Integrals containing the infinite product $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{b+n}\right)^3\right]$

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We study several integrals that contain the infinite product $\prod\limits_{n=0}^{\infty}\left[1+\left(\frac{x}{b+n}\right)^3\right]$ in the denominator of their integrand. These considerations lead to closed form evaluation $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{\left(e^x+e^{-x}+e^{ix\sqrt{3}}\right)^2}=\frac{1}{3}$ and to some other formulas.

1. The infinite product

$$\prod_{n=0}^{\infty}\left[1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right]$$

and more general products have been studied in the literature (see [1], ch. 16). In this paper we consider integrals of the form

$$\int_0^{\infty} P_b(x)f(x)dx, \tag{1}$$

where

$$P_b(x)=\frac{1}{\prod\limits_{k=0}^{\infty}\left(1+\frac{x^3}{(k+b)^3}\right)}. \tag{2}$$

Several such integrals will be evaluated in closed form. However while others do not have a closed form will allow us to evaluate some integrals of elementary functions.

Note that the infinite product in (2) can be written in terms of Gamma functions [2]

$$P_b(x)=\frac{\Gamma(b+x)\Gamma(b+\omega x)\Gamma(b+x/\omega)}{\Gamma^3(b)},\quad \omega=e^{\frac{2\pi i}{3}}.$$

The notation $\omega=e^{\frac{2\pi i}{3}}$ for third root of unity will be used throughout the paper.

2. Consider the contour integral

$$\int_C P_b(z)\frac{dz}{z}. \tag{3}$$

along the contour depicted in Fig.1. We assume that $b>0$. The most interesting cases considered in this paper correspond to $b=1$ and $b=1/2$.

Inside the contour of integration, the integrand $h(z)=P_b(z)/z$ has simple poles at $z=-(k+b-1)/\omega,\ k\in\mathbb{N}$, with residues

$$\frac{(-1)^k}{(k-1)!}\frac{|\Gamma(b-\omega(k+b-1))|^2}{(k+b-1)\Gamma^3(b)},$$

and no poles on the contour of integration if we choose $R=N+b-1/2$ for some large natural number N. Also $h(z)dz$ is symmetric under the change $z\to\omega z$, and as a consequence the integrals along straight lines cancel each other out. Let's denote the integrals along Γ_R and C_ε as I_R and I_ε respectively. Then

$$\lim_{\varepsilon\to 0} I_\varepsilon=-\frac{2\pi i}{3},$$

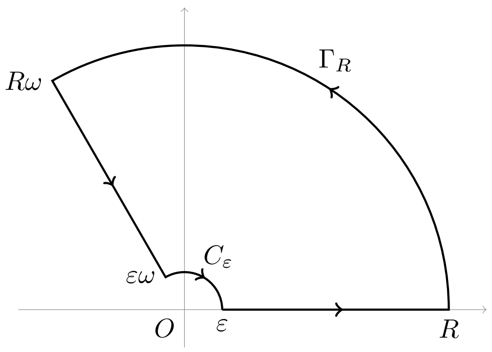

Fig.1

and (Appendix A)

$$\lim_{R\to+\infty} I_R = 0.$$

Using residue theorem we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{|\Gamma(b-\omega(n+b))|^2}{n+b} = \frac{1}{3}\Gamma^3(b). \tag{4}$$

The integral 3.985.1 from [3]

$$\int_{-\infty}^{\infty} \frac{e^{iax}dx}{\cosh^{\nu}\beta x} = \frac{2^{\nu-1}}{\beta\Gamma(\nu)}\Gamma\left(\frac{\nu}{2}+\frac{ai}{2\beta}\right)\Gamma\left(\frac{\nu}{2}-\frac{ai}{2\beta}\right) \tag{5}$$

allows to write (4) as an integral of a hypergeometric function

$$\int_{-\infty}^{\infty} \frac{e^{ib\sqrt{3}x}}{\cosh^{3b} x} {}_2F_1\left(\left.\begin{matrix} b, 3b \\ b+1 \end{matrix}\right| -\frac{e^{i\sqrt{3}x}}{2\cosh x}\right) dx = 2^{3b-1}\frac{b}{3}\frac{\Gamma^3(b)}{\Gamma(3b)}. \tag{6}$$

