Article

Concerning the Dirac γ-Matrices Under a Lorentz Transformation of the Dirac Equation

G. G. Nyambuya[1]
National University of Science and Technology,
Faculty of Applied Sciences – Department of Applied Physics,
P. O. Box 939, Ascot, Bulawayo,
Republic of Zimbabwe.

Abstract

We embolden the idea that the Dirac 4×4 γ-matrices are four-vectors where the space components (γ^i) represent spin and the forth component (γ^0) should likewise represent the time component of spin in the usual four-vector formalism of the Special Theory of Relativity. With the γ-matrices as four-vectors, it is seen that the Dirac equation admits two kinds of wavefunctions – (1) the usual four component Dirac bispinor ψ and (2) a scalar four component bispinor ϕ. Realizing this, and knowing forehand of the existing mystery as to why Leptons and Neutrinos come in pairs, we seize the moment and make the suggestion that the pair (ψ, ϕ) can be used as a starting point to explain mystery of why in their three generations $[(e^{\pm}, \nu_{\mathrm{e}}), (\mu^{\pm}, \nu_{\mu}), (\tau^{\pm}, \nu_{\tau})]$, Leptons and Neutrinos come in doublets. In this suggestion, the scalar-bispinor ϕ can be thought of as the Neutrino while the usual Dirac bispinor ψ can be thought of as the Lepton.

Keywords: Coulomb gauge, Maxwell's equations, Gamma-Ray Bursts.

1 Introduction

As taught to physics students through the plethora of textbooks available on our planet (*e.g.*, Zee 2010, Itzykson & Zuber 1980, Sakurai 1967, Messiah 1962, Schweber 1961), the Dirac 4×4 γ-matrices (γ^{μ}) are usually presented as objects that undergo a transformation during a Lorentz transformation of the Dirac (1928*a*,*b*) equation. This issue of the transformation of these γ-matrices is not well represented in the literature (cf., Nikolić 2014). There thus is a need to clear the air around this issue regarding the proper transformation properties of these matrices. To that end, we here argue in favour of these matrices as physical four-vectors and as such, they must under a Lorentz transformation transform as four-vectors. In-fact, it is well known that the γ^i-matrices ($i = 1, 2, 3$) represent spin (*i.e.*, $\boldsymbol{\mathcal{S}} = \frac{1}{2}\hbar\gamma^1\hat{\boldsymbol{\imath}} + \frac{1}{2}\hbar\gamma^2\hat{\boldsymbol{\jmath}} + \frac{1}{2}\hbar\gamma^3\hat{\boldsymbol{k}}$) because, together with the angular momentum operator ($\boldsymbol{\mathcal{L}}$), their sum total of the orbital angular momentum and spin ($\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{S}}$) commutes with the Dirac Hamiltonian ($\mathcal{H}_D$), *i.e.* ($[\boldsymbol{\mathcal{J}}, \mathcal{H}_D] = 0$), implying that $\boldsymbol{\mathcal{J}}$ is a constant of motion.

For a particle whose rest-mass and wave-function are m_0 and ψ respectively, the corresponding Dirac equation is given by:

$$\imath\hbar\gamma^{\mu}\partial_{\mu}\psi = \mathrm{m}_0 c\psi, \tag{1.1}$$

[1]Correspondence: E-mail: physicist.ggn@gmail.com

where:

$$\gamma^0 = \begin{pmatrix} \mathcal{I}_2 & 0 \\ 0 & -\mathcal{I}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.2}$$

are the 4×4 Dirac γ-matrices where $\mathcal{I}_2$ and 0 are the $2{\times}2$ identity and null matrices respectively, and $|\psi\rangle$ is the four component Dirac wave-function, $\hbar$ is the normalized Planck constant, c is the speed of light in vacuum, $\imath = \sqrt{-1}$, and:

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{1.3}$$

is the Dirac 4×1 four component wavefunction and ψ_L and ψ_R are the Dirac bispinors that are defined such that:

$$\psi_L = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix}. \tag{1.4}$$

Throughout this reading – unless otherwise specified; the Greek indices will *here-and-after* be understood to mean $(\mu, \nu, \,... = 0, 1, 2, 3)$ and the lower case English alphabet indices $(i, j, k \,... = 1, 2, 3)$.

