THE CHAMELEON EFFECT, THE BINOMIAL THEOREM AND BEAL'S CONJECTURE

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Abstract. In psychology, the Chameleon Effect describes how an animal's behaviour can adapt to, or mimic, its environment through non-conscious mimicry. In the first part of this paper, we show how $a^x - b^y$ can be expressed as a binomial expansion (with an upper index, z) that, like a chameleon, mimics a standard binomial formula (to the power z) without its own value changing even when z itself changes. In the second part we will show how this leads to a proof for the Beal Conjecture. We finish by outlining how this method can be applied to a more generalised form of the equation.

Introduction In the 17th century, Pierre de Fermat asserted that $A^n = B^n + C^n$ has no solutions for values of n greater than 2, where A, B and C are co-prime integers. This became known as Fermat's Last Theorem (FLT). Up until 1995, when Annals of Mathematics published a proof by Andrew Wiles,[1] FLT was technically a conjecture. But mathematicians so far have been unable to prove it using only the mathematical knowledge available to Fermat himself.

In 1993, a Texan number theory enthusiast named Andrew Beal began to suspect that co-prime bases for the equation $A^x = B^y + C^z$ might be impossible for values of x, y, z greater than 2, where A, B, C are co-prime integers. This has become known commonly as Beal's Conjecture (BC)[2] or sometimes the Mauldin Conjecture or the Tijdeman-Zagier Conjecture. It is a generalised form of FLT and states that if $A^x = B^y + C^z$, where A, B, C, are positive co-prime integers and x, y, z are all positive co-prime integers greater than 2, then A, B, C, must have a common prime factor.

The problem with the early attempts to prove FLT was that mathematicians proceeded by proving the cases for values of n one by one in what might be called a horizontal approach. After Fermat's proof using the method of infinite descent for the case $n = 4$ it became clear that only cases for prime values of n needed to be proved. So after Euler's proof for $n = 3$ in 1770, Gustav Lejeune-Dirilecht and Adrien-Marie Legendre both independently proved the case for $n = 5$ in 1825, and in 1839 Gabriel Lame succeeded in the case for $n = 7$. Although their methods might have yielded important mathematical insights, frustratingly it would take forever to prove all values of prime n, given the infinite set of primes, unless a proof could be found to tackle all the primes in one swoop.

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[1] Andrew Wiles, "Modular Elliptic Curves and Fermat's Last Theorem", Annals of Mathematics, Second Series, Vol. 141, No.3 (May, 1995), pp. 443-551.

[2] See "http://www.bealconjecture.com". Last accessed 14.3.14.

A vertical approach to the problem using, for example, proof by infinite descent, would seem to be the most productive one, simply on the grounds that it would deal with all values of n including all prime numbers. One of the inherent problems, however, of proving the theorem using a vertical approach appeared to be that when one assumed that a counterexample existed and then tried to derive a contradiction, the very properties that created the contradiction in the first place appeared to belong also to the non-counterexample.[3] With the additional parameters imposed by Beal's Conjecture the problems only seem to multiply.
Here, we demonstrate how $a^x - b^y$ can be reconfigured and expressed as a binomial expansion, containing not only the standard factors for a single power but also an additional non-standard factor. We will then investigate how this might lead to a simple proof for Beal's Conjecture. We finish by outlining how this method can be applied to a more generalised form of the equation. Without loss of integrity, we will rearrange the equation as $a^x - b^y = c^z$.

Definition 0.1. For the equation $a^x - b^y = c^z$, we define $a, b, c,$ as square-free integers (of which one at most must be even); and x, y, z are positive integers (of which one at most may be even), and $\gcd(x, y, z) = 1$,

Lemma 0.2. *To demonstrate that $a^x - b^y$ can be expressed as a binomial formula.*

We first observe that by adding $[ab(a^{x-2} - b^{y-2}) - b^y]$ to a^x and b^y respectively, and then rearranging, it is possible to reconfigure the expression such that:

$$a^x - b^y = (a+b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2}). \tag{0.1}$$

Now, since $a^x - b^y = (a+b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2})$, we can repeat the process for the components $(a^{x-1} - b^{y-1})$ and $(a^{x-2} - b^{y-2})$, and expand the right hand side of this equation as follows:

$$(a+b)[(a+b)(a^{x-2}-b^{y-2})-ab(a^{x-3}-b^{y-3)})]-ab[(a+b)(a^{x-3}-b^{y-3})-ab(a^{x-4}-b^{y-4)})] \tag{0.2}$$

$$= (a+b)^2(a^{x-2} - b^{y-2}) - 2ab(a+b)(a^{x-3} - b^{y-3}) + (ab)^2(a^{x-4} - b^{y-4}). \tag{0.3}$$

