# A Poincaré-Hopf type formula for a pair of vector fields

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**Abstract**

We extend the reslut about Poincaré-Hopf type formula for the difference of the Chern character numbers (cf.[3]) to the non-isolated singularities, and establish a Poincaré-Hopf type formula for a pair of vector field with the function $h^{T_{\mathbb{C}}M}(\cdot,\cdot)$ has non-isolated zero points over a closed, oriented smooth manifold of dimension $2n$.
**Keyword** Chern character; Signature $\mathbb{Z}_2$-graded; Poincaré-Hopf type formula; non-isolated zero points

## 1 Introduction

Let $M$ be a closed, oriented smooth manifold of dimension $2n$. Let $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ be the complexification of $TM$. Let $g^{TM}$ be a the Riemannian metric on $M$, it induces canonically a complex symmetric bilinear form on $T_{\mathbb{C}}M$, denoted by $h^{T_{\mathbb{C}}M}$(cf.[2]). Let any $K \in \Gamma(T_{\mathbb{C}}M)$ be the section of $T_{\mathbb{C}}M$, then $K = \xi + \sqrt{-1}\eta$, where $\xi$ and $\eta$ be vector field, we define

$$h^{T_{\mathbb{C}}M}(K,K) = |\xi|^2_{g^{TM}} - |\eta|^2_{g^{TM}} + 2\sqrt{-1}\langle \xi, \eta\rangle_{g^{TM}}$$

Surely, $h^{T_{\mathbb{C}}M}(K,K)$ is a smooth function on $M$, we denoted the set of zero pionts of this function by $Zero(K)$. The Euler number of manifold $M$ is denoted by $\chi(M)$. In [5], Jacobowitz established the following result: if $Zero(K) = \emptyset$, then $\chi(M) = 0$.

In the end of [5], Jacobowitz asked a question like that: Is there a counting formula for $\chi(M)$ of Poincaré-Hopf type, when $Zero(K) \neq \emptyset$?

In [3], Huitao Feng, Weiping Li and Weiping Zhang establish a Poincaré-Hopf formula for the difference of the Chern character numbers of two vector bundles with $Zero(K)$ is isolated, and use the formula they get a Poincaré-Hopf type formula to the set of $Zero(K)$ consists of a finite number of points on a spin manifold $M$. This result is an answers of the question asked by Jacobowitz in [5].

In [2], we establish a Poincaré-Hopf type formula for a pair of sections of an oriented real vector bundle of rank $2n$ over a closed, oriented manifold of dimension $2n$, with isolated zero points, which generalized the corresponding result in [3].

In this article, we will extend the Poincaré-Hopf type formula for the difference of the Chern character numbers of two complex vector bundles to $Zero(K)$ is non-isolated.

**Theorem 1.**

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [M]\rangle = \sum_X \langle \frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_X)}, [X]\rangle$$

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By use of the Poincaré-Hopf type formula for the difference of the Chern character numbers of two complex vector bundles with $Zero(K)$ is non-isolated, we establish a Poincaré-Hopf type formula for a pair of vector field with the function $h^{T_{\mathbb{C}}M}(\cdot,\cdot)$ has non-isolated zero points over a closed, oriented smooth manifold of dimension $2n$.

**Theorem 2.** *Let $M$ be a closed, oriented smooth manifold of dimension $2n$ and $n \geq 2$. Let $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ be the complexification of $TM$. Let $g^{TM}$ be a the Riemannian metric on $M$, it induces canonically a complex symmetric bilinear form on $T_{\mathbb{C}}M$, denoted by $h^{T_{\mathbb{C}}M}$. Let $K = \xi + \sqrt{-1}\eta$, where $\xi$ and $\eta$ are vector fields, $K \in \Gamma(T_{\mathbb{C}}M)$ be the section of $T_{\mathbb{C}}M$. Let $X$ is the connected component of $Zero(K)$, and $\mathcal{N}_X$ be the normal bundle of the connected component $X$, then*

$$\chi(M) = \frac{1}{(-2)^n}\sum_X \langle \frac{\mathrm{ch}(\Lambda_+(T^*M\otimes\mathbb{C})) - \mathrm{ch}(\Lambda_-(T^*M\otimes\mathbb{C}))}{e(\mathcal{N}_X)}, [X]\rangle$$

# 2 The difference of the Chern character numbers

Let $E_+$, $E_-$ be two complex vector bundles over $M$, and $E = E_+ \oplus E_-$ be the $Z_2$-graded complex vector bundle over $M$. Let $\nabla^{E_+}$,$\nabla^{E_-}$ be the connection about $E_+$ and $E_-$, and $\nabla^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix}$ be the $Z_2$-graded connection on E.

