A Remark on the Localization formulas about two Killing vector fields

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Abstract

In this article, we will discuss a localization formulas of equlvariant cohomology about two Killing vector fields on the set of zero points $\mathrm{Zero}(X_M - \sqrt{-1}Y_M) = \{x \in M \mid |Y_M(x)| = |X_M(x)| = 0\}$. As application, we use it to get formulas about characteristic numbers and to get a Duistermaat-Heckman type formula on symplectic manifold.

The localization theorem for equivariant differential forms was obtained by Berline and Vergne(see [3]). They discuss on the zero points of a Killing vector field, the localization formula expresses the integral of an equivariantly closed differential form as an integral over the set of zeros of the Killing vector field. The de Rham model for equivariant cohomology give a deeper understanding of equivariant differential forms(see [1]). In [6], we introduce the equlvariant cohomology about two Killing vector fields and to establish a localization formulas on the set of zero points

$$\mathrm{Zero}(X_M + \sqrt{-1}Y_M) = \{x \in M \mid \langle X_M(x), Y_M(x)\rangle = 0, |Y_M(x)| = |X_M(x)|\}.$$

For gaining a deeper understanding of equlvariant cohomology about two Killing vector fields, we introduce the Cartan model for equlvariant cohomology about two Killing vector fields(see [7]).

In this article, we will to establish a localization formulas of equlvariant cohomology about two Killing vector fields on the set of zero points

$$\mathrm{Zero}(X_M - \sqrt{-1}Y_M) = \{x \in M \mid |Y_M(x)| = |X_M(x)| = 0\}.$$

We will see that the set of zero points $\mathrm{Zero}(X_M - \sqrt{-1}Y_M)$ is smaller and more basic. As application, we use the localization formulas to get formulas about characteristic numbers and to get a Duistermaat-Heckman type formula on symplectic manifold.

1 Equlvariant cohomology by two Killing vector fields

First, let us review the definition of equlvariant cohomology about two Killing vector fields. Let M be a smooth closed oriented manifold. Let G be a compact Lie group acting smoothly on M, and let $\mathfrak{g}$ be its Lie algebra. Let g^{TM} be a G-invariant metric on TM. Let $\Omega^*(M)$ be the space of smooth differetial forms on M, the de Rham complex is $(\Omega^*(M), d)$.

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Let $\Omega^*(M)\otimes_{\mathbb{R}}\mathbb{C}$ be the space of smooth complex-valued differetial forms on M. If $X, Y\in\mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M given by

$$(X_M f)(x) = \frac{d}{dt} f(\exp(-tX)\cdot x)\mid_{t=0} .$$

If $X, Y\in\mathfrak{g}$, then X_M, Y_M are Killing vector field. Let L_{X_M} be the Lie derivative of X_M on $\Omega^*(M)$, i_{X_M} be the interior multiplication induced by the contraction of X_M.

Set

$$L_{X_M+\sqrt{-1}Y_M} \doteq L_{X_M} + \sqrt{-1}L_{Y_M}$$

be the operator on $\Omega^*(M)\otimes_{\mathbb{R}}\mathbb{C}$.

Set

$$i_{X_M+\sqrt{-1}Y_M} \doteq i_{X_M} + \sqrt{-1}i_{Y_M}$$

be the interior multiplication induced by the contraction of $X_M+\sqrt{-1}Y_M$. It is also a operator on $\Omega^*(M)\otimes_{\mathbb{R}}\mathbb{C}$.

Set

$$d_{X+\sqrt{-1}Y} = d + i_{X_M+\sqrt{-1}Y_M}.$$

So

$$d^2_{X+\sqrt{-1}Y} = L_{X_M} + \sqrt{-1}L_{Y_M} = L_{X_M+\sqrt{-1}Y_M}.$$

Let

$$\Omega^*_{X_M+\sqrt{-1}Y_M}(M) = \{\omega\in\Omega^*(M)\otimes_{\mathbb{R}}\mathbb{C} : L_{X_M+\sqrt{-1}Y_M}\omega = 0\}$$

be the space of smooth $(X_M+\sqrt{-1}Y_M)$-invariant forms on M. Then we get a complex $(\Omega^*_{X_M+\sqrt{-1}Y_M}(M), d_{X+\sqrt{-1}Y})$. We call a form ω is $d_{X+\sqrt{-1}Y}$-closed if $d_{X+\sqrt{-1}Y}\omega = 0$. The corresponding cohomology group

$$H^*_{X+\sqrt{-1}Y}(M) = \frac{\mathrm{Ker} d_{X+\sqrt{-1}Y}|_{\Omega^*_{X+\sqrt{-1}Y}(M)}}{\mathrm{Im} d_{X+\sqrt{-1}Y}|_{\Omega^{*-1}_{X+\sqrt{-1}Y}(M)}}$$

is called the equivariant cohomology associated with $X_M+\sqrt{-1}Y_M$. By the same way, we can define the equivariant cohomology about two vector fields (not Killing vector fields). We can see that, if we set $Y_M = 0$, then we get the equivariant cohomology as normal.