And again, repeating the process for $(a^{x-2} - b^{y-2})$, $(a^{x-3} - b^{y-3})$ and $(a^{x-4} - b^{y-4})$, we get:

$$(a+b)^3(a^{x-3}-b^{y-3})-3ab(a+b)^2(a^{x-4}-b^{y-4})+3(ab)^2(a+b)(a^{x-5}-b^{y-5})-(ab)^3(a^{x-6}-b^{y-6}). \tag{0.4}$$

If we continue this process we can expand $(a^x - b^y)$ *ad infinitum*, and using the binomial formula we can generalise it, for all $z \in \mathbb{Z}$, in the following formula:

$$a^x - b^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} - b^{y-z-k}). \tag{0.5}$$

By comparison, the *standard form* of the binomial theorem to the power z is:

$$(p-q)^z = \sum_{k=0}^{z} \binom{z}{k} p^{z-k}(-q)^k. \tag{0.6}$$

So the first three factors of the sum in (0.5), namely, $\binom{z}{k}$, $(a+b)^{z-k}$, and $(-ab)^k$ all obviously conform to the conventional forms of a standard binomial formula. But the last factor, $(a^{x-z-k} - b^{y-z-k})$, does not but may do.

[3]Peter Schorer "Is There a "Simple" Proof of Fermat's Last Theorem? Part (1) Introduction and Several New Approaches", 2014, www.occampress.com/fermat.pdf last accessed 14.3.14.

Remark 0.3. It is at this point that we can see the Chameleon Effect working. This new binomial formula, with an upper index z, behaves like a chameleon, that mimics a standard binomial expansion to the power z without changing its own value, so that regardless of the value we give to z, the value of $(a^x - b^y)$ never changes. It is this property that allows us to compare the two sides of the Beal equation more easily.

This begs the question whether the whole right hand side of (0.5) can be represented in the form of a binomial expansion to the power z? But before proceeding we need to observe that in the same way we expanded $(a^x - b^y)$, we can also expand $(a^x + b^y)$, such that

$$a^x + b^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (a^{x-z-k} + b^{y-z-k}). \tag{0.7}$$

We will return to this equation, but simply to notice again that, as in (0.5), the first three factors of the sum all conform to the conventional forms of a standard binomial formula, while the last may or may not. Now let us therefore examine this finding in the light of Beal's Conjecture.

Theorem 0.4. *To prove that, for the equation $a^x - b^y = c^z$, integer solutions only exist for the values of x or y or $z = 1, 2$, but not for values of $x, y, z > 2$.*

Proof. We first assume that there exists a solution for the equation $a^x - b^y = c^z$ for values of $x, y, z > 2$. So if $c^z = a^x - b^y$ then it follows, from (0.5), that:

$$c^z = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (a^{x-z-k} - b^{y-z-k}). \tag{0.8}$$

Now, before we even consider trying to find contradictions, we must establish the circumstances under which the right hand side conforms to a standard form of binomial expansion to a power z.

Lemma 0.5. *To establish the circumstances under which the right hand side conforms to a standard form of binomial expansion to a power z.*

Bearing in mind that the right hand side of the equation in (0.8) already possesses standard factors, it follows that the last factor, $(a^{x-z-k} - b^{y-z-k})$, must be equal to $|s|^{z-k}|t|^k$, for all $s, t \in \mathbb{Z}$ except the trivial case of $s = (-ab)$ and $t = (a+b)$. We must use absolute values of s and t to avoid limiting the possible values of c.