3. Here we specialize b in (6) so that the hypergeometirc function can be written in terms of elementary functions. This happens when $b = 1+3n$ or $b = 1/2+3n$, where n is a non-negative integer. Only the two cases with $n = 0$ are considered below:

Let $b = 1/2$, then the hypergeometric function becomes $\sqrt{\frac{2\cosh x}{2\cosh x + e^{i\sqrt{3}x}}}$ and we get

$$\int_{-\infty}^{\infty} \frac{\text{sech} x \; e^{\frac{i\sqrt{3}}{2}x} dx}{\sqrt{e^x+e^{-x}+e^{i\sqrt{3}x}}} = \frac{\pi}{3}. \tag{7}$$

If $b = 1$ then the hypergeometric function becomes $\frac{2\cosh x\left(4\cosh x + e^{i\sqrt{3}x}\right)}{\left(2\cosh x + e^{i\sqrt{3}x}\right)^2}$ and we get

$$\int_{-\infty}^{\infty} \frac{e^{i\sqrt{3}x}}{\cosh^2 x} \frac{4\cosh x + e^{i\sqrt{3}x}}{\left(2\cosh x + e^{i\sqrt{3}x}\right)^2} dx = \frac{2}{3},$$

which due to $\frac{4\cosh x + e^{i\sqrt{3}x}}{\cosh^2 x\left(2\cosh x + e^{i\sqrt{3}x}\right)^2} = -\frac{4}{\left(2\cosh x + e^{i\sqrt{3}x}\right)^2} + \frac{1}{\cosh^2 x}$ can be simplified further as

$$\int_{-\infty}^{\infty} \frac{dx}{\left(e^x+e^{-x}+e^{ix\sqrt{3}}\right)^2} = \frac{1}{3}. \tag{8}$$

It is interesting to note that there is another way to write the sum (4) with $b=1$ as an integral

$$\int_{-\infty}^{\infty}\frac{e^{i\sqrt{3}x}\cosh x}{\left(e^x+e^{-x}+e^{i\sqrt{3}x}\right)^2}dx=\frac{1}{12}. \tag{9}$$

One might observe how the $2\pi/3$ rotation symmetry of the product $\prod\limits_{k=1}^{\infty}\left(1+\frac{x^3}{k^3}\right)$ manifests itself in (8) and (9): The set of roots of the equation $e^x+e^{-x}+e^{i\sqrt{3}x}=0$ has the same $2\pi/3$ rotation symmetry (see Appendix B).

4. The last integral in section 3 gives

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{i\sqrt{3}x}\sinh x\,dx}{\left(2\cosh x+e^{i\sqrt{3}x}\right)^2}&=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{i\sqrt{3}x}\,d(2\cosh x+e^{i\sqrt{3}x})}{\left(2\cosh x+e^{i\sqrt{3}x}\right)^2}-\frac{i\sqrt{3}}{2}\int_{-\infty}^{\infty}\frac{e^{2i\sqrt{3}x}dx}{\left(2\cosh x+e^{i\sqrt{3}x}\right)^2}\\&=\frac{i\sqrt{3}}{2}\left(\int_{-\infty}^{\infty}\frac{e^{i\sqrt{3}x}dx}{2\cosh x+e^{i\sqrt{3}x}}-\int_{-\infty}^{\infty}\frac{e^{2i\sqrt{3}x}dx}{\left(2\cosh x+e^{i\sqrt{3}x}\right)^2}\right)\\&=i\sqrt{3}\int_{-\infty}^{\infty}\frac{e^{i\sqrt{3}x}\cosh x}{\left(e^x+e^{-x}+e^{i\sqrt{3}x}\right)^2}dx=\frac{i\sqrt{3}}{12}.\end{aligned}$$

Thus we have

$$\int_{-\infty}^{\infty}\frac{e^{i\sqrt{3}x}\sinh x\,dx}{\left(2\cosh x+e^{i\sqrt{3}x}\right)^2}=\frac{i\sqrt{3}}{12}. \tag{10}$$

5. After the substitution $t=e^{2x}$ equation (8) becomes

$$\int_0^{\infty}\frac{dt}{(1+t+t^{\alpha})^2}=\frac{2}{3},\quad \alpha=\frac{1+i\sqrt{3}}{2}, \tag{11}$$

while combining (9) and (10) we find two analogous representations

$$\int_0^{\infty}\frac{t^{\alpha}dt}{(1+t+t^{\alpha})^2}=\frac{\alpha}{3}, \tag{12}$$

$$\int_0^{\infty}\frac{t^{\alpha-1}dt}{(1+t+t^{\alpha})^2}=\frac{1}{3\alpha}. \tag{13}$$

(13) is related to (12) by complex conjugation and change of variable $t\to 1/t$.