For any $\omega\in\Omega^*(M)\otimes_{\mathbb{R}}\mathbb{C}$, we can write it by $\xi+\sqrt{-1}\eta$, where $\xi,\eta\in\Omega^*(M)$. By the definition of $d_{X+\sqrt{-1}Y}$-closed forms, we have $\omega=\xi+\sqrt{-1}\eta$ is $d_{X+\sqrt{-1}Y}$-closed if and only if $d\xi + i_{X_M}\xi - i_{Y_M}\eta = 0$ and $d\eta + i_{X_M}\eta + i_{Y_M}\xi = 0$. For a special case, we have the following result

Lemma 1. *$\omega=\xi+\sqrt{-1}\eta\in\Omega^*(M)\otimes_{\mathbb{R}}\mathbb{C}$ with ξ,η are m-forms, then ω is $d_{X+\sqrt{-1}Y}$-closed if and only if $d\xi=0, d\eta=0$ and $i_{X_M}\xi = i_{Y_M}\eta,\ i_{X_M}\eta = -i_{Y_M}\xi$.*

Proof. For $\omega=\xi+\sqrt{-1}\eta$, by the definition of $d_{X+\sqrt{-1}Y}$-closed forms, we have

$$d\xi + i_{X_M}\xi - i_{Y_M}\eta = 0,\quad d\eta + i_{X_M}\eta + i_{Y_M}\xi = 0,$$

and because ξ,η are m-forms, they have the same degree, so $d\xi = 0, d\eta = 0$ and $i_{X_M}\xi = i_{Y_M}\eta,\ i_{X_M}\eta = -i_{Y_M}\xi$.

If $d\xi = 0, d\eta = 0$ and $i_{X_M}\xi = i_{Y_M}\eta,\ i_{X_M}\eta = -i_{Y_M}\xi$; then we have

$$d\xi + i_{X_M}\xi - i_{Y_M}\eta = 0,\quad d\eta + i_{X_M}\eta + i_{Y_M}\xi = 0,$$

so $\omega=\xi+\sqrt{-1}\eta$ is $d_{X+\sqrt{-1}Y}$-closed forms. $\square$

The condition $i_{X_M}\xi = i_{Y_M}\eta,\ i_{X_M}\eta = -i_{Y_M}\xi$ looks like the Cauchy-Riemann condition about holomorhpic functions.

Example 1. *If $f = u + \sqrt{-1}v$ is a holomorhpic functions on $\mathbb{C}$, by the Cauchy-Riemann condition one have*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Set $M = \mathbb{C}$, let $X_M = \frac{\partial}{\partial x}, Y_M = \frac{\partial}{\partial y}$, so by the Cauchy-Riemann condition we have

$$i_{\frac{\partial}{\partial x}}du = i_{\frac{\partial}{\partial y}}dv,\ i_{\frac{\partial}{\partial x}}dv = -i_{\frac{\partial}{\partial y}}du$$

Then by Lemma 1., df is a $d_{X+\sqrt{-1}Y}$-closed forms.

2 Some special $d_{X+\sqrt{-1}Y}$-closed forms

In this section, we will give four special $d_{X+\sqrt{-1}Y}$-closed forms, $d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y')$, $d_{X+\sqrt{-1}Y}(Y' - \sqrt{-1}X')$, $d_{X+\sqrt{-1}Y}(X' - \sqrt{-1}Y')$ and $d_{X+\sqrt{-1}Y}(Y' + \sqrt{-1}X')$.

Lemma 2. *If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M, X', Y' be the 1-form on M which is dual to X_M, Y_M by the metric g^{TM}, then*

$$L_{X_M}Y' + L_{Y_M}X' = 0$$

Proof. Because

$$(L_{X_M}\omega)(Z) = X_M(\omega(Z)) - \omega([X_M, Z])$$

here $Z \in \Gamma(TM)$, So we get

$$(L_{X_M}Y')(Z) = X_M\langle Y_M, Z\rangle - \langle [X_M, Z], Y_M\rangle$$

$$(L_{Y_M}X')(Z) = Y_M\langle X_M, Z\rangle - \langle [Y_M, Z], X_M\rangle.$$

Because X_M, Y_M are Killing vector fields, so(see [11])

$$\begin{aligned} X_M\langle Y_M, Z\rangle &= \langle L_{X_M}Y_M, Z\rangle + \langle Y_M, L_{X_M}Z\rangle \\ &= \langle [X_M, Y_M], Z\rangle + \langle Y_M, [X_M, Z]\rangle \end{aligned}$$

$$\begin{aligned} Y_M\langle X_M, Z\rangle &= \langle L_{Y_M}X_M, Z\rangle + \langle X_M, L_{Y_M}Z\rangle \\ &= \langle [Y_M, X_M], Z\rangle + \langle X_M, [Y_M, Z]\rangle \end{aligned}$$

then we get

$$(L_{X_M}Y' + L_{Y_M}X')(Z) = \langle [X_M, Y_M], Z\rangle + \langle [Y_M, X_M], Z\rangle = 0$$

□

Lemma 3. *If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M, X', Y' be the 1-form on M which is dual to X_M, Y_M by the metric g^{TM}, then*

1) $d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y')$

2) $d_{X+\sqrt{-1}Y}(Y' - \sqrt{-1}X')$

are the $d_{X+\sqrt{-1}Y}$-closed forms.