So let $(a^{x-z-k} - b^{y-z-k}) = |s|^{z-k}|t|^k$, such that, from (0.8):

$$c^z = \sum_{k=0}^{z} \binom{z}{k} [(a+b)|s|]^{z-k}[(-ab)|t|]^k. \tag{0.9}$$

And thus:

$$c^z = [(a+b)|s| + (-ab)|t|]^z. \tag{0.10}$$

The equation in (0.9) is now in the standard form for a binomial expansion. But from these absolute values of s and t, four possibilities arise.
When $|s| = s$ and $|t| = t$ it follows that:

$$[(a+b)s + (-ab)t]^z = c^z. \tag{0.11}$$

When $|s| = (-s)$ and $|t| = (-t)$

$$[(a+b)(-s) + (-ab)(-t)]^z = (-c)^z. \tag{0.12}$$

However, when $|s|, |t|$ have opposite signs (from each other), it creates a new power to z, namely c_1^z, where c_1 is square-free and $\gcd(a, b, c_1) = 1$.
So when $|s| = s$ and $|t| = (-t)$ we get:

$$[(a+b)s + (-ab)(-t)]^z = c_1^z. \tag{0.13}$$

Likewise, when $|s| = (-s)$ and $|t| = t$ it follows that:

$$[(a+b)(-s) + (-ab)t]^z = (-c_1)^z. \tag{0.14}$$

We first consider the new power in (0.13) and (0.14). At first glance, this change in signs for s or t might be thought inevitably to affect the signs in the terms of our binomial expansion in (0.8) thereby compromising its overall integrity, which would undermine our whole strategy. However, we also noted in (0.7) that:

$$a^x + b^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k} (-ab)^k (a^{x-z-k} + b^{y-z-k}). \tag{0.15}$$

Since the first three factors in this equation (and their respective signs) are exactly the same, the changes of sign in s and t affect only the final factor, in fact by reversing the sign in the brackets. We can therefore, more precisely, let $(a^{x-z-k} - b^{y-z-k}) = s^{z-k}t^k$, such that:

$$c^z = \sum_{k=0}^{z} \binom{z}{k} [(a+b)s]^{z-k} [(-ab)t]^k, \tag{0.16}$$

and let $(a^{x-z-k} + b^{y-z-k}) = s^{z-k}(-t)^k$, such that:

$$c_1^z = \sum_{k=0}^{z} \binom{z}{k} [(a+b)s]^{z-k} [(-ab)(-t)]^k. \tag{0.17}$$

Notice that whenever s is negative, only the sign of c changes. The absolute value of c remains the same. So whether our power is c^z or c_1^z now depends on whether z is odd or even. So from (0.11), when z is odd, we get:

$$-(a^x + b^y) = -(c)^z$$

$$\Rightarrow c^z = a^x + b^y. \tag{0.18}$$

The integrity of the initial equation is thus preserved. Also from (0.11), when z is even, using our parallel assumption, we get:

$$-(a^x + b^y) = (c)^z$$

$$\Rightarrow a^x + b^y + c^z = 0. \tag{0.19}$$

Clearly, in this case, there can be no integer solutions.

Now, from (0.13), when z is odd, we get:

$$-(c_1)^z = -(a^x + b^y)$$

$$\Rightarrow (c_1^z) = a^x + b^y. \tag{0.20}$$

The integrity of the initial equation is thus preserved.

Likewise, from (0.13) when z is even, it follows that:

$$(c_1^z) = -(a^x + b^y)$$
$$\Rightarrow a^x + b^y + c_1^z = 0. \tag{0.21}$$

Again, in this case, there can be no integer solutions.

From all this, we now have a secondary proof, parallel to the statement of the main proof we saw earlier, that accommodates the change of sign. So we must now also assume that there exists a solution for the equation $a^x + b^y = c_1^z$ for values of $x, y, z > 2$. So if $c_1^z = a^x + b^y$ then it follows, from (0.7), that:

$$c_1^z = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} + b^{y-z-k}). \tag{0.22}$$

We now have two equations to consider, one for c^z and one for c_1^z. From (0.8) and (0.9), since both equations are equal to c^z, we have the following:

(0.23)
$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k s^{z-k} t^k = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} - b^{y-z-k}).$$

Likewise, we also have an equation for c_1^z:

(0.24)
$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k s^{z-k} (-t)^k = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} + b^{y-z-k}).$$

But since we have defined c_1 in the same way we defined c, we can combine these two equations without loss of integrity as follows:

(0.25)
$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k |s|^{z-k} |t|^k = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} \pm b^{y-z-k}).$$

As we will see, the change of sign in the brackets makes no difference to the proof. For this equation to have solutions it is necessary that the last factor on the right hand side, $(a^{x-z-k} \pm b^{y-z-k})$, must correspond in each and every counterpart (k^{th}) term with the last factors on the right hand side, $(|s|^{z-k}|t|^k)$, for any given value of z. If they do, then the whole of the right hand side will be a power to z (as we know the left hand side is), and the Beal equation will have solutions. But if just one term of the sum exists where $(a^{x-z-k} \pm b^{y-z-k})$ does not equal $(|s|^{z-k}|t|^k)$, then not only will the integrity of that particular k^{th} term be compromised as a valid binomial term, but also the whole expression as an expansion of a power to z. In the latter scenario, no solutions will exist.