Let

$$v \in \Gamma(\mathrm{Hom}(E_+, E_-))$$

be a homomorphism between $E_+$ and $E_-$. Let

$$v^* \in \Gamma(\mathrm{Hom}(E_-, E_+))$$

be the adjoint of $v$ with respect to the Hermitian metrics on $E_\pm$ respectively. And

$$V = \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} \in \Gamma(\mathrm{Hom}(E, E))$$

Let $Z(v)$ denote the set of the points at which $v$ is noninvertible.We always assume that $Z(v)$ is the compact submanifold of $M$, the connected components of $Z(v)$ is denoted by $X$, and $Z(v) = \bigcup X$. Let $U_X$ be the tubular neighborhood of the connected component $X$. Let $\mathcal{N}_X$ be the normal bundle of the connected component $X$. By the tubular neighborhood theorem(cf.[7]),$U_X$ is diffeomorphic to the total space of the normal bundle $\mathcal{N}_X$.

**Lemma 1.** *The following identity holds,*

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [M]\rangle = \sum_X \langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [U_X]\rangle$$

*Proof.*

$$\begin{aligned}\int_M (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) &= \int_{M\setminus\bigcup U_X} (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) + \int_{\bigcup U_X} (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \\ &= \int_{M\setminus\bigcup U_X} (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) + \sum_X \int_{U_X} (\mathrm{ch}(E_+) - \mathrm{ch}(E_-))\end{aligned}$$

Because $\mathrm{ch}E = \mathrm{ch}(E_+) - \mathrm{ch}(E_-)$ is independent of the choice of the connection $\nabla^E$, we need to construct a special connection on $E$(cf.[4]). By

$$[\nabla^E, V] = \nabla^E \cdot V - V \cdot \nabla^E = \begin{pmatrix} 0 & \nabla^{E_+}v^* - v^*\nabla^{E_-} \\ \nabla^{E_-}v - v\nabla^{E_+} & 0 \end{pmatrix}$$

$$\begin{aligned} V[\nabla^E, V] &= \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} \begin{pmatrix} 0 & \nabla^{E_+}v^* - v^*\nabla^{E_-} \\ \nabla^{E_-}v - v\nabla^{E_+} & 0 \end{pmatrix} \\ &= \begin{pmatrix} v^*(\nabla^{E_-}v - v\nabla^{E_+}) & 0 \\ 0 & v(\nabla^{E_+}v^* - v^*\nabla^{E_-}) \end{pmatrix} \end{aligned}$$

then we can construct two connection on $E$.

$$\nabla_1^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix} + \begin{pmatrix} v^*(\nabla^{E_-}v - v\nabla^{E_+}) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\nabla_2^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & v(\nabla^{E_+}v^* - v^*\nabla^{E_-}) \end{pmatrix}.$$

Here we only use $\nabla_1^E$. Because $v$ is invertible on $M \setminus Z(v)$, so we can choose $v^* = v^{-1}$ on $M \setminus Z(v)$. And use $\nabla_1^E$ to construct a new connection

$$\widetilde{\nabla}_1^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix} + \begin{pmatrix} \rho v^{-1}(\nabla^{E_-}v - v\nabla^{E_+}) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\rho$ is a truncating function with $\rho(x) = 1, x \in M \setminus \bigcup U_X$ and $\rho(x) = 0, x \in X$.