Proof.

$$\begin{aligned} d^2_{X+\sqrt{-1}Y}(X^{'}+\sqrt{-1}Y^{'}) &= (L_{X_M}+\sqrt{-1}L_{Y_M})(X^{'}+\sqrt{-1}Y^{'}) \\ &= L_{X_M}X^{'} - L_{Y_M}Y^{'} + \sqrt{-1}(L_{X_M}Y^{'} + L_{Y_M}X^{'}) \\ &= 0 \end{aligned}$$

So $d_{X+\sqrt{-1}Y}(X^{'}+\sqrt{-1}Y^{'})$ is the $d_{X+\sqrt{-1}Y}$-closed form;

$$\begin{aligned} d^2_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}) &= (L_{X_M}+\sqrt{-1}L_{Y_M})(Y^{'}-\sqrt{-1}X^{'}) \\ &= L_{X_M}Y^{'} + L_{Y_M}X^{'} + \sqrt{-1}(L_{Y_M}Y^{'} - L_{X_M}X^{'}) \\ &= 0 \end{aligned}$$

So $d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'})$ is the $d_{X+\sqrt{-1}Y}$-closed form. □

Lemma 4. *If $X, Y \in \mathfrak{g}$ and with $[X, Y] = 0$, then $[X_M, Y_M] = 0$.*

Proof. Because $[X, Y] = 0$, so we have

$$\exp(-tX)\exp(-sY) = \exp(-sY)\exp(-tX)$$

where $s, t \in \mathbb{R}$, and for any $f \in C^\infty(M)$

$$([X_M, Y_M]f)(p) = (X_M(Y_Mf) - Y_M(X_Mf))(p)$$

$$= \frac{\partial^2}{\partial t\partial s}\bigg|_{s=t=0} f\Big(\exp(-tX)\exp(-sY)\cdot p\Big) - \frac{\partial^2}{\partial s\partial t}\bigg|_{s=t=0} f\Big(\exp(-sY)\exp(-tX)\cdot p\Big) = 0$$

So we get $[X_M, Y_M] = 0$. □

Lemma 5. *If $X, Y \in \mathfrak{g}$ with $[X, Y] = 0$, let X_M, Y_M be the corresponding smooth vector field on M, $X^{'}, Y^{'}$ be the 1-form on M which is dual to X_M, Y_M by the metric g^{TM}, then*

$$L_{X_M}Y^{'} = 0, \ L_{Y_M}X^{'} = 0.$$

Proof. Because $[X, Y] = 0$, by Lemma 4., $[X_M, Y_M] = 0$, then

$$(L_{X_M}Y^{'})(Z) = \langle [X_M, Y_M], Z\rangle + \langle Y_M, [X_M, Z]\rangle - \langle [X_M, Z], Y_M\rangle = 0,$$

$$(L_{Y_M}X^{'})(Z) = \langle [Y_M, X_M], Z\rangle + \langle X_M, [Y_M, Z]\rangle - \langle [Y_M, Z], X_M\rangle = 0.$$

□

Lemma 6. *If $X, Y \in \mathfrak{g}$ with $[X, Y] = 0$, let X_M, Y_M be the corresponding smooth vector field on M, $X^{'}, Y^{'}$ be the 1-form on M which is dual to X_M, Y_M by the metric g^{TM}, then*

1) $d_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'})$

2) $d_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'})$

are the $d_{X+\sqrt{-1}Y}$-closed forms.

Proof. Because $[X,Y]=0$, by Lemma 5., we have $L_{X_M}Y^{'}=0,\ L_{Y_M}X^{'}=0$;

$$\begin{aligned}d^2_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'}) &= (L_{X_M}+\sqrt{-1}L_{Y_M})(X^{'}-\sqrt{-1}Y^{'})\\ &= L_{X_M}X^{'}+L_{Y_M}Y^{'}+\sqrt{-1}(L_{Y_M}X^{'}-L_{X_M}Y^{'})\\ &= 0\end{aligned}$$

So $d_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'})$ is the $d_{X+\sqrt{-1}Y}$-closed form.

$$\begin{aligned}d^2_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'}) &= (L_{X_M}+\sqrt{-1}L_{Y_M})(Y^{'}+\sqrt{-1}X^{'})\\ &= L_{X_M}Y^{'}-L_{Y_M}X^{'}+\sqrt{-1}(L_{X_M}X^{'}+L_{Y_M}Y^{'})\\ &= 0\end{aligned}$$

So $d_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'})$ is the $d_{X+\sqrt{-1}Y}$-closed form.

□

3 The set of zero points

In [6], we have get that for any $\eta\in H^*_{X+\sqrt{-1}Y}(M)$ and $s\geq 0$, we have

$$\int_M \eta=\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X^{'}+\sqrt{-1}Y^{'}))\}\eta.$$

Here we will give the same results about $d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'})$, $d_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'})$ and $d_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'})$.

Lemma 7. *For any $\eta\in H^*_{X+\sqrt{-1}Y}(M)$ and $s\geq 0$, we have*

1) $\int_M\eta=\int_M\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta$,

2) *When* $[X,Y]=0$, *then* $\int_M\eta=\int_M\exp\{-s(d_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'}))\}\eta$,

3) *When* $[X,Y]=0$, *then* $\int_M\eta=\int_M\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'}))\}\eta$.

