, $\Gamma(a,x)$ is the incomplete gamma function.

$$I = 1 + \frac{2}{3\sqrt[3]{2}}\Gamma\left(\frac{1}{3}\right)\sum_{n=1}^{\infty} e^{-2n}\sqrt[3]{\frac{1}{n}} - \frac{2}{3\sqrt[3]{2}}\sum_{n=1}^{\infty} e^{-4n} f(n) \tag{10}$$

$$f(n) = \cfrac{1}{2n + \cfrac{2/3}{1 + \cfrac{1}{2n + \cfrac{5/3}{1 + \cfrac{2}{2n + \cfrac{8/3}{1+\dots}}}}}} = \cfrac{1}{2n + \cfrac{2}{3 + \cfrac{3}{2n + \cfrac{5}{3 + \cfrac{6}{2n + \cfrac{8}{3+\dots}}}}}} \tag{11}$$

$$I = \frac{e^4+1}{e^4-1}\sum_{n=0}^{\infty} c_n \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{3k+1} \tag{12}$$

$$c_0 = 1 \quad , c_n = \frac{2^n}{(e^4+1)n!} + \frac{1}{(e^4-1)}\sum_{k=1}^{n}\frac{2^k}{k!}c_{n-k} \quad ,n\in\mathbb{N} \tag{13}$$

References

1. Boros, G. and Moll, V.H.: Irresistible Integrals, Cambridge University Press: UK, USA,2004.
2. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products. 5th. ed., ed. Alan Jeffrey. Academic Press, 1994.