What we must now proceed to show, therefore, is whether (for any given value of $z > 2$), $|s|^{z-k}|t|^k$ is equal to $(a^{x-z-k} \pm b^{y-z-k})$ for all k terms of the sum *simultaneously* (from $k = 0$ to $k = z$). In other words, we must show whether:

when $z = 1$, $|s|^1|t|^0 = (a^{x-1} \pm b^{y-1})$ (for $k = 0$), *and that* $|s|^0|t|^1 = (a^{x-2} \pm b^{y-2})$ (for $k = 1$);

and when $z = 2$, $|s|^2|t|^0 = (a^{x-2} \pm b^{y-2})$, (for $k = 0$), *and that* $|s|^1|t|^1 = (a^{x-3} \pm b^{y-3})$ (for $k = 1$), *and that* $|s|^0|t|^2 = (a^{x-4} \pm b^{y-4})$ (for $k = 2$);

and when $z = 3$, $|s|^3|t|^0 = (a^{x-3} \pm b^{y-3})$ (for $k = 0$), *and that* $|s|^2|t|^1 = (a^{x-4} \pm b^{y-4})$ (for $k = 1$), *and that* $|s|^1|t|^2 = (a^{x-5} \pm b^{y-5})$ (for $k = 2$), *and that* $|s|^0|t|^3 = (a^{x-6} \pm b^{y-6})$ (for $k = 3$).

And so on. But without testing for every value of z one by one *ad infinitum* we will test for all values of $z > 2$ in one go by using the generalised equation:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k = \sum_{k=0}^{z} (a^{x-z-k} \pm b^{y-z-k}) \tag{0.26}$$

So what we will now show is that when $z > 2$ an inequality arises in at least one of the corresponding k^{th} terms, but that when $z = 1, 2$ every corresponding k^{th} term will be equal. We will do this using proof by contradiction.

Lemma 0.6. *To prove that when $z > 2$ an inequality arises in at least one of the corresponding k^{th} terms, but that when $z = 1, 2$ every corresponding k^{th} term will be equal.*

First, then, we assume that

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k = \sum_{k=0}^{z} (a^{x-z-k} \pm b^{y-z-k}). \tag{0.27}$$

But how can we show this, when the number of equations we need to compare increases as the values of z and k increase (*ad infinitum*)? We shall do this in three steps. First, from the first and last terms (where s and t occur on their own) we will establish the respective values of $|s|, |s|^z, |s|^{z-1}, |t|, |t|^z$, and $|t|^{z-1}$ (in terms of a and b). Secondly we will use these results to evaluate what the *second* and *penultimate* terms should be, and equating them with the second and penultimate terms directly derived from $(a^{x-z-k} \pm b^{y-z-k})$. Thirdly, we will substitute like-terms to reveal the contradictions when they occur. [We will not need to go further than finding the second and penultimate terms since they are sufficient to reveal the contradiction.]

STEP 1

Using the equation in (0.26), we can establish the respective values of $|s|, |s|^z, |s|^{z-1}, |t|, |t|^z$, and $|t|^{z-1}$, using first and last terms (i.e. $k = 0$ and $k = z$). So when $k = 0$, the first term in the binomial series is $|s|^z$, such that:

$$|s|^z = \pm(a^{x-z} \pm b^{y-z}), \tag{0.28}$$

from which it follows that:

$$|s| = \pm(a^{x-z} \pm b^{y-z})^{1/z}, \tag{0.29}$$

and

$$|s|^{z-1} = \pm(a^{x-z} \pm b^{y-z})^{(z-1)/z}. \tag{0.30}$$

Likewise, when $k = z$, the last term in the binomial series is $|t|^z$, such that:

$$|t|^z = \pm(a^{x-2z} \pm b^{y-2z}), \tag{0.31}$$

from which it follows that:

$$|t| = \pm(a^{x-2z} \pm b^{y-2z})^{1/z}, \tag{0.32}$$

and

$$|t|^{z-1} = \pm(a^{x-2z} \pm b^{y-2z})^{(z-1)/z}. \tag{0.33}$$

STEP 2

Using these different values of $|s|$ and $|t|$, we are now in a position to work out what the *second* and *penultimate* terms (in terms of a and b). Thus, from (0.29) and (0.31), it follows that the *second* term, $|s|^{z-1}|t|$, is:

$$\pm(a^{x-z} \pm b^{y-z})^{(z-1)/z}(a^{x-2z} \pm b^{y-2z})^{1/z}. \tag{0.34}$$

And since we know, from the right hand side of the equation in (0.26), that the second term in the binomial expansion is $\pm(a^{x-z-1} \pm b^{y-z-1})$, i.e. when $k = 1$, it follows that:

$$\pm(a^{x-z-1} \pm b^{y-z-1}) = \pm(a^{x-z} \pm b^{y-z})^{(z-1)/z}(a^{x-2z} \pm b^{y-2z})^{1/z}. \tag{0.35}$$

Dividing both sides by $\pm(a^{x-z} \pm b^{y-z})^{(z-2)/z}$ we get:

$$\frac{\pm(a^{x-z-1} \pm b^{y-z-1})}{\pm(a^{x-z} \pm b^{y-z})^{(z-2)/z}} = \pm(a^{x-z} \pm b^{y-z})^{1/z}(a^{x-2z} \pm b^{y-2z})^{1/z}. \tag{0.36}$$

Similarly, it follows from (0.28) and (0.32), that the *penultimate* term, $|s||t|^{z-1}$, is:

$$\pm(a^{x-z} \pm b^{y-z})^{1/z}(a^{x-2z} \pm b^{y-2z})^{(z-1)/z}. \tag{0.37}$$

And since we know, from the right hand side of the equation in (0.26), that the penultimate term in the binomial expansion is $\pm(a^{x-2z+1} \pm b^{y-2z+1})$, i.e. when $k = z - 1$, it follows that:

$$\pm(a^{x-2z+1} \pm b^{y-2z+1}) = \pm(a^{x-z} \pm b^{y-z})^{1/z}(a^{x-2z} \pm b^{y-2z})^{(z-1)/z}. \tag{0.38}$$

Dividing both sides by $(a^{x-2z} \pm b^{y-2z})^{(z-2)/z}$ we get:

$$\frac{\pm(a^{x-2z+1} \pm b^{y-2z+1})}{\pm(a^{x-2z} \pm b^{y-2z})^{(z-2)/z}} = \pm(a^{x-z} \pm b^{y-z})^{1/z}(a^{x-2z} \pm b^{y-2z})^{1/z}. \tag{0.39}$$

STEP 3

Thirdly, we are in a position to substitute like-terms. For the right hand sides of the equations in (0.35) and (0.38) are exactly the same. Therefore by substituting like-terms we get:

$$\frac{\pm(a^{x-z-1} \pm b^{y-z-1})}{\pm(a^{x-z} \pm b^{y-z})^{(z-2)/z}} = \frac{\pm(a^{x-2z+1} \pm b^{y-2z+1})}{\pm(a^{x-2z} \pm b^{y-2z})^{(z-2)/z}} \tag{0.40}$$

We raise both sides by the power of z and rearrange to get:

$$\pm\left(\frac{a^{x-z-1} \pm b^{y-z-1}}{a^{x-2z+1} \pm b^{y-2z+1}}\right)^{z} = \pm\left(\frac{a^{x-z} \pm b^{y-z}}{a^{x-2z} \pm b^{y-2z}}\right)^{(z-2)} \tag{0.41}$$

We will return shortly to the case of $z = 1, 2$, but for now (still assuming that $x, y, z > 2$) we can say that solutions will exist either a) if, in (0.40), the bracketed factors on each side of the equation have a value of 1 (since the main outer exponents are not equal), or b) in (0.39) if $\pm(a^{x-z-1} \pm b^{y-z-1})^z = \pm(a^{x-z} - b^{y-z})^{(z-2)}$ *and* simultaneously if $\pm(a^{x-2z+1} \pm b^{y-2z+1})^z = \pm(a^{x-2z} \pm b^{y-2z})^{(z-2)}$ (and if both sides have equal signs).

