A NEW BINOMIAL FORMULA FOR THE SUM OF TWO POWERS

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Abstract. In this paper, we reveal a new binomial formula that expresses the sum of, or difference between two powers, $a^x \pm b^y$, as a binomial expansion of a single power, z. Like the standard binomial formula it includes the normal binomial coefficients, factors and indices, but includes an additional non-standard factor. The new formula (with an upper index z) mimics a standard binomial formula (to the power z) without the value of the binomial expansion changing even when z itself changes. This has exciting implications for certain diophantine equations. This short paper simply highlights its existence.

Introduction The standard binomial formula expands a *single* power of a binomial expression. Here we reveal a new binomial formula that expands the sum/difference of *two* powers, $a^x \pm b^y$, as if it were a single power. The critical thing about this new formula is that it has an upper index, z, which can have any value without changing the overall value of the sum itself. It thus mimics a standard binomial formula (to the power z) without its own value changing even when z itself changes.

1. The binomial formula for the sum/difference between two powers

The new formula for the difference between two powers, for all $z \in \mathbb{Z}$, is:

$$a^x \pm b^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k} (-ab)^k (a^{x-z-k} \pm b^{y-z-k}).$$

The last factor, $(a^{x-z-k} \pm b^{y-z-k})$, is non-standard. By comparison, the *standard* binomial formula for a single power to z (as a difference between two powers) is:

$$(p \pm q)^z = \sum_{k=0}^{z} \binom{z}{k} p^{z-k} (\pm q)^k.$$

So from our new formula, we say that $a^x \pm b^y$ is equal to:

$$(a+b)(a^{x-1} \pm b^{y-1}) - ab(a^{x-2} \pm b^{y-2})$$

$$(a+b)^2(a^{x-2} \pm b^{y-2}) - 2ab(a+b)(a^{x-3} \pm b^{y-3}) + (ab)^2(a^{x-4} \pm b^{y-4})$$

$$(a+b)^3(a^{x-3} \pm b^{y-3}) - 3ab(a+b)^2(a^{x-4} \pm b^{y-4}) + 3(ab)^2(a+b)(a^{x-5} \pm b^{y-5}) - (ab)^3(a^{x-6} \pm b^{y-6})$$

And so on, *ad infinitum*.

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