So on $M \setminus \bigcup U_X$,

$$\widetilde{\nabla}_1^E = \begin{pmatrix} v^{-1}\nabla^{E_-}v & 0 \\ 0 & \nabla^{E_-} \end{pmatrix},$$

$$\widetilde{R}_1^E = (\widetilde{\nabla}_1^E)^2 = \begin{pmatrix} v^{-1}R^{E_-}v & 0 \\ 0 & R^{E_-} \end{pmatrix}.$$

By the definition of Chern character form (cf.[8] or [1])

$$\mathrm{ch}(E, \widetilde{\nabla}_1^E) = \mathrm{tr_s}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}\widetilde{R}_1^E\right)\right] = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}v^{-1}R^{E_-}v\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E_-}\right)\right]$$

then

$$\int_{M\setminus\bigcup U_X} (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) = \int_{M\setminus\bigcup U_X} \mathrm{ch}(E, \widetilde{\nabla}_1^E) = 0.$$

So

$$\int_M (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) = \sum_X \int_{U_X} (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)).$$

□

**Lemma 2.** *The following identity holds,*

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [U_X] \rangle = \langle \frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_X)}, [X] \rangle$$

*Proof.* Let $\mathcal{N}_X$ be the normal bundle over $X$, consider the maps $\pi : \mathcal{N}_X \to X$ and $i : X \to \mathcal{N}_X$ where $\pi$ is the bundle projection and $i$ denotes inclusion as the zero section. Let $\pi_!$ be the integration over the fibre, and $i_!$ be the Thom isomorphism of $\mathcal{N}_X$(cf.[6],chapter III.§12.). By assumption $X$ is compact, then we know(cf.[6],chapter III. lemma 12.2.)

$$i^* i_!(u) = e(\mathcal{N}_X) \cdot u$$

for all $u \in H^*(X) = H^*_{cpt}(X)$, where $e(\mathcal{N}_X)$ is the Euler class of $\mathcal{N}_X$. If $u = \pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}]$, then

$$i^* i_!(\pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}]) = e(\mathcal{N}_X) \cdot \pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}]$$

so

$$\pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}] = \frac{i^* i_!(\pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}])}{e(\mathcal{N}_X)},$$

because

$$i^* i_!(\pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}]) = i^*[\mathrm{ch}(E_+ - E_-) \mid_{\mathcal{N}_X}] = \mathrm{ch} i^*[(E_+ - E_-) \mid_{\mathcal{N}_X}] = \mathrm{ch}(E_+ - E_-) \mid_X .$$

So

$$\pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}] = \frac{\mathrm{ch}(E_+ - E_-) \mid_X}{e(\mathcal{N}_X)} = \frac{(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_X}{e(\mathcal{N}_X)},$$

$$\langle \pi_![(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)) \mid_{\mathcal{N}_X}], [X] \rangle = \langle \frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_X)}, [X] \rangle$$

then

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [U_X] \rangle = \langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [\mathcal{N}_X] \rangle = \langle \frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_X)}, [X] \rangle$$

□

# 3 The proof of Theorem 1.

By Lemma 1. and Lemma 2. we get the result in Theorem 1.,

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [M] \rangle = \sum_X \langle \frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_X)}, [X] \rangle$$

**Corollery 1** (Huitao Feng, Weiping Li and Weiping Zhang)**.**

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [M] \rangle = (-1)^{n-1} \sum_p \deg(v_p)$$

*Proof.* By Theorem 1., if $X = p$ is the isolated zero points, then $\frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_p)} = \frac{0}{0}$. By Lemma 2.

$$\frac{\mathrm{ch}(E_+) - \mathrm{ch}(E_-)}{e(\mathcal{N}_p)} = \langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [U_p] \rangle.$$

Let $\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}_1^E$, by transgression formula

$$\begin{aligned}\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [U_p]\rangle &= -\frac{\sqrt{-1}}{2\pi}\int_{U_p} d\int_0^1 \mathrm{tr_s}\left[\frac{d\nabla_t^E}{dt}\exp(\frac{\sqrt{-1}}{2\pi}R_t^E)\right]dt \\ &= -\frac{\sqrt{-1}}{2\pi}\int_{\partial U_p}\int_0^1 \mathrm{tr_s}\left[\frac{d\nabla_t^E}{dt}\exp(\frac{\sqrt{-1}}{2\pi}R_t^E)\right]dt\end{aligned}$$

because we can choose $\nabla^E = \begin{pmatrix} d & 0 \\ 0 & d\end{pmatrix}$, then

$$\nabla_t^E = (1-t)\begin{pmatrix} d & 0 \\ 0 & d\end{pmatrix} + t\left[\begin{pmatrix} d & 0 \\ 0 & d\end{pmatrix} + \begin{pmatrix} v^{-1}(dv) & 0 \\ 0 & 0\end{pmatrix}\right]$$

so

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi}\int_{\partial U_p}\int_0^1 \mathrm{tr_s}\left[\frac{d\nabla_t^E}{dt}\exp(\frac{\sqrt{-1}}{2\pi}R_t^E)\right]dt \\ &= \frac{\sqrt{-1}}{2\pi}\int_{\partial U_p}\int_0^1 \mathrm{tr}\left[v^{-1}(dv)\frac{1}{(n-1)!}\left(\frac{\sqrt{-1}}{2\pi}t(1-t)(v^{-1}(dv))^2\right)^{n-1}\right]dt \\ &= (\frac{\sqrt{-1}}{2\pi})^n\int_0^1 \frac{t^{n-1}(1-t)^{n-1}}{(n-1)!}dt\int_{\partial U_p}\mathrm{tr}\left((v^{-1}(dv))^{2n-1}\right) \\ &= (\frac{\sqrt{-1}}{2\pi})^n\frac{(n-1)!}{(2n-1)!}\int_{\partial U_p}\mathrm{tr}\left((v^{-1}(dv))^{2n-1}\right)\end{aligned}$$

Then we get

$$\langle \mathrm{ch}(E_+) - \mathrm{ch}(E_-), [U_p]\rangle = -(\frac{\sqrt{-1}}{2\pi})^n\frac{(n-1)!}{(2n-1)!}\int_{\partial U_p}\mathrm{tr}\left((v^{-1}(dv))^{2n-1}\right) = (-1)^{n-1}\deg(v_p)$$

□

# 4 The proof of Theorem 2.

Let $M$ be a closed, oriented smooth manifold of dimension $2n$, $E$ be a oriented real vector bundle on $M$ with rank $2n$. Let $E_{\mathbb{C}} = E\otimes\mathbb{C}$ denote the complexification of the vector bundle $E$. Let any $K\in\Gamma(E_{\mathbb{C}})$ be the section of $E_{\mathbb{C}}$, then $K = \xi + \sqrt{-1}\eta$, where $\xi$ and $\eta$ be smooth section of $E$. Let $g^E$ be a Euclidian inner product on $E$, then it induces canonically a complex symmetric bilinear form $h^{E_{\mathbb{C}}}$ on $E_{\mathbb{C}}$, such that

$$h^{E_{\mathbb{C}}}(K,K) = |\xi|_{g^E}^2 - |\eta|_{g^E}^2 + 2\sqrt{-1}\langle\xi,\eta\rangle_{g^E}.$$

The zero points of the smooth function $h^{E_{\mathbb{C}}}(K,K)$ is denoted by $Zero(K)$.

Let $E^*$ be the dual bundle of $E$, set $\Lambda(E^*\otimes\mathbb{C})$ be the exterior algebra bundle with complex valued. For any $e\in\Gamma(E)$, Clifford element $c(e)$ acting on $\Lambda(E^*\otimes\mathbb{C})$ is defined by $c(e) = e^*\wedge - i_e$, where $e^*$ corresponds to $e$ via $g^E$, $e^*\wedge$ and $i_e$ are the standard notation for exterior and interior multiplications.

Let $e_1, e_2, \cdots, e_{2n}$ be the local orthonormal basis of $E$, set

$$\tau = (\sqrt{-1})^n c(e_1)c(e_2)\cdots c(e_{2n})$$

we known that $\tau^2 = 1$ and $\tau$ does not depend on the choice of the orthonormal basis. Then $\tau$ is a bundle homomorphism on $\Lambda(E^* \otimes \mathbb{C})$, it give the $\mathbb{Z}_2$-grading on $\Lambda(E^* \otimes \mathbb{C})$,

$$\Lambda(E^* \otimes \mathbb{C}) = \Lambda_+(E^* \otimes \mathbb{C}) \oplus \Lambda_-(E^* \otimes \mathbb{C})$$

where $\Lambda_\pm(E^* \otimes \mathbb{C})$ is corresponds to the characteristic subbundle with characteristic value $\pm$ of the operator $\tau$. So $\Lambda(E^* \otimes \mathbb{C})$ is a super vector bundle. The $\mathbb{Z}_2$-grading is called Signature $\mathbb{Z}_2$-graded.

