The closed-form theory of tuned mass damper with hysteretic friction

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The classical theory of lumped mechanical systems employs the viscous friction mechanisms (dashpots), while the loss factors of most solid structures are controlled by hysteresis. This paper presents analysis of the forced vibration of 2-DOF in-series systems with hysteretic friction where one of the partial 1-DOF systems plays the role of tuned (or auxiliary) mass damper (TMD). The assumption of hysteretic damping is acceptable if the loss factor remains about stable at least in the frequency range containing the resonance peaks. The closed-form simple relationships for the transmissibility at the resonance frequencies are derived in the "nearby" case where (1) the natural undamped frequencies of 2-DOF in-series system are most close to each other and (2) the loss factors of the 2-DOF system become similar and equal to the arithmetic average of the partial loss factors. The independent parameters are the mass ratio and partial loss factors. The relationships become very simple if the mass ratio is high or low compared to the square of each partial loss factor. In cases where the ratio of natural undamped frequencies of 2-DOF system are moderately lower or higher than in the "nearby case", the transmissibility peak magnitudes are about similar to those for the "nearby" case. The results can be utilized for the noise and vibration control in machinery and buildings.

Keywords: noise and vibration control, 2-DOF mechanical system, hysteresis, tuned mass damper, closed-form theory, machinery, buildings.

1. INTRODUCTION

The vibration effects in 1-DOF and 2-DOF mechanical systems with viscous friction are well known [1-3 *etc.*]. However, most solid structures exhibit non-viscous damping mechanisms: hysteresis, structural losses (caused by energy leaks to the adjacent structures), and coulomb friction [4-15 *etc*.]. All the damping mechanisms can be simulated via a loss factor η calculated from energy considerations. The loss factor increases with frequency in case of viscous friction, does not depend on frequency for hysteresis, and can reduce with frequency if the attenuation is caused by the by energy leaks to the adjacent structures.

If it is not feasible to increase the vibration energy dissipation in a main structure, the damping can be provided by a second 1-DOF system attached to the main structure. Such a system is referred to as tuned (or auxiliary) mass damper. The theory of 2-DOF systems incorporating both main 1-DOF system and 1-DOF tuned mass damper is well known in case of viscous friction [1, 15-19 *etc*.]. The goal of this paper is to build a similar theory in case of hysteretic friction and develop closed-form relationships which are easy for physical interpretation and noise and vibration control of machinery and buildings. Here, the assumption of hysteretic damping is acceptable if the loss factor remains about stable at least in the frequency range containing the resonance peaks.

2. MATHEMATICAL MODEL OF 2-DOF IN-SERIES MECHANICAL SYSTEM WITH HYSTERETIC FRICTION

2.1. Description of the mathematical model

Consider a 2-DOF in-series system incorporating two rigid bodies and two springs with hysteretic damping (Fig. 1). Here, the masses of the first and second bodies are m_1 and m_2, the spring constants are k_1 and k_2 , so the partial undamped natural angular frequencies are given by the equations $\omega_{p1} = \sqrt{k_1/m_1}$ and $\omega_{p2} = \sqrt{k_2/m_2}$. The first spring is attached to the rigid base vibrating with the displacement $Y_0 = y_0 \exp(i \omega t)$ where $i = \sqrt{-1}$ is the imaginary unit, ω is the angular frequency and y_0 is the displacement amplitude. Hence, the stationary vibration displacements of the first and second masses can be expressed as

$Y_1 = y_1 \exp(i\,\omega\,t)$ and $Y_2 = y_2 \exp(i\,\omega\,t)$ where y_1 and y_2 are the relevant displacement amplitudes. The complex spring constants can be defined as

$K_1 = k_1\left(1 + i\,\eta_{p1}\right)$ and $K_2 = k_2\left(1 + i\,\eta_{p2}\right)$ where η_{p1} and η_{p2} are the partial loss factors associated with the first and second springs [5]. Here, the second 1-DOF system plays the role of a passive tuned mass damper [1, 15-19]. In most real structures, the loss factor is below 0.1, so it makes sense to suggest $\eta_{p1} = 0.05$ for estimation purposes. The loss factors of common tuned mass dampers used to be much over 0.05 but still well below 1. But sometimes the role of TMD is played by a system with similar loss factor (like in automotive vehicles with the auxiliary cooling module attached to the main radiator via vibration isolators).

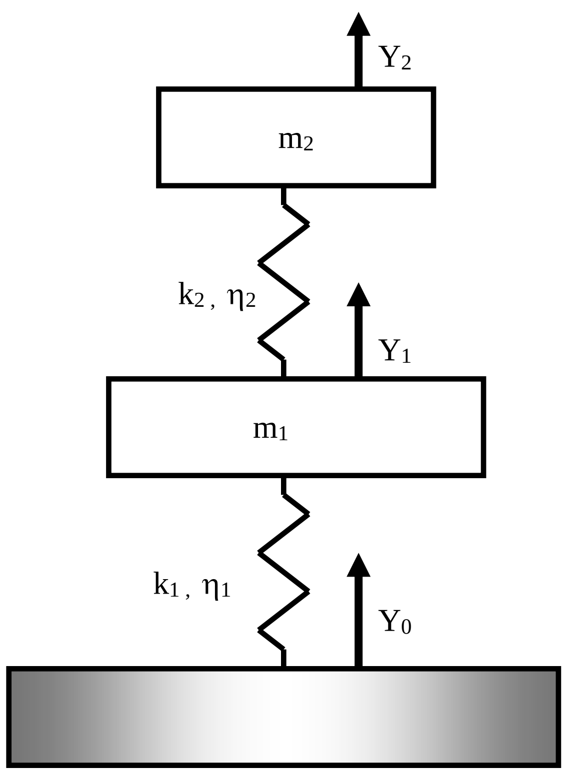

Fig. 1. 2-DOF in-series system with the hysteresis damping (the lowest body simulates a shaker, the dampers are not shown).

Basing on the above, consider

$$0.05 \le \eta_{p2} \le 0.25 \tag{1}$$

The differential equations of motion can be written in the form

$$\begin{cases} m_1 \ddot{Y}_1 + K_1 Y_1 + K_2 (Y_1 - Y_2) = K_1 Y_0, \\ m_2 \ddot{Y}_2 + K_2 (Y_2 - Y_1) = 0. \end{cases} \tag{2}$$

The characteristic equation for this dynamic system is quadratic relative to the unknown ω^2

$$\begin{aligned} m_1 m_2 \Big\{ \omega^4 - \omega^2 \big[\omega_{p1}^2 (1 + i\,\eta_{p2}) + \omega_{p2}^2 (1 + i\,\eta_{p2}) (1+\mu) \big] + \\ + \omega_{p1}^2\, \omega_{p2}^2 (1 + i\,\eta_{p1}) (1 + i\,\eta_{p2}) \Big\} = 0 \end{aligned} \tag{3}$$

and has two roots

$$\tilde{\Omega}_{1,2}^2 = \Omega_{1,2}^2 \left(1 + i\,\eta_{1,2} \right) \tag{4}$$

where Ω_1 and Ω_2 are the undamped natural angular frequencies of the 2-DOF system,

η_1 and η_2 are the relevant loss factors, and the mass ratio

$$\mu = \frac{m_2}{m_1} \tag{5}$$

The undamped natural frequencies $\Omega_{1,2}$ and loss factors $\eta_{1,2}$ of a 2-DOF in-series system could be calculated using the quadratic formula [20] in Eq. (3) but the approximate expressions derived in the sections 2.2 – 2.4 are simpler and still quite accurate.

2.2. Undamped natural frequencies of 2-DOF in-series system

Applying the Vieta's formulas [20] to Eq. (3) and using Eq. (4), obtain

$$\begin{cases} \omega_{p1}^2\, \omega_{p2}^2 \, (1 + i\,\eta_{p1})(1 + i\,\eta_{p2}) = \Omega_1^2\, \Omega_2^2 \,(1 + i\,\eta_1)\,(1 + i\,\eta_2), \\ \omega_{p1}^2 (1 + i\,\eta_{p1}) + \omega_{p2}^2 (1 + i\,\eta_{p2}) (1+\mu) = \Omega_1^2 (1 + i\,\eta_1) + \Omega_2^2 (1 + i\,\eta_2). \end{cases} \tag{6}$$

The products $\eta_{p1}\eta_{p2}$ and $\eta_1\eta_2$ are small enough to be ignored compared to 1 (in particular, if $\eta_{p1} = 0.05$ and $\eta_{p2} = 0.25$ then $\eta_{p1}\eta_{p2} \approx 0.01 << 1$), so the first of Eqs (6) can be reduced to the form

$$\omega_{p1}^2\,\omega_{p2}^2\;[1+i\,(\eta_{p1}+\eta_{p2})] = \Omega_1^2\,\Omega_2^2\,[1+i\,(\eta_1+\eta_2)]. \qquad (7)$$

Equating the real and imaginary parts on both sides of Eq. (7), obtain:

$$\begin{cases} \omega_{p1}^2\,\omega_{p2}^2 = \Omega_1^2\,\Omega_2^2\,, \\ \eta_{p1}+\eta_{p2} = \eta_1+\eta_2. \end{cases} \qquad (8)$$

Equating the real and imaginary parts on both sides of the second of Eqs (6), obtain

$$\begin{cases} \omega_{p1}^2 + \omega_{p2}^2\,(1+\mu) = \Omega_1^2 + \Omega_2^2\,, \\ \omega_{p1}^2\eta_{p1} + \omega_{p2}^2\,(1+\mu)\,\eta_{p2} = \Omega_1^2\eta_1 + \Omega_2^2\eta_2. \end{cases} \qquad (9)$$

Using the firsts of Eqs (8) and Eqs (9), calculate the undamped natural angular frequencies

$$\Omega_{2,1} = \omega_{p1}\sqrt{D_{2,1}} \qquad (10)$$

where $\Omega_2 > \Omega_1$ and

$$D_{2,1} = \frac{1+b}{2}\left(1 \pm \sqrt{1-\ r}\ \right), \qquad (11)$$

$$r = \frac{1}{1+\mu}\;\frac{4\,b}{[1+b]^2}\,, \qquad (12)$$

$$b\ = p^2\left(1+\mu\right), \qquad (13)$$

$$p = \frac{\omega_{p2}}{\omega_{p1}}. \qquad (14)$$

Using Eqs (11)-(14), express the ratio of the undamped natural frequencies Ω_1 and Ω_2 as

$$q = \frac{\Omega_1}{\Omega_2} = \sqrt{\frac{1-\sqrt{1-r}}{1+\sqrt{1-r}}} = \sqrt{\frac{1}{r}} - \sqrt{\frac{1}{r}-1} \tag{15}$$

where the parameter r is defined by Eq. (12).

2.3. The "nearby" case where the undamped natural frequencies of 2-DOF in-series system are most close to each other

According to the well-known inequality $(u+v)^2 \geq 4uv$ for the positive values u and v, the parameter r defined by Eq. (12) attains its maximum

$$r_{max} = \frac{1}{1+\mu} \tag{16}$$

for every given μ if

$$b = 1. \tag{17a}$$

Using Eq. (13), transform Eq. (17a) to the form

$$p = \frac{1}{\sqrt{1+\mu}} \tag{17b}$$

where p is the ratio of the partial undamped natural frequencies given by Eq. (14). Using Eq. (15), calculate the derivative

$$\frac{dq}{dr} = \frac{1}{2\,r^{3/2}}\left(\frac{1}{\sqrt{1-r}} - 1\right)$$

which is positive, so the parameter q grows with the value r and attain its maximum

$$q_{max} = \sqrt{1+\mu} - \sqrt{\mu} \tag{18}$$

at the maximum value of parameter r. Therefore, under the condition defined by Eq. (17b) the undamped natural frequencies of 2-DOF in-series system are most close to each other for the every given mass ratio μ. The “nearby” case was studied earlier [13, 14] but only to analyze a relatively high vibration of the auxiliary mass.

Using Eqs (10), (11) and (16), express the undamped natural frequencies in the “nearby” case as

$$\Omega_{2,1} = \omega_{p1}\sqrt{1 \pm \sqrt{\frac{\mu}{1+\mu}}} \, . \tag{19}$$

The relationships between the ratios $q = \Omega_1/\Omega_2$ and $p = \omega_{p2}/\omega_{p1}$ for various values of the mass ratio $\mu = m_2/m_1$ are computed and plotted for comparison in Fig. 2 where the magnitude and location of the maxima are in good agreement with Eqs (17b) and (18).

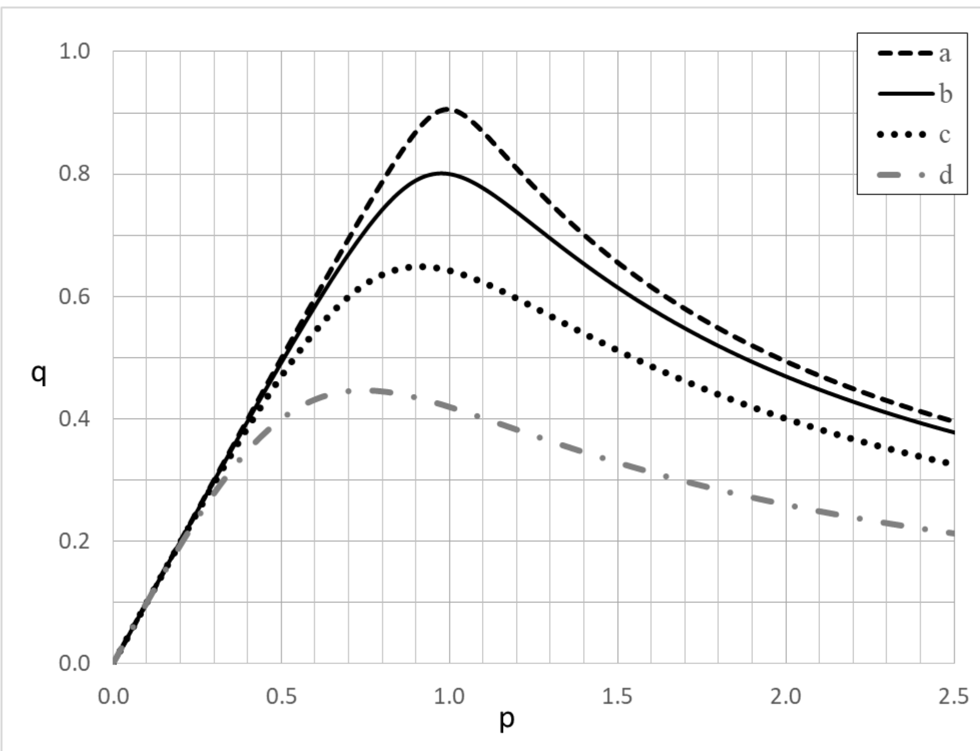

Fig. 2. Relationship between the parameters $q = \Omega_1/\Omega_2$ and $p = \omega_{p2}/\omega_{p1}$ for various values of $\mu = m_2/m_1$: (a) $\mu = 0.01$, (b) $\mu = 0.05$, (c) $\mu = 0.20$, (d) $\mu = 0.80$.

2.4. Loss factors of 2-DOF in-series system

Substituting Eqs (10)-(13) into the second of Eqs (9), obtain

$$\left(1-\sqrt{1-r}\right)\eta_1+\left(1+\sqrt{1-r}\right)\eta_2=\frac{2}{b+1}\left(\eta_{p1}+b\,\eta_{p2}\right).$$

Solving this equation together with the second of Eqs (8), calculate the loss factors of 2-DOF in-series systems

$$\eta_{2,1}=\eta\ \pm\ \frac{(b-1)\left(\eta_{p1}-\eta_{p2}\right)}{2\,(b+1)\sqrt{1-r}} \qquad (20)$$

where the parameter r is defined by Eq. (12), the average loss factor

$$\eta=\frac{\eta_{p1}+\eta_{p2}}{2}=\frac{\eta_1+\eta_2}{2}. \qquad (21)$$

As follows from Eq. (20):

if $b\to 0$ then $\eta_1\to\eta_{p2}$ and $\eta_2\to\eta_{p1}$,

if $b\to\infty$ then $\eta_1\to\eta_{p1}$ and $\eta_2\to\eta_{p2}$,

if $b=1$ then $\eta_1=\eta_2=\eta$.

Thus, in the “nearby” case both loss factors of 2-DOF in-series system equal the average loss factor defined by Eq. (21). For illustration, the loss factors η_1 and η_2 described by Eqs (20) are plotted as functions of the independent parameter $p=\omega_{p2}/\omega_{p1}$ for $\eta_{p1}=0.05$ and $\eta_{p2}=0.25$, and two values of the mass ratio $\mu=m_2/m_1$: 0.05 and 0.40 (Fig. 3).

3. TRANSMISSIBILITY FUNