On radicals , polynomials , pi

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abstract
This note presents some formulas for pi

The radical r :

$$r=\sqrt{\sqrt{2}+1}+\sqrt{2}+i\sqrt{\sqrt{2}-1} \quad ,i=\sqrt{-1} \tag{1}$$

r is root of the equation:

$$r^8-16r^6+72r^4+64r^2+16=0 \tag{2}$$

related radicals:

$$\sqrt{2}\sqrt{2+i+2\sqrt{1+i}}=\sqrt{\sqrt{2}+1}+\sqrt{2}+i\sqrt{\sqrt{2}-1} \tag{3}$$

$$\sqrt{2}\sqrt{2-i+2\sqrt{1-i}}=\sqrt{\sqrt{2}+1}+\sqrt{2}-i\sqrt{\sqrt{2}-1} \tag{4}$$

$$\sqrt{2}\sqrt{2+i-2\sqrt{1+i}}=\sqrt{\sqrt{2}+1}-\sqrt{2}+i\sqrt{\sqrt{2}-1} \tag{5}$$

$$\sqrt{2}\sqrt{2-i-2\sqrt{1-i}}=\sqrt{\sqrt{2}+1}-\sqrt{2}-i\sqrt{\sqrt{2}-1} \tag{6}$$

Pi formulas:

$$\pi=32\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n}}{2n+1}\operatorname{Im}\left(i\left(\sqrt{\sqrt{2}+1}+\sqrt{2}+i\sqrt{\sqrt{2}-1}\right)^{-4n-2}\right) \tag{7}$$

$$\pi=8\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{-2n}}{2n+1}\operatorname{Im}\left(-i\left(\sqrt{\sqrt{2}+1}-\sqrt{2}+i\sqrt{\sqrt{2}-1}\right)^{4n+2}\right) \tag{8}$$

The radical s :

$$s=\sqrt{2\sqrt{3}+3}+\sqrt{6}+i\sqrt{2\sqrt{3}-3} \tag{9}$$

s is root of the equation:

$$s^8 - 48s^6 + 600s^4 + 576s^2 + 144 = 0 \tag{10}$$

related radicals:

$$\sqrt{2}\sqrt{6+i\sqrt{3}+2\sqrt{9+3i\sqrt{3}}} = \sqrt{2\sqrt{3}+3}+\sqrt{6}+i\sqrt{2\sqrt{3}-3} \tag{11}$$

$$\sqrt{2}\sqrt{6-i\sqrt{3}+2\sqrt{9-3i\sqrt{3}}} = \sqrt{2\sqrt{3}+3}+\sqrt{6}-i\sqrt{2\sqrt{3}-3} \tag{12}$$

$$\sqrt{2}\sqrt{6+i\sqrt{3}-2\sqrt{9+3i\sqrt{3}}} = \sqrt{2\sqrt{3}+3}-\sqrt{6}+i\sqrt{2\sqrt{3}-3} \tag{13}$$

$$\sqrt{2}\sqrt{6-i\sqrt{3}-2\sqrt{9-3i\sqrt{3}}} = \sqrt{2\sqrt{3}+3}-\sqrt{6}-i\sqrt{2\sqrt{3}-3} \tag{14}$$

Pi formulas:

$$\pi = 48\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n 12^n}{2n+1}\operatorname{Im}\left(i\left(\sqrt{2\sqrt{3}+3}+\sqrt{6}+i\sqrt{2\sqrt{3}-3}\right)^{-4n-2}\right) \tag{15}$$

$$\pi = 4\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n 12^{-n}}{2n+1}\operatorname{Im}\left(-i\left(\sqrt{2\sqrt{3}+3}-\sqrt{6}+i\sqrt{2\sqrt{3}-3}\right)^{4n+2}\right) \tag{16}$$

The polynomial $p(z)$:

$$p(z) = 5z^6 - 6z^5 + 39z^4 - 20z^3 + 51z^2 - 6z + 1 \quad , z \in \mathbb{C} \tag{17}$$

roots

$$p(z) = 0 \Rightarrow z = \{z_1, z_2, z_3, z_4, z_5, z_6\} \tag{18}$$

$$z_1 = \frac{\sqrt[3]{4}-\sqrt[3]{2}+i\left(\sqrt{3}-1\right)}{\sqrt[3]{4}+\sqrt[3]{2}+\sqrt{3}+1} \quad , \quad z_2 = \overline{z}_1 \tag{19}$$

$$z_3 = \frac{\sqrt[3]{4}-\sqrt[3]{2}+i\left(\sqrt{3}+1\right)}{\sqrt[3]{4}+\sqrt[3]{2}-\sqrt{3}+1} \quad , \quad z_4 = \overline{z}_3 \tag{20}$$

$$z_5 = \frac{3144+2498\sqrt[3]{2}+1941\sqrt[3]{4}+2i\sqrt{a+b\sqrt[3]{2}+c\sqrt[3]{4}}}{5\left(1628+1276\sqrt[3]{2}+1017\sqrt[3]{4}\right)} \quad , \quad z_6 = \overline{z}_5 \tag{21}$$

$$a = 273032228 \, , b = 216705226 \, , c = 171999417 \tag{22}$$

Pi formula:

$$\pi = 24\sum_{n=0}^{\infty}\frac{1}{2n+1}\operatorname{Im}\left(\left(\frac{\sqrt[3]{4}-\sqrt[3]{2}+i\left(\sqrt{3}-1\right)}{\sqrt[3]{4}+\sqrt[3]{2}+\sqrt{3}+1}\right)^{2n+1}\right) \tag{23}$$

other formulas:

$$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\operatorname{Im}\left(\left(\sqrt{\frac{\sqrt{2}+1}{2}}-1+i\sqrt{\frac{\sqrt{2}-1}{2}}\right)^{n}\right) \tag{24}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{2^{-n}}{n}\operatorname{Im}\left(\left(2-\sqrt{\sqrt{2}+1}+i\sqrt{\sqrt{2}-1}\right)^{n}\right) \tag{25}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\operatorname{Im}\left(\left(\frac{\sqrt{3}+1}{2\sqrt[3]{2}}-1+i\frac{\sqrt{3}-1}{2\sqrt[3]{2}}\right)^{n}\right) \tag{26}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{\sqrt{3}+1}{2\sqrt[3]{4}}+i\frac{\sqrt{3}-1}{2\sqrt[3]{4}}\right)^{n}\right) \tag{27}$$

Remark: $\operatorname{Im}(z)$ imaginary part of z , $\bar{z}$ conjugate of z .

References

1. Adamchik, V. and Wagon, S.: A Simple Formula for π . Amer. Math. Monthly 104,1997.
2. Backhouse, N.: Note 79.36. Pancake Functions and Approximations to π . Math. Gaz. 79 , 1995.
3. Beck, G. and Trott, M.: Calculating Pi: From Antiquity to Moderns Times. http://documents.wolfram.com/mathematica/Demos/Notebooks/CalculatingPi.html.
4. Beckmann, P.: A History of Pi , 3rd ed. New York: Dorset Press, 1989.
5. Berndt, B.C.: Ramanujan's Notebooks, Part IV. New York: Springer-Verlag, 1994.
6. Blatner, D.: The Joy of Pi. New York: Walker, 1997.
7. Castellanos, D.: The Ubiquitous Pi. Part I. Math. Mag. 61, 1988.
8. Castellanos, D.: The Ubiquitous Pi. Part II. Math. Mag. 61, 1988.
9. Gurdon, X. and Sebah, P.: Collection of Series for π . http://numbers.computation.free.fr/Constants/Pi/piSeries.html.