For any $e \in \Gamma(E)$, we have $c(e)\tau = -\tau c(e)$, so $c(e)$ is a bundle homomorphism from $\Lambda_\pm(E^* \otimes \mathbb{C})$ to $\Lambda_\mp(E^* \otimes \mathbb{C})$. Then for any $\xi, \eta \in \Gamma(E)$, we can construct a bundle homomorphism

$$v_K = \tau c(\xi) + \sqrt{-1}c(\eta) : \Lambda_+(E^* \otimes \mathbb{C}) \to \Lambda_-(E^* \otimes \mathbb{C}).$$

Let $v_K$ extend to an endomorphism of $\Lambda(E^* \otimes \mathbb{C})$ by acting as zero on $\Lambda_-(E^* \otimes \mathbb{C})$, with the notation unchanged. Let $v_K^*$ be the adjoint of $v_K$ with respect to the metrics on $\Lambda_\pm(E^* \otimes \mathbb{C})$ respectively. Set $V = v_K + v_K^*$. Then $V$ is an odd endomorphism of $\Lambda(E^* \otimes \mathbb{C})$. We use $Z(v_K)$ to denoted the noninvertible points of $v_K$. $V^2$ is fiberwise positive over $M \setminus Z(v_K)$ (cf. [3]).

**Lemma 3.** *Let $M$ be a closed, oriented smooth manifold of dimension $2n$,*

**1)** *If $n \geq 2$, then $Z(v_K) = Zero(K)$.*

**2)** *If $n = 1$, then $Z(v_K) = Zero(K) \backslash Z_+$*
*where $Z_+ = \{p \in Zero(K) | \xi(p), \eta(p)$ form a oriented frame on $E_p\}$.*

*Proof.* Please see [2] or [3]. □

We always assume that $Z(v_K)$ is the compact submanifold of $M$, the connected components of $Z(v_K)$ is denoted by $X$.

**Lemma 4.** *The following identity holds,*

$$\langle \mathrm{ch}(\Lambda_+(E^* \otimes \mathbb{C})) - \mathrm{ch}(\Lambda_-(E^* \otimes \mathbb{C})), [M] \rangle = (-2)^n \chi(E)$$

*Proof.* This is a well known result(cf.[6]), Please see [2] for a proof from differential geometry. □

**Corollery 2.**

$$\langle \mathrm{ch}(\Lambda_+(T^*M \otimes \mathbb{C})) - \mathrm{ch}(\Lambda_-(T^*M \otimes \mathbb{C})), [M] \rangle = (-2)^n \chi(M)$$

*Proof.* By Lemma 4., if $E = TM$ so we get the result. □

Now we can give the proof of the Theorem 2. By Theorem 1. and Corollery 2., we have

$$\begin{aligned} (-2)^n \chi(M) &= \langle \mathrm{ch}(\Lambda_+(T^*M \otimes \mathbb{C})) - \mathrm{ch}(\Lambda_-(T^*M \otimes \mathbb{C})), [M] \rangle \\ &= \sum_X \langle \frac{\mathrm{ch}(\Lambda_+(T^*M \otimes \mathbb{C})) - \mathrm{ch}(\Lambda_-(T^*M \otimes \mathbb{C}))}{e(\mathcal{N}_X)}, [X] \rangle \end{aligned}$$

So

$$\chi(M) = \frac{1}{(-2)^n} \sum_X \langle \frac{\mathrm{ch}(\Lambda_+(T^*M \otimes \mathbb{C})) - \mathrm{ch}(\Lambda_-(T^*M \otimes \mathbb{C}))}{e(\mathcal{N}_X)}, [X] \rangle.$$

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