6. If we apply the approach of section 2 to

$$\int_C P_b(z)dz,$$

instead, then the integrals over straight lines no longer cancel out. However, there is nevertheless a simplification: in this case the sum analogous to (4) reduces to an integral of elementary function for all b so that in this case we find the transformation

$$\int_0^{\infty}\frac{dx}{\prod\limits_{k=0}^{\infty}\left(1+\frac{x^3}{(k+b)^3}\right)}=\frac{4\pi\Gamma(3b)}{\Gamma^3(b)\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{ixb\sqrt{3}}\,dx}{\left(e^x+e^{-x}+e^{ix\sqrt{3}}\right)^{3b}}. \tag{14}$$

7. In principle integrals (7) and (8) can be written in terms of real-valued functions by calculating the real part of the integrand. The resulting formulas are cumbersome and therefore omitted. However there is another way to get a compact integral of a real valued function, at least for (8). First of all, all the roots of the function $e^z+e^{-z}+e^{iz\sqrt{3}}$ lie on the three rays $z=ir\omega^k,\ r>0,\ (k=0,1,2)$ (see Appendix A). If one bends the contour of integration so that it never crosses these zeroes then the integral (8) will not change. Since the integrand decreases exponentially when $z\to\infty$, $0<\arg z<\pi/6$ or $5\pi/6<\arg z<\pi$ we have

$$\frac{1}{\beta}\int_0^\infty \frac{dx}{\left(e^{-x/\beta}+e^{x/\beta}+e^{-ix\sqrt{3}/\beta}\right)^2}+\beta\int_0^\infty \frac{dx}{\left(e^{\beta x}+e^{-\beta x}+e^{i\beta x\sqrt{3}}\right)^2}=\frac{1}{3},\quad \beta=e^{\pi i/6},$$

and after elementary simplifications

$$\int_0^\infty \frac{e^{x\sqrt{3}}\cos\left(\frac{\pi}{6}-x\right)}{\left(2\cos x+e^{x\sqrt{3}}\right)^2}dx=\frac{1}{6}. \tag{15}$$

Similarly, for the case $b=1$ of (14)

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^3}{1^3}\right)\left(1+\frac{x^3}{2^3}\right)\left(1+\frac{x^3}{3^3}\right)\dots}=8\pi\int_0^\infty \frac{e^{x\sqrt{3}}\,dx}{\left(2\cos x+e^{x\sqrt{3}}\right)^3}. \tag{16}$$

8. It turns out that (8) has a parametric extention. Consider the contour integral

$$\int_{C'} P_1(z)\frac{e^{az}dz}{z}, \tag{17}$$

where the contour C' is a circle of radius $R=N+1/2$ for large natural N. Since $|P_1(z)|$ decreases exponentially with N on the circle C' (Appendix A), (17) will be zero in the limit $N\to\infty$ for sufficiently small $|a|$. Therefore the sum of residues of the integrand over three sets of simple poles $z=-ke^{2\pi ij/3}$, $k\in\mathbb{N}$, $(j=0,1,2)$ plus a simple pole at the origin, will be 0 according to residue theorem. As a result one will obtain three sums similar to (4) and then convert them to integrals of the type (8). However there is a trick that allows to avoid these calculations. Note that the factor e^{az} in (17) will introduce additional factors $\exp\left(-ake^{\frac{2\pi ij}{3}}\right)$ in the sum over residues. When converted to an integral via (5) these factors have the effect of multiplying $e^{ix\sqrt{3}}$ by $\exp\left(-ae^{\frac{2\pi ij}{3}}\right)$:

$$\sum_{j=1}^{3}\int_{-\infty}^{\infty}\frac{dx}{\left(e^x+e^{-x}+\exp\left(-ae^{\frac{2\pi ij}{3}}\right)e^{ix\sqrt{3}}\right)^2}=1,$$

or equivalently

$$\int\limits_{-\infty}^{\infty}\frac{dx}{\left(e^x+e^{-x}+e^{a+ix\sqrt{3}}\right)^2}+\int\limits_{-\infty}^{\infty}\frac{e^a dx}{\left(e^{a+x}+e^{-x}+e^{ix\sqrt{3}}\right)^2}+\int\limits_{-\infty}^{\infty}\frac{e^a dx}{\left(e^{a+x}+e^{-x}+e^{-ix\sqrt{3}}\right)^2}=1, \tag{18}$$

where $|a|$ is sufficiently small.