Proof. For 1), because

$$\frac{\partial}{\partial s}\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta$$

$$=-\int_M(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta$$

and by assumption we have

$$d_{X+\sqrt{-1}Y}\eta=0$$
$$d_{X+\sqrt{-1}Y}\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}=0$$

So we get

$$\begin{aligned}&(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta\\ &=d_{X+\sqrt{-1}Y}[(Y^{'}-\sqrt{-1}X^{'})\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta]\end{aligned}$$

and by Stokes formula we have

$$\frac{\partial}{\partial s}\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta = 0$$

Then we get

$$\int_M \eta = \int_M \exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}))\}\eta$$

For 2) and 3), when $[X,Y]=0$, we have

$$d_{X+\sqrt{-1}Y}\exp\{-s(d_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'}))\} = 0,$$

$$d_{X+\sqrt{-1}Y}\exp\{-s(d_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'}))\} = 0,$$

so by the same way as in 1), we can get the results. □

For

$$d_{X+\sqrt{-1}Y}(Y^{'}-\sqrt{-1}X^{'}) = d(Y^{'}-\sqrt{-1}X^{'}) + \langle X_M+\sqrt{-1}Y_M, Y_M-\sqrt{-1}X_M\rangle$$

and

$$\langle X_M+\sqrt{-1}Y_M, Y_M-\sqrt{-1}X_M\rangle = 2\langle X_M, Y_M\rangle + \sqrt{-1}(|Y_M|^2-|X_M|^2)$$

Set

$$\mathrm{Zero}(Y_M-\sqrt{-1}X_M) = \{x\in M \mid \langle X_M(x)+\sqrt{-1}Y_M(x), Y_M(x)-\sqrt{-1}X_M(x)\rangle = 0\}.$$

We can see that

$$\mathrm{Zero}(Y_M-\sqrt{-1}X_M) = \{x\in M \mid \langle X_M(x), Y_M(x)\rangle = 0, |Y_M(x)| = |X_M(x)|\},$$

This set of zero points $\mathrm{Zero}(Y_M-\sqrt{-1}X_M)$ is the same as in [6]. This set of zero points is first discussed by H.Jacobowitz (see [8] and [9]).

For

$$d_{X+\sqrt{-1}Y}(X^{'}-\sqrt{-1}Y^{'}) = d(X^{'}-\sqrt{-1}Y^{'}) + \langle X_M+\sqrt{-1}Y_M, X_M-\sqrt{-1}Y_M\rangle$$

and

$$\langle X_M+\sqrt{-1}Y_M, X_M-\sqrt{-1}Y_M\rangle = |X_M|^2+|Y_M|^2,$$

set

$$\mathrm{Zero}(X_M-\sqrt{-1}Y_M) = \{x\in M \mid \langle X_M(x)+\sqrt{-1}Y_M(x), X_M(x)-\sqrt{-1}Y_M(x)\rangle = 0\}.$$

For

$$d_{X+\sqrt{-1}Y}(Y^{'}+\sqrt{-1}X^{'}) = d(Y^{'}+\sqrt{-1}X^{'}) + \langle X_M+\sqrt{-1}Y_M, Y_M+\sqrt{-1}X_M\rangle$$

and

$$\langle X_M+\sqrt{-1}Y_M, Y_M+\sqrt{-1}X_M\rangle = \sqrt{-1}(|X_M|^2+|Y_M|^2),$$

set

$$\mathrm{Zero}(Y_M+\sqrt{-1}X_M) = \{x\in M \mid \langle X_M(x)+\sqrt{-1}Y_M(x), Y_M(x)+\sqrt{-1}X_M(x)\rangle = 0\}.$$

We can see that

$$\text{Zero}(X_M - \sqrt{-1}Y_M) = \text{Zero}(Y_M + \sqrt{-1}X_M) = \{x \in M \mid |X_M(x)| = |Y_M(x)| = 0\}.$$

So we get two kinds of zero points, the one is

$$\{x \in M \mid \langle X_M(x), Y_M(x)\rangle = 0, |Y_M(x)| = |X_M(x)|\},$$

the other one is

$$\{x \in M \mid |X_M(x)| = |Y_M(x)| = 0\};$$

obviously

$$\{x \in M \mid |X_M(x)| = |Y_M(x)| = 0\} \subset \{x \in M \mid \langle X_M(x), Y_M(x)\rangle = 0, |Y_M(x)| = |X_M(x)|\}.$$

Corollery 1. *For any* $\eta \in H^*_{X+\sqrt{-1}Y}(M)$ *with* $[X,Y] = 0$ *and* $s \geq 0$*, we have*

1)

$$\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta$$
$$= \int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X' - \sqrt{-1}Y'))\} \exp\{-s(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta,$$

2)

$$\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(Y' - \sqrt{-1}X'))\}\eta$$
$$= \int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X' - \sqrt{-1}Y'))\} \exp\{-s(d_{X+\sqrt{-1}Y}(Y' - \sqrt{-1}X'))\}\eta,$$

3)

$$\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(Y' + \sqrt{-1}X'))\}\eta$$
$$= \int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X' - \sqrt{-1}Y'))\} \exp\{-s(d_{X+\sqrt{-1}Y}(Y' + \sqrt{-1}X'))\}\eta.$$

Proof. Because $\exp\{-s(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta \in H^*_{X+\sqrt{-1}Y}(M)$, $\exp\{-s(d_{X+\sqrt{-1}Y}(Y' - \sqrt{-1}X'))\}\eta \in H^*_{X+\sqrt{-1}Y}(M)$, $\exp\{-s(d_{X+\sqrt{-1}Y}(Y' + \sqrt{-1}X'))\}\eta \in H^*_{X+\sqrt{-1}Y}(M)$. So by Lemma 7., we get the result. □

Lemma 8. *For any* $\eta \in H^*_{X+\sqrt{-1}Y}(M)$ *and* $[X,Y] = 0$*, if* $\text{Zero}(X_M - \sqrt{-1}Y_M) = \emptyset$*, then* $\int_M \eta = 0$.

Proof. Because

$$d_{X+\sqrt{-1}Y}(X' - \sqrt{-1}Y') = d(X' - \sqrt{-1}Y') + |X_M|^2 + |Y_M|^2$$

so

$$\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X'+\sqrt{-1}Y'))\}\eta = \int_M \exp\{-s(|X_M|^2+|Y_M|^2)\} \exp\{-sd(X'-\sqrt{-1}Y')\}\eta.$$

By $\text{Zero}(X_M - \sqrt{-1}Y_M) = \emptyset$, we can see easily that when $s \to +\infty$, the right hand side of the above equality is of exponential decay and so the result follows. □

4 Localization formula on $\mathrm{Zero}(X_M - \sqrt{-1}Y_M)$

In the following section we denote $\mathrm{Zero}(X_M - \sqrt{-1}Y_M)$ by M_0. For simplicity, we assume that M_0 is the connected submanifold of M, and $\mathcal{N}$ is the normal bundle of M_0 about M.

Set E is a G-equivariant vector bundle, if ∇^E is a connection on E which commutes with the action of G on $\Omega(M, E)$, we see that

$$[\nabla^E, L_X^E] = 0$$

for all $X \in \mathfrak{g}$. Then we can get a moment map by

$$\mu^E(X) = L_X^E - [\nabla^E, i_X] = L_X^E - \nabla_X^E$$

We known that if y be the tautological section of the bundle $\pi^* E$ over E, then the vertical component of X_E may be identified with $-\mu^E(X)y$(see [2] proposition 7.6). For the normal bundle $\mathcal{N}$ of M_0, the vector fields $X_{\mathcal{N}}$ and $Y_{\mathcal{N}}$ are vertical and are given at the point $(x, y) \in M_0 \times \mathcal{N}_x$ by the vectors $-\mu^{\mathcal{N}}(X)y, -\mu^{\mathcal{N}}(Y)y \in \mathcal{N}_x$.

If E is the tangent bundle TM and ∇^{TM} is Levi-Civita connection, then we have

$$\mu^{TM}(X)Y = L_X Y - \nabla_X^{TM} Y = -\nabla_Y^{TM} X$$

We known that for any Killing vector field X, $\mu^{TM}(X)$ as linear endomorphisms of TM is skew-symmetric, $-\mu^{TM}(X)$ annihilates the tangent bundle TM_0 and induces a skew-symmetric automorphism of the normal bundle $\mathcal{N}$(see [10] chapter II, proposition 2.2 and theorem 5.3). The restriction of $\mu^{TM}(X)$ to $\mathcal{N}$ coincides with the moment endomorphism $\mu^{\mathcal{N}}(X)$.

Now we construct a one-form α on $\mathcal{N}$:

$$Z \in \Gamma(T\mathcal{N}) \to \alpha(Z) = \langle -\mu^{\mathcal{N}}(X)y, \nabla_Z^{\mathcal{N}} y\rangle - \sqrt{-1}\langle -\mu^{\mathcal{N}}(Y)y, \nabla_Z^{\mathcal{N}} y\rangle$$

Let $Z_1, Z_2 \in \Gamma(T\mathcal{N})$, we known $d\alpha(Z_1, Z_2) = Z_1\alpha(Z_2) - Z_2\alpha(Z_1) - \alpha([Z_1, Z_2])$, so

$$\begin{aligned} d\alpha(Z_1, Z_2) = &\langle -\nabla_{Z_1}^{\mathcal{N}}\mu^{\mathcal{N}}(X)y, \nabla_{Z_2}^{\mathcal{N}} y\rangle - \langle -\nabla_{Z_2}^{\mathcal{N}}\mu^{\mathcal{N}}(X)y, \nabla_{Z_1}^{\mathcal{N}} y\rangle \\ &- \sqrt{-1}\langle -\nabla_{Z_1}^{\mathcal{N}}\mu^{\mathcal{N}}(Y)y, \nabla_{Z_2}^{\mathcal{N}} y\rangle + \sqrt{-1}\langle -\nabla_{Z_2}^{\mathcal{N}}\mu^{\mathcal{N}}(Y)y, \nabla_{Z_1}^{\mathcal{N}} y\rangle \\ &+ \langle -\mu^{\mathcal{N}}(X)y, R^{\mathcal{N}}(Z_1, Z_2)y\rangle - \sqrt{-1}\langle -\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}(Z_1, Z_2)y\rangle \end{aligned}$$

Recall that $\nabla^{\mathcal{N}}$ is invariant under L_X for all $X \in \mathfrak{g}$, so that $[\nabla^{\mathcal{N}}, \mu^{\mathcal{N}}(X)] = 0$, $[\nabla^{\mathcal{N}}, \mu^{\mathcal{N}}(Y)] = 0$. And by X, Y are Killing vector field, we have $d\alpha$ equals

$$2\langle -(\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot, \cdot\rangle + \langle -\mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}} y\rangle$$

And by $|X_{\mathcal{N}}|^2 = \langle \mu^{\mathcal{N}}(X)y, \mu^{\mathcal{N}}(X)y\rangle$, $|Y_{\mathcal{N}}|^2 = \langle \mu^{\mathcal{N}}(Y)y, \mu^{\mathcal{N}}(Y)y\rangle$. So We can get

$$\begin{aligned} d_{X_{\mathcal{N}}+\sqrt{-1}Y_{\mathcal{N}}}(X_{\mathcal{N}}' - \sqrt{-1}Y_{\mathcal{N}}') &= d(X_{\mathcal{N}}' - \sqrt{-1}Y_{\mathcal{N}}') + \langle X_{\mathcal{N}} - \sqrt{-1}Y_{\mathcal{N}}, X_{\mathcal{N}} + \sqrt{-1}Y_{\mathcal{N}}\rangle \\ &= -2\langle(\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot, \cdot\rangle \\ &+ \langle -\mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y + R^{\mathcal{N}} y\rangle \end{aligned}$$

Theorem 1. *Let M be a smooth closed oriented manifold, G be a compact Lie group acting smoothly on M. For any $\eta \in H_{X+\sqrt{-1}Y}^*(M)$, $[X, Y] = 0$, the following identity hold:*

$$\int_M \eta = \int_{M_0} \frac{\eta}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y) + R^{\mathcal{N}}}{2\pi}]}$$

Proof. Here we use the method come from [5]. Set $s = \frac{1}{2t}$, so by Lemma 7. we get

$$\int_M \eta = \int_M \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X^{'} - \sqrt{-1}Y^{'}))\}\eta$$

Let V is a neighborhood of M_0 in $\mathcal{N}$. We identify a tubular neighborhood of M_0 in M with V. Set $V^{'} \subset V$. When $t \to 0$, because

$$\langle X_M(x) + \sqrt{-1}Y_M(x), X_M(x) - \sqrt{-1}Y_M(x)\rangle = |X_M|^2 + |Y_M|^2 \neq 0$$

out of M_0, so we have

$$\int_M \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X^{'} - \sqrt{-1}Y^{'}))\}\eta \sim \int_{V'} \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X^{'} - \sqrt{-1}Y^{'}))\}\eta.$$

Because

$$\int_{V'} \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X^{'} - \sqrt{-1}Y^{'}))\}\eta = \int_{V'} \exp\{-\frac{1}{2t}(d_{X_{\mathcal{N}}+\sqrt{-1}Y_{\mathcal{N}}}(X^{'}_{\mathcal{N}} - \sqrt{-1}Y^{'}_{\mathcal{N}}))\}\eta$$

then

$$\int_{V'} \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X^{'} - \sqrt{-1}Y^{'}))\}\eta =$$

$$\int_{V'} \exp\{\frac{1}{t}\langle(\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot,\cdot\rangle + \frac{1}{2t}\langle\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y\rangle\}\eta$$

$$+\int_{V'} \exp\{-\frac{1}{2t}\langle-\mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y\rangle\}\eta$$

By making the change of variables $y = \sqrt{t}y$, we find that the above formula is equal to

$$t^n \int_{V'} \exp\{\frac{1}{t}\langle(\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot,\cdot\rangle + \frac{1}{2}\langle\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y\rangle\}\eta$$

$$+\int_{V'} \exp\{-\frac{1}{2}\langle-\mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y\rangle\}\eta$$

we known that

$$\frac{(\frac{\langle(\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot,\cdot\rangle}{t})^n}{n!} = (\mathrm{Pf}(\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})))\mathrm{dy}$$

here dy is the volume form of the submanifold M_0. Because

$$(\mathrm{Pf}(\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})))^2 = \det(\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})),$$

let n be the dimension of M_0, then we get

$$= \int_{V'} \exp\{\frac{1}{2}\langle\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y\rangle\}\eta[\det(\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y))]^{\frac{1}{2}}dy_1 \wedge ... \wedge dy_n$$

$$+\int_{V'} \exp\{-\frac{1}{2}\langle-\mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y\rangle\}\eta$$

Because by $[X,Y]=0$ we have $[\mu^{TM}(X),\mu^{TM}(Y)]=0$. And by $-\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y)$, $R^{\mathcal{N}}$ are skew-symmetric, so we get

$$= \int_{V'} \exp\{-\frac{1}{2}\langle -\mu^{\mathcal{N}}(X)y+\sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y-\sqrt{-1}\mu^{\mathcal{N}}(Y)y+R^{\mathcal{N}}y\rangle\}dy_1\wedge ...\wedge dy_n$$

$$\cdot[\det(\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y))]^{\frac{1}{2}}\eta$$

$$= \int_{M_0} (2\pi)^n[\det(\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y))]^{-\frac{1}{2}}[\det(-\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y)+R^{\mathcal{N}})]^{-\frac{1}{2}}$$

$$\cdot[\det(\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y))]^{\frac{1}{2}}\eta$$

$$= \int_{M_0} (2\pi)^n[\det(-\mu^{\mathcal{N}}(X)-\sqrt{-1}\mu^{\mathcal{N}}(Y)+R^{\mathcal{N}})]^{-\frac{1}{2}}\eta$$

$$= \int_{M_0} \frac{\eta}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X})-\sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})+\mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

□

By Theorem 1.,we can get the localization formulas of Berline and Vergne(see [2] or [3]).

Corollery 2 (N.Berline and M.Vergne)**.** *Let M be a smooth closed oriented manifold, G be a compact Lie group acting smoothly on M. For any $\eta \in H_X^*(M)$, the following identity hold:*

$$\int_M \eta = \int_{M_0} \frac{\eta}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X})+\mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. By Theorem 1., we set $Y=0$, then we get the result. □

5 Application in Characteristic Numbers

As in [6], we will give the application of the localization formula about two Killing vector field in characteristic numbers. So let's recall the Chern-Weil theory(see [12]) about equivariant connection and equivariant curvature without proof(see [6] for proof) .

Let M be an even dimensional compact oriented manifold without boundary, G be a compact Lie group acting smoothly on M and $\mathfrak{g}$ be its Lie algebra. Let g^{TM} be a G-invariant Riemannian metric on TM, ∇^{TM} is the Levi-Civita connection associated to g^{TM}. Here ∇^{TM} is a G-invariant connection, we see that $[\nabla^{TM}, L_{X_M}]=0$ for all $X\in\mathfrak{g}$.

The equivariant connection $\widetilde{\nabla}^{TM}$ is the operator on $\Omega^*(M,TM)$ corresponding to a G-invariant connection ∇^{TM} is defined by the formula

$$\widetilde{\nabla}^{TM} = \nabla^{TM} + i_{X_M+\sqrt{-1}Y_M}$$

here X_M, Y_M be the smooth vector field on M corresponded to $X, Y \in \mathfrak{g}$.

Lemma 9. *The operator $\widetilde{\nabla}^{TM}$ preserves the space $\Omega^*_{X_M+\sqrt{-1}Y_M}(M,TM)$ which is the space of smooth $(X_M+\sqrt{-1}Y_M)$-invariant forms with values in TM.*

We will also denote the restriction of $\widetilde{\nabla}^{TM}$ to $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, TM)$ by $\widetilde{\nabla}^{TM}$.

The equivariant curvature $\widetilde{R}^{TM}$ of the equivariant connection $\widetilde{\nabla}^{TM}$ is defined by the formula(see [2])

$$\widetilde{R}^{TM} = (\widetilde{\nabla}^{TM})^2 - L_{X_M} - \sqrt{-1}L_{Y_M}$$

It is the element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM))$. We see that

$$\begin{aligned}\widetilde{R}^{TM} &= (\nabla^{TM} + i_{X_M+\sqrt{-1}Y_M})^2 - L_{X_M} - \sqrt{-1}L_{Y_M} \\ &= R^{TM} + [\nabla^{TM}, i_{X_M+\sqrt{-1}Y_M}] - L_{X_M} - \sqrt{-1}L_{Y_M} \\ &= R^{TM} - \mu^{TM}(X) - \sqrt{-1}\mu^{TM}(Y)\end{aligned}$$

Lemma 10. *The equivariant curvature $\widetilde{R}^{TM}$ satisfies the equvariant Bianchi formula*

$$\widetilde{\nabla}^{TM}\widetilde{R}^{TM} = 0$$

Now we construct the equivariant characteristic forms by $\widetilde{R}^{TM}$. If $f(x)$ is a polynomial in the indeterminate x, then $f(\widetilde{R}^{TM})$ is an element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM))$. We use the trace map

$$\mathrm{Tr} : \Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM)) \to \Omega^*_{X_M+\sqrt{-1}Y_M}(M)$$

to obtain an element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M)$, which we call an equivariant characteristic form.

Lemma 11. *The equivariant differential form $\mathrm{Tr}(f(\widetilde{R}^{TM}))$ is $d_{X_M+\sqrt{-1}Y_M}$-closed, and its equivariant cohomology class is independent of the choice of the G-invariant connection ∇^{TM}.*

As an application of Theorem 1., we can get the following localization formulas for characteristic numbers

Theorem 2. *Let M be an $2m$-dim compact oriented manifold without boundary, G be a compact Lie group acting smoothly on M and $\mathfrak{g}$ be its Lie algebra. Let $X, Y \in \mathfrak{g}$, and X_M, Y_M be the corresponding smooth vector field on M, $M_0 = \mathrm{Zero}(X_M - \sqrt{-1}Y_M)$. If $f(x)$ is a polynomial, then we have*

$$\int_M \mathrm{Tr}(f(\widetilde{R}^{TM})) = \int_{M_0} \frac{\mathrm{Tr}(f(\widetilde{R}^{TM}))}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X})-\sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})+\mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. By Lemma 10., we have $\mathrm{Tr}(f(\widetilde{R}^{TM}))$ is $d_{X_M+\sqrt{-1}Y_M}$-closed. And by Theorem 1., we get the result. □

Now we use the detaminate map

$$\det : \Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM)) \to \Omega^*_{X_M+\sqrt{-1}Y_M}(M)$$

to obtain an element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M)$.

Lemma 12. *The equivariant differential form $\mathrm{Pf}(-\widetilde{R}^{TM})$ is $d_{X_M+\sqrt{-1}Y_M}$-closed, and its equivariant cohomology class is independent of the choice of the G-invariant connection ∇^{TM}.*

Proof. Because $\det A = \exp(\mathrm{Tr}(\log(A)))$, so

$$\det(-\widetilde{R}^{TM}) = \exp(\mathrm{Tr}(\log(-\widetilde{R}^{TM}))).$$

and we know that $\det(-\widetilde{R}^{TM}) = (\mathrm{Pf}(-\widetilde{R}^{TM}))^2$, by Lemma 11., we get the result. □

Theorem 3. *Let M be an $2m$-dim compact oriented manifold without boundary, G be a compact Lie group acting smoothly on M and $\mathfrak{g}$ be its Lie algebra. Let $X, Y \in \mathfrak{g}$, and X_M, Y_M be the corresponding smooth vector field on M, $M_0 = \mathrm{Zero}(X_M - \sqrt{-1}Y_M)$. Then we have*

$$\int_M \mathrm{Pf}(-\widetilde{R}^{TM}) = \int_{M_0} \frac{\mathrm{Pf}(-\widetilde{R}^{TM})}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y}) + \mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. Because $\mathrm{Pf}(-\widetilde{R}^{TM})$ is $d_{X_M+\sqrt{-1}Y_M}$-closed and by Theorem 1., we get the result. □

6 Application in Symplectic Manifolds

Let (M, ω) be a smooth closed symplectic manifold, ω is a closed nondegenerate 2-form with $d\omega = 0$(see [4]). Let G be a connected compact Lie group acting on M via symplectomorphism, i.e.

$$L_X \omega = 0$$

for $\forall X \in \mathfrak{g}$, here $\mathfrak{g}$ be its Lie algebra. If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M given by

$$(X_M f)(x) = \frac{d}{dt} f(\exp(-tX) \cdot x) \mid_{t=0} .$$

By the symplectic form ω there is a isomorphism between vector fields and 1-form on M, i.e.

$$\Gamma(TM) \to \Omega^1(M) : X_M \mapsto i_{X_M}\omega$$

For $H \in C^\infty(M)$, then a vector field X^H on M is called a Hamiltonian vector field with the energy function H, if for X^H we have $i_{X^H}\omega = dH$.

We can also define the equivariant cohomology associated with $X + \sqrt{-1}Y$ on symplectic manifold in the same way as in Section 1.

Here we define the equivariant extension of the symplectic form by

$$\omega - H_X - \sqrt{-1}H_Y$$

where $dH_X = i_{X_M}\omega$, $dH_Y = i_{Y_M}\omega$.

Lemma 13. *The equivariant symplectic form $\omega - H_X - \sqrt{-1}H_Y$ is a $d_{X+\sqrt{-1}Y}$-closed form.*

Proof.

$$\begin{aligned} d_{X+\sqrt{-1}Y}(\omega - H_X - \sqrt{-1}H_Y) &= (d + i_{X_M} + \sqrt{-1}i_{Y_M})(\omega - H_X - \sqrt{-1}H_Y) \\ &= d\omega - dH_X - \sqrt{-1}dH_Y + i_{X_M}\omega + \sqrt{-1}i_{Y_M}\omega \\ &= d\omega \\ &= 0 \end{aligned}$$

□

Since $d(H_X + \sqrt{-1}H_Y) = i_{X_M}\omega + \sqrt{-1}i_{Y_M}\omega$, the set of points where the one-form $d(H_X + \sqrt{-1}H_Y)$ vanishes coincides with the zero set $\mathrm{Zero}(X_M - \sqrt{-1}Y_M)$.

Theorem 4. *Let (M, ω) be a compact symplectic manifold, and let G be a connected compact Lie group acting on M and $\mathfrak{g}$ be its Lie algebra. Also assume M be a Riemannian manifold with G-invariant Riemannian metric g^{TM}. Let $X, Y \in \mathfrak{g}$, and X_M, Y_M be the corresponding smooth vector field on M, $M_0 = \mathrm{Zero}(X_M - \sqrt{-1}Y_M)$. Then we have*

$$\int_M \exp(-H_X - \sqrt{-1}H_Y)\frac{\omega^n}{n!} = \int_{M_0} \frac{\exp(\omega)}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y}) + \mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. By Lemma 13., $\omega - H_X - \sqrt{-1}H_Y$ is a $d_{X+\sqrt{-1}Y}$-closed form; and

$$\int_M \exp(\omega - H_X - \sqrt{-1}H_Y) = \int_M \exp(-H_X - \sqrt{-1}H_Y)\exp(\omega) = \int_M \exp(-H_X - \sqrt{-1}H_Y)\frac{\omega^n}{n!}.$$

Note that $\exp(-H_X - \sqrt{-1}H_Y) = 1$ on M_0. Then by Theorem 1., we get the result. □

Obviously, this is a Duistermaat-Heckman type formula.

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