Taking these in turn:
a) since $(a^{x-z-1} \pm b^{y-z-1}) \neq (a^{x-2z+1} \pm b^{y-2z+1})$, and $(a^{x-2z} \pm b^{y-2z}) \neq (a^{x-z} - b^{y-z})$, neither side in (0.41) has a value of 1. Thus we can eliminate this possibility.
b) *even without* exponents $(a^{x-2z+1} \pm b^{y-2z+1})$ is greater than $(a^{x-2z} \pm b^{y-2z})$; *with* exponents the inequality is even greater. So it follows that $(a^{x-2z+1} \pm b^{y-2z+1})^z \neq (a^{x-2z} \pm b^{y-2z})^{(z-2)}$.
We have thus eliminated all the possibilities. So, for all values of $x, y, z > 2$ it follows that:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k \neq \sum_{k=0}^{z}(a^{x-z-k} \pm b^{y-z-k}). \tag{0.42}$$

But this contradicts our second equation in (0.26). And so, our initial assumption that, for any value of $x, y, z > 2$, solutions exist for the equation $c^z = a^x - b^y$ is also false. Thus, Beal's Conjecture is true. □

We have now proved BC, but the question remains about the cases of $z = 1, 2$. Well, from (0.40), when $z = 1$ (and there is equal polarity of signs) it follows that:

$$\left(\frac{a^{x-2} \pm b^{y-2}}{a^{x-1} \pm b^{y-1}}\right)^1 = \left(\frac{a^{x-1} \pm b^{y-1}}{a^{x-2} \pm b^{y-2}}\right)^{-1} \tag{0.43}$$

$$\Rightarrow 1 = 1. \tag{0.44}$$

Thus, when the signs are equal on both sides, there is no contradiction. And again from (0.40), when $z = 2$ (and there is equal polarity of signs), it follows that:

$$\left(\frac{a^{x-3} \pm b^{y-3}}{a^{x-3} \pm b^{y-3}}\right)^2 = \left(\frac{a^{x-2} \pm b^{y-2}}{a^{x-4} \pm b^{y-4}}\right)^0 \tag{0.45}$$

$$\Rightarrow 1 = 1. \tag{0.46}$$

Again, when both signs are equal, no contradiction. So in both cases, when $z = 1$ and when $z = 2$, the standard rules of binomial expansion can be applied to our non-standard binomial expression without contradiction such that $(a^{x-z-k} \pm b^{y-z-k})$ is equal to $|s|^{z-k}|t|^k$, and therefore that in these cases solutions to the original equation exist.

Finally, it is worth mentioning the obvious point that we can apply the same method to Fermat's Last Theorem with the same result. But we can also apply it to a more generalised theorem, that there are no integer solutions for the equation $Pa^x - Qb^y = Rc^z$, where a, b, c, P, Q, R are square-free integers (of which one of Pa, Qb, Rc at most must be even), and $\gcd(a, b, c, P, Q, R) = 1$, and where integer solutions only exist for the values of x or y or $z = 1, 2$, but not for values of $x, y, z > 2$, where x, y, z are square-free integers (of which one at most may be even), and $gcd(x, y, z) = 1$.

Without giving a full proof, I simply observe that $Pa^x \pm Qb^y$ expands to

$$(a + b)(Pa^{x-1} \pm Qb^{y-1}) - ab(Pa^{x-2} \pm Qb^{y-2}), \tag{0.47}$$

which in turn expands *ad infinitum* to create the binomial formula:

$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (Pa^{x-z-k} \pm Qb^{y-z-k}). \tag{0.48}$$

And like the proof above, we can suggest that if this is itself an exponent to a single power, say Rc^z, then the following equation must have solutions:

$$\sum_{k=0}^{z} (\tfrac{1}{R})(a+b)^{z-k}(-ab)^k (Pa^{x-z-k} \pm Qb^{y-z-k}) = \sum_{k=0}^{z} (\tfrac{1}{R})|s|^{z-k}|t|^k. \tag{0.49}$$

This, similarly, leads to the final step where we can show that no integer solutions for the values of $x, y, z > 2$ exist for the equation:

$$\sum_{k=0}^{z} (\tfrac{1}{R})(Pa^{x-z-k} \pm Qb^{y-z-k}) = \sum_{k=0}^{z} (\tfrac{1}{R})|s|^{z-k}|t|^k. \tag{0.50}$$

Following the same logic as above, we find that the values of P, Q and R are all eventually cancelled out, leading to the same contradictions we saw in the proof for Beal's Conjecture.

References

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