9. There is a similarity between (4) with $b=1$ and the identity due to Ramanujan ([4], p. 309)

$$e^{ay}=\frac{-a}{2ci}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{-a+k(ci-b)}{2ci}\right)(-2ie^{-by}\sin cy)^k}{\Gamma\left(\frac{-a-k(ci-b)}{2ci}+1\right)k!}.$$

After equating the coefficents of a^1 in Taylor series expansion of both sides into powers of a and transforming the Gamma function in the denominator via Euler's reflection formula

$$y=\frac{1}{2\pi ic}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(\frac{k}{2}-\frac{ikb}{2c}\right)\Gamma\left(\frac{k}{2}+\frac{ikb}{2c}\right)\sin\pi\left(\frac{k}{2}+\frac{ikb}{2c}\right)(-2ie^{-by}\sin cy)^k. \tag{19}$$

To make this similarity more exact we differentiate (19) with respect to y, divide by $(c\cot cy-b)$, repeat this procedure one more time and then set $c=1$, $b=\sqrt{3}$

$$\frac{2\pi i\sin y}{(\cos y-\sqrt{3}\sin y)^3}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}|\Gamma\left(1-\omega k\right)|^2\cdot\sin\left(\pi ke^{\pi i/3}\right)(-2ie^{-\sqrt{3}y}\sin y)^k. \tag{20}$$

This series converges when $|2e^{-\sqrt{3}y}\sin y|<e^{-\frac{\pi}{2\sqrt{3}}}$ [4]. It will be convenient to use another variable α related to y by

$$e^{-\alpha}=2e^{-\sqrt{3}y}\sin y.$$

The condition that the series (20) converges now takes a very simple form $\operatorname{Re}\alpha>\frac{\pi}{2\sqrt{3}}$. In the following it will be assumed for simplicity that $\alpha>\frac{\pi}{2\sqrt{3}}$.

Is it possible that (20) leads to evaluation of integrals with infinite product $\prod\limits_{k=1}^{\infty}\left(1+\frac{z^3}{k^3}\right)$? Consider the contour integral

$$\int\limits_C\frac{\left(-ie^{-\alpha}\right)^{-\omega z}\sin\pi z}{z\prod\limits_{k=1}^{\infty}\left(1+\frac{z^3}{k^3}\right)}\,dz, \tag{21}$$

where C is the contour in Fig.2. Due to the asymptotics

$$|\left(-ie^{-\alpha}\right)^{-\omega z}\sin\pi z|\sim\frac{1}{2}\exp\left[-\left(\frac{\alpha}{2}+\frac{\pi\sqrt{3}}{4}\right)x+\left(\frac{5\pi}{4}-\frac{\alpha\sqrt{3}}{2}\right)y\right],\quad 0<\arg z<\frac{2\pi}{3},$$

and the result of Appendix A, the integral over circular arc Γ_R vanishes in the limit $R\to\infty$ if

$$\begin{cases}-\left(\frac{\alpha}{2}+\frac{11\pi}{4\sqrt{3}}\right)x+\left(\frac{5\pi}{4}-\frac{\alpha\sqrt{3}}{2}\right)y<0, & 0<\arg z<\frac{\pi}{3},\\ \left(\frac{\pi}{2\sqrt{3}}-\alpha\right)(x+y\sqrt{3})<0, & \frac{\pi}{3}<\arg z<\frac{2\pi}{3}.\end{cases}$$

When $\alpha>\frac{\pi}{2\sqrt{3}}$ these conditions are satisfied automatically.

The same approach as in section 2 yields

$$\begin{aligned}&\int_0^{\infty}\frac{\left(-ie^{-\alpha}\right)^{-\omega x}\sin\pi x-\left(-ie^{-\alpha}\right)^{-x/\omega}\sin\pi\omega x}{x\prod\limits_{k=1}^{\infty}\left(1+\frac{x^3}{k^3}\right)}\,dx\\ &=2\pi i\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}|\Gamma\left(1-\omega k\right)|^2\cdot\sin\left(\pi ke^{\pi i/3}\right)\left(-ie^{-\alpha}\right)^k\\ &=\frac{4\pi^2\sin y}{(\cos y-\sqrt{3}\sin y)^3}.\end{aligned}$$

For real α one can decompose the function in the numerator of this integral into real and imaginary parts

$$\left(-ie^{-\alpha}\right)^{-\omega x}\sin\pi x-\left(-ie^{-\alpha}\right)^{-x/\omega}\sin\pi\omega x=f(x,\alpha)+ig(x,\alpha)$$

where

$$f(x,\alpha)=\frac{1}{2}e^{-\sqrt{3}\pi x/4-\alpha x/2}\left(e^{\sqrt{3}\pi x}\sin\frac{\left(\pi-2\sqrt{3}\alpha\right)x}{4}+2\sin\frac{\left(2\sqrt{3}\alpha+3\pi\right)x}{4}-\sin\frac{\left(2\sqrt{3}\alpha-5\pi\right)x}{4}\right),$$

$$g(x,\alpha)=\frac{1}{2}e^{-\sqrt{3}\pi x/4-\alpha x/2}\left(\cos\frac{\left(2\sqrt{3}\alpha-5\pi\right)x}{4}-e^{\sqrt{3}\pi x}\cos\frac{\left(\pi-2\sqrt{3}\alpha\right)x}{4}\right).$$

As a result

$$\int_0^\infty \frac{g(x,\alpha)\ dx}{x\prod\limits_{k=1}^{\infty}\left(1+\frac{z^3}{k^3}\right)}=0, \tag{22}$$

$$\int_0^\infty \frac{f(x,\alpha)\ dx}{x\prod\limits_{k=1}^{\infty}\left(1+\frac{z^3}{k^3}\right)}=\frac{8\pi^2\sin y}{\left(\sqrt{3}\sin y-\cos y\right)^3}, \tag{23}$$

where y is the root of the equation $2e^{-y\sqrt{3}}\sin y=e^{-\alpha}$ near $y=0$.

These formulas simplify when $\alpha=\frac{5\pi}{2\sqrt{3}}$

$$\int\limits_0^\infty \frac{\left(1-e^{\pi\sqrt{3}x}\cos\pi x\right)e^{-\frac{2\pi}{\sqrt{3}}x}\ dx}{x\left(1+\frac{x^3}{1^3}\right)\left(1+\frac{x^3}{2^3}\right)\left(1+\frac{x^3}{3^3}\right)\ldots}=0, \tag{24}$$

$$\int\limits_0^\infty \frac{\sin\pi x\left(4\cos\pi x-e^{\pi\sqrt{3}x}\right)e^{-\frac{2\pi}{\sqrt{3}}x}dx}{x\left(1+\frac{x^3}{1^3}\right)\left(1+\frac{x^3}{2^3}\right)\left(1+\frac{x^3}{3^3}\right)\ldots}=\frac{8\pi^2\sin y}{\left(\sqrt{3}\sin y-\cos y\right)^3}, \tag{25}$$

where $y=0.0054167536\ldots$ is the root of the equation $2e^{-y\sqrt{3}}\sin y=e^{-\frac{5\pi}{2\sqrt{3}}}$.

10. The hyperbolic log-trigonometric integral

$$\operatorname{Im}\int_0^\infty \frac{dt}{\left(it\sqrt{3}+\ln(2\sinh t)\right)^2}=0, \tag{26}$$

or in terms of real valued functions

$$\int_0^\infty \frac{t\ln\left(2\sinh t\right)}{\left[3t^2+\ln^2\left(2\sinh t\right)\right]^2}\,dt=0, \tag{27}$$

is also related to the infinite product in the title. Indeed

$$\begin{aligned}\int_0^\infty \frac{\sin\pi x\ dx}{x\prod\limits_{k=1}^{\infty}\left(1-\frac{x^3}{k^3}\right)}&=\int_0^\infty \frac{\sin\pi x}{x}\Gamma(1-x)\Gamma(1-\omega x)\Gamma(1-x/\omega)dx\\ &=\pi\int_0^\infty \frac{\Gamma(1-\omega x)\Gamma(1-x/\omega)}{\Gamma(1+x)}dx\\ &=\pi\int_0^\infty B(1-\omega x,1-x/\omega)dx\\ &=\pi\int_0^\infty dx\int_0^\infty \frac{t^{-\omega x}}{(1+t)^{x+1}}dt\end{aligned} \tag{28}$$