2 Lorentz Transformation of the Dirac as Usually Presented

To prove Lorentz Invariance (Covariance) two conditions must be satisfied:

1. The first condition is that: given any two inertial observers O and O$'$ anywhere in spacetime, if in the frame O we have:

$$[i\hbar\gamma^\mu\partial_\mu - \mathrm{m}_0 c]\psi(x) = 0, \tag{2.1}$$

as the Dirac equation for the particle ψ, then:

$$[i\hbar\gamma^{\mu'}\partial_{\mu'} - \mathrm{m}_0 c]\psi'(x') = 0, \tag{2.2}$$

is the equation describing the same state but in the frame O$'$.

2. The second condition is that: given that $\psi(x)$ is the wavefunction as measured by observer O, there must be a prescription for observer O$'$ to compute $\psi'(x')$ from $\psi(x)$ where $\psi'(x')$ describes to O$'$ the same physical state as that measured by O. The conserve must be true as-well, that is: there must exist a prescription such that starting from equation (2.2), one can arrive at (2.1).

In simpler mathematical terms, the above two requirements are saying that: starting from equation (2.1), there must exist some physically legitimate transformations within the framework of Lorentz transformations that can take us from this equation (2.1) to equation (2.2) and *vice-versa*. If we can find these, then, the Dirac equation is said to be Lorentz Invariant (Covariant).

Now, since the Lorentz transformations are linear, it is to be required or expected of the transformations between $\psi(x)$ and $\psi'(x')$ to be linear too, that is:

$$\psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x'), \tag{2.3}$$

where $S(\Lambda)$ is a 4×4 matrix which depends only on the relative velocities of O and O$'$ and Λ is the Lorentz transformation matrix. $S(\Lambda)$ has an inverse if O $\to$ O$'$ and also O$'$ $\to$ O. The inverse is:

$$\psi(x) = S^{-1}(\Lambda)\psi'(x') = S^{-1}(\Lambda)\psi'(\Lambda x), \tag{2.4}$$

or we could write:

$$\psi(x) = S(\Lambda^{-1})\psi'(\Lambda x) \Longrightarrow S(\Lambda^{-1}) = S^{-1}(\Lambda). \tag{2.5}$$

We can now write (2.1), as:

$$\left[i\hbar\gamma^{\mu}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\partial_{\mu'} - \mathrm{m}_0 c\right] S^{-1}(\Lambda)\psi'(x') = 0, \tag{2.6}$$

and multiplying this from the left by $S(\Lambda)$, we have:

$$S(\Lambda)\left[i\hbar\gamma^{\mu}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\partial_{\mu'} - \mathrm{m}_0 c\right] S^{-1}(\Lambda)\psi'(x') = 0, \tag{2.7}$$

and hence:

$$\left[i\hbar S(\Lambda)\gamma^{\mu}\frac{\partial x^{\mu'}}{\partial x^{\mu}}S^{-1}(\Lambda)\partial_{\mu'} - \mathrm{m}_0 c\right]\psi'(x') = 0. \tag{2.8}$$

Therefore, for the above equation to be identical to equation (2.2) (hence Lorentz Invariant), the requirement is that:

$$\gamma^{\mu'} = S(\Lambda)\gamma^{\mu}\frac{\partial x^{\mu'}}{\partial x^{\mu}}S^{-1}(\Lambda), \tag{2.9}$$

hence, we have shown that – for as long as $S^{-1}(\Lambda)$ exists, equation (2.1) is Lorentz Invariant.

3 Dirac 4×4 γ-Matrices as a Four-Vector

The Dirac equation (1.1) can be re-written in the Schrödinger formulation as $(\mathscr{H}\psi = \mathscr{E}\psi)$ where $\mathscr{H}$ and $\mathscr{E}$ are the energy and Hamiltonian operators respectively. In this Schrödinger formulation, $\mathscr{H}$, will be such that it is given by:

$$\mathscr{H} = \gamma^0 \mathrm{m}_0 c^2 - \imath\hbar c\gamma^0\gamma^j\partial_j, \tag{3.1}$$

and $(\mathcal{E} = i\hbar\partial/\partial t)$.

Now, according to the quantum mechanical equation governing the evolution of any quantum operator $\mathcal{Q}$, we have:

$$\imath\hbar\frac{\partial \mathcal{Q}}{\partial t} = \mathcal{Q}\mathcal{H} - \mathcal{H}\mathcal{Q} = [\mathcal{Q}, \mathcal{H}]\,. \tag{3.2}$$

If:

$$[\mathcal{Q}, \mathcal{H}] \equiv 0, \tag{3.3}$$

then, the quantum mechanical observable corresponding to the operator $\mathcal{Q}$ is a conserved physical quantity.

With this [equation (3.2)] in mind, Dirac asked himself the natural question – what the "strange" new γ-matrices appearing in his equation really represent. What are they? In-order to answer this question, he decided to have a "look" at the quantum mechanical orbital angular momentum operator:

$$\mathcal{L}_i = (\boldsymbol{r} \times \boldsymbol{p})_i = -\imath\hbar\epsilon_{ijk}x_j\partial_k, \tag{3.4}$$

where, ϵ_{ijk} is the completely-antisymmetric three dimensional *Levi-Civita* tensor. In the above definition of $\mathcal{L}_i$ the momentum operator $\boldsymbol{p}$ is the usual quantum mechanical operator, *i.e.*:

$$\boldsymbol{p} = -\imath\hbar\boldsymbol{\nabla} \qquad \Rightarrow \qquad p_i = \imath\hbar\partial_i. \tag{3.5}$$

From this definition of $\mathcal{L}_i$ given in (3.4), it follows from (3.2) that $\imath\hbar\partial\mathcal{L}_i/\partial t = [\mathcal{L}_i, \mathcal{H}]$, will be such that:

$$\imath\hbar\frac{\partial \mathcal{L}_i}{\partial t} = -\imath\hbar \mathrm{m}_0 c^2\epsilon_{ijk}\left[x_j\partial_k, \gamma^0\right] + \hbar^2 c\epsilon_{ijk}\left[x_j\partial_k, \gamma^0\gamma^l\partial_l\right]. \tag{3.6}$$

Now, because – the term $\gamma^0\mathrm{m}_0c^2$ is a constant containing no term in p_i, it follows from this fact that $(\epsilon_{ijk}\left[x_j\partial_k, \gamma^0\right] \equiv 0)$, hence (3.6) will reduce to:

$$\imath\hbar\frac{\partial \mathcal{L}_i}{\partial t} = \hbar^2 c\epsilon_{ijk}\gamma^0\gamma^l\left[x_j\partial_k, \partial_l\right] = \hbar^2 c\epsilon_{ijk}\gamma^0\gamma^l\left(x_j\partial_k\partial_l - \partial_l x_j\partial_k\right). \tag{3.7}$$

From the commutation relation of position (x_i) and momentum $(-\imath\hbar\partial_j)$ due to the Heisenberg (1927) uncertainty principle, namely $(-\imath\hbar\left[x_i, \partial_j\right] = -\imath\hbar\delta_{ij})$ where δ_{ij} is the usual Kronecker-delta function, it follows that if in (3.7), we substitute $(\partial_l x_j = x_j\partial_l + \delta_{lj})$, this equation is going to reduce to:

$$\imath\hbar\frac{\partial \mathcal{L}_i}{\partial t} = \hbar^2 c\epsilon_{ijk}\gamma^0\gamma^l\underbrace{\left(x_j\partial_k\partial_l - x_j\partial_l\partial_k\right)} + \hbar^2 c\epsilon_{ijk}\gamma^0\gamma^l\delta_{lj}\partial_k. \tag{3.8}$$

The term with the under-brace vanishes identically, that is to say: $(x_j\partial_k\partial_l - x_j\partial_l\partial_k \equiv 0)$; and $(\epsilon_{ijk}\gamma^0\gamma^l\delta_{lj} = \epsilon_{ilk}\gamma^0\gamma^l)$, it follows that (3.8) will reduce to:

$$\imath\hbar\frac{\partial \mathcal{L}_i}{\partial t} = \hbar^2 c\epsilon_{ilk}\gamma^0\gamma^l\partial_k. \tag{3.9}$$

Since this result (3.9) is non-zero, it follows from the dynamical evolution theorem (3.3) of Quantum Mechanics (QM) that none of the angular momentum components $\mathcal{L}_i$ are – *for the Dirac particle* – going to be constants of motion. This result obviously bothered the great and agile mind of Paul Dirac. For example, a non-conserved angular momentum would mean spiral orbits *i.e.*, Dirac particles do not move in fixed and well defined orbits as happens with electrons of the Hydrogen atom for example; at the very least, this is very disturbing because it does not tally with observations. The miniature beauty that Dirac had – had the rare privilege to discover and, the first human being to "see" with his beautiful and great mind – this – had to be salvaged[2] somehow.

Enter spin! Dirac figured that *Subtle Nature* must conserve something redolent with orbital angular momentum, and he considered adding something to $\mathcal{L}_i$ that would satisfy the desired conservation criterion, *i.e.*: call this unknown, mysterious and arcane quantity $\mathcal{S}_i$ and demand that:

$$\imath\hbar\frac{\partial\left(\mathcal{L}_i+\mathcal{S}_i\right)}{\partial t}\equiv 0. \tag{3.10}$$

This means that this strange quantity $\mathcal{S}_i$ must be such that:

$$\imath\hbar\frac{\partial\mathcal{S}_i}{\partial t}=\left[\mathcal{S}_i,\mathcal{H}\right]=-\hbar^2 c\epsilon_{ilk}\gamma^0\gamma^l\partial_k. \tag{3.11}$$

Solving (3.11) for $\mathcal{S}_i$, Dirac arrived at:

$$\mathcal{S}_i=\frac{1}{2}\hbar\left(\begin{array}{cc}\sigma_i & 0\\ 0 & \sigma_i\end{array}\right)=\frac{1}{2}\hbar\gamma^5\gamma^i, \tag{3.12}$$

where $(\gamma^5=\imath\gamma^0\gamma^1\gamma^2\gamma^3)$, is the usual Dirac gamma-5 matrix.

Now, realising that:

1. The matrices σ_i are Pauli matrices and they had been *ad hocly* introduced into physics to account for the spin of the Electron (Uhlenbeck & Goudsmit 1925);
2. His equation – when taken in the non-relativistic limit, it would account for the then unexplained gyromagnetic ratio $(\mathrm{g}=2)$ of the Electron and this same equation emerged with σ_i explaining the Electron's spin;

The agile Dirac seized the golden moment and forthwith identified $\mathcal{S}_i$ with the ψ-particle's spin. The factor $\frac{1}{2}\hbar$ in $\mathcal{S}_i$ implies that the Dirac particle carries spin $1/2$, hence, the Dirac equation (1.1) is an equation for a particle with spin $1/2$!

While in this way (*i.e.*, as demonstrated above) Dirac was able to explain and "demystify" Wolfgang Pauli $(1900-1958)$'s strange spin concept which at the time had only been inserted into physics by "*the sleight of hand*" out of unavoidable necessity, what bothers us (*i.e.*, myself) the most is how it comes about that we (physicists) have had issues to do with the transformational properties of the γ-matrices? Why? Really – why? The fact that orbital angular momentum $\boldsymbol{\mathcal{L}}$ is a vector, it follows that $\boldsymbol{\mathcal{S}}$ is vector as-well because we can only add vectors to vectors. If $\boldsymbol{\mathcal{S}}$ is a vector, then the matrices γ^i must be components of a 3-vector, so must the matrix γ^0 be the component of the time-vector in the usual four-vector formalism, hence γ^μ must be a four-vector. So, right from the word go, it must have been clear that the γ-matrices must be four-vectors.

[2]Such is the indispensable attitude of the greatest theoretical physicists that ever graced the face of planet Earth – beauty must and is to be preserved; this is an ideal for which they will live for, and if needs be, it is an ideal for which they will give-up their life by taking a gamble to find that unknown quantity that restores the beauty glimpsed!

4 Dirac Equation with the γ-Matrices as a Four-Vector

With γ-matrices now taken as a four-vector, the object $\gamma^\mu\partial_\mu$ is a scalar, the meaning of which is that the Dirac equation will now accommodate two types of spinors "the usual Dirac bispinor" and a new "scalar-bispinor", *i.e.*:

1. A spinor that is a scalar. Let us call this a scalar-bispinor and let us denote it with the symbol ϕ and because of its scalar nature – under a Lorentz transformation, we will have $(\phi' = \phi)$. Just like the ordinary Dirac wavefunction ψ is a 4×1 component object, ϕ is also a 4×1 object, *i.e.*:

$$\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_L \\ \\ \phi_R \end{pmatrix}, \tag{4.1}$$

where ϕ_L and ϕ_R are the scalar-spinors – which like the ordinary left and right handed Dirac spinors (ψ_L, ψ_R); ψ_L and ϕ_R are defined:

$$\phi_L = \begin{pmatrix} \phi_0 \\ \\ \phi_1 \end{pmatrix} \text{ and } \phi_R = \begin{pmatrix} \phi_2 \\ \\ \phi_3 \end{pmatrix}. \tag{4.2}$$

Consideration of the scalar-bispinor has been made in the past by others (*e.g.*, Chapman & Leiter 1976).

2. The ordinary Dirac bispinor ψ: that transforms linearly under a Lorentz transformation *i.e.* $(\psi' = S\psi)$, where Lorentz Invariance (Covariance) requires that $S = S(x^\mu, \dot{x}^\mu)$ be such that:

$$\gamma^{\mu'}\partial_{\mu'} S = \gamma^\mu \partial_\mu S = 0, \tag{4.3}$$

and:

$$\gamma^\mu = S^{-1}\gamma^\mu S \quad \Rightarrow \quad [S, \gamma^\mu] = 0. \tag{4.4}$$

Now, we certainly must ask "*What does this all mean*". That is to say, the fact that the Dirac equation allows for the existence of the usual Dirac bispinor ψ and in addition to that – a scalar-bispinor ϕ? Taken at the same level of understanding that the Dirac equation's prediction of the existence of antimatter is premised on the Dirac equation being symmetric under charge conjugation – on that very same level of understanding, this fact that the Dirac equation in its most natural and un-tempered state as presented herein – it, allows for the existence of the usual Dirac bispinor ψ and a scalar-bispinor ϕ; on this very same train of logic – the said fact on the Dirac bispinor ψ and a scalar-bispinor ϕ, naturally implies that for every Dirac bispinor ψ, there must exist a corresponding scalar-bispinor ϕ. That is, the Dirac bispinor ψ and the scalar-bispinor ϕ must come in pairs. There is no escape from this train of logic.

If we are thinking of Leptons and Neutrinos, the above pair-picture of (ψ, ϕ) makes perfect sense. Based on this picture, we can write the Dirac equation for this pair (ψ, ϕ) as:

$$\imath\hbar\gamma^\mu\partial_\mu \begin{pmatrix} \psi \\ \\ \phi \end{pmatrix} = \mathrm{m}_0 c \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} \psi \\ \\ \phi \end{pmatrix}, \tag{4.5}$$

where η is a scalar-constant that we have introduced so as to accommodate the possibility that the particle-pair (ψ, ϕ), may have different masses. In this way, one can begin to entertain ideas on how to explain the Lepton-Neutrino pairing $[(e^{\pm}, \nu_{\mathrm{e}}), (\mu^{\pm}, \nu_{\mu}), (\tau^{\pm}, \nu_{\tau})]$. We have no intention of doing this or going any deeper on this matter but merely to point out – as we have just done – that, this idea may prove a viable avenue of research to those seeking an explanation of why this mysterious pairing occurs in nature.

5 General Discussion

We must categorically state that – what we have presented herein is not new at all. All we have endeavoured is to make bold the point that the γ-matrices constitute a four-vector. Perhaps the only novelty there is – in the present contribution – is the suggestion that we have made – namely that, the resulting scalar-bispinor (ϕ) and the usual Dirac bispinor (ψ) can be used as a starting point to explain the currently open problem of the three generation Lepton-Neutrino pairing $(e^{\pm}, \nu_{\mathrm{e}})$, $(\mu^{\pm}, \nu_{\mu})$ and $(\tau^{\pm}, \nu_{\tau})$; where the scalar-bispinor can be assumed to be the Neutrino while the usual Dirac bispinor can be thought of the Lepton.

References

Chapman, T. C. & Leiter, D. J. (1976), 'On the Generally Covariant Dirac Equation', *American Journal of Physics* **44**(9), 858–862.

Dirac, P. A. M. (1928*a*), 'The Quantum Theory of the Electron', *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **117**(778), 610–624.

Dirac, P. A. M. (1928*b*), 'The Quantum Theory of the Electron. Part II', *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **118**(779), 351–361.

Heisenberg, W. (1927), 'Ueber den anschaulichen Inhalt der quantentheoretischen Kinematik and Mechanik', *Zeitschrift für Physik* **43**, 172–198. *Engilish Translation*: Wheeler, J. A. and Zurek, W. H. (eds) (1983) Quantum Theory and Measurement (Princeton NJ: Princeton University Press), pp. 62-84.

Itzykson, C. & Zuber, J.-B. (1980), *Quantum Mechanics II*, McGraw-Hill, New York, USA.

Messiah, A. (1962), *Quantum Mechanics II*, 2^{st} edn, North-Holland Publishing Company, Amsterdam, Netherland.

Nikolić, H. (2014), 'How (Not) To Teach Lorentz Covariance of the Dirac Equation', *European Journal of Physics* **35**(3), 035003(10pp).

Sakurai, J. J. (1967), *Advanced Quantum Mechanics*, Addison-Wesley Publishing Company, Massachusetts, USA.

Schweber, S. S. (1961), *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson and Company, New York, USA.

Uhlenbeck, G. E. & Goudsmit, S. (1925), *Naturwissenschaften* **33**(47), 953.

Zee, A. (2010), *Quantum Field Theory in a Nutshell*, 2^{st} edn, Princeton University Press, USA.