Changing the order of integration and calculating the integral over x we get

$$\int_0^\infty \frac{\sin \pi x \; dx}{x \prod\limits_{k=1}^{\infty} \left(1 - \frac{x^3}{k^3}\right)} = -2\pi \int_0^\infty \frac{dt}{\left(it\sqrt{3} + \ln(2\sinh t)\right)^2}. \tag{29}$$

(26) is the statement of the fact that the integral on the RHS of (29) is real. Of course by replacing in (28) ω with any complex number of unit argument one gets other integrals like (26).

It is known that Laplace transform of the digamma function leads to some log-triginometric integrals [5–7] that contain the expression $x^2 + \ln^2(2e^{-a}\cos x)$ in the denominator. This expression should be compared to the expression $3t^2 + \ln^2(2\sinh t)$ in the denominator of (27).

Appendix A: Asymptotics of the product of gamma functions

Due to the asymptotic relation

$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + O(1), \quad |\arg z| < \pi,$$

one has

$$\ln\{\Gamma(b+z)\Gamma(b+\omega z)\Gamma(b+z/\omega)\} = 3\left(b - \frac{1}{2}\right)\ln z - \frac{2\pi}{\sqrt{3}} z + O(1), \quad |\arg z| < \frac{\pi}{3}.$$

From this it follows that

$$|P_b(z)| = C|z|^{3b-3/2} \cdot \begin{cases} e^{-\frac{2\pi}{\sqrt{3}}x}, & 0 < \arg z < \frac{\pi}{3}, \\ e^{\frac{\pi}{\sqrt{3}}x - \pi y}, & \frac{\pi}{3} < \arg z < \frac{2\pi}{3}, \end{cases}$$

where $z = x + iy$.

Appendix B: Roots of the equation $e^{i\sqrt{3}z} + 2\cosh z = 0$

The fact that the roots of the equation $e^{i\sqrt{3}z} + 2\cosh z = 0$ are symmetric under $z \to \omega z$ is easy to check directly.

Since $\frac{1}{2}e^{-\pi\sqrt{3}/2} = 0.0329...$ is quite small the equation $e^{i\sqrt{3}z} + 2\cosh z = 0$ will have roots close to $\pi i \left(n + \frac{1}{2}\right)$, where n is a non-negative integer. Below it is shown that these are the only roots in the upper half plane.

Let $f(z) = e^{i\sqrt{3}z}$, $g(z) = 2\cosh z$. Obviously, on the real axis $|f(z)| < |g(z)|$. Now consider $f(z)$ and $g(z)$ on the closed contour C depicted in Fig.2

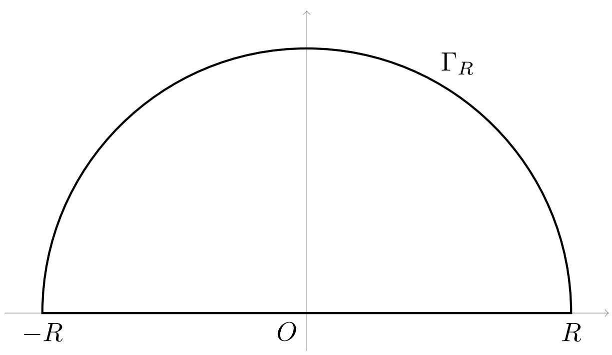

Fig.2

Here Γ_R is a semicircle of radius $R = \pi N$ for some large natural N. We have for $z = x + iy \in \Gamma_R$

$$|f(z)| = e^{-\sqrt{3}y} \leq 1,$$

$$|g(z)| = 2\sqrt{\sinh^2 x + \cos^2 \sqrt{\pi^2 N^2 - x^2}} \geq 2.$$

Thus $|f(z)| < |g(z)|$ on the contour C. According to Rouche's theorem this means that the function $f(z) + g(z)$ has the same number of roots inside the contour C as the function $g(z)$, as required.

This analysis shows that the roots of $e^{i\sqrt{3}z} + 2\cosh z = 0$ are located on the three rays $z = ir\omega^k,\ k = 0, 1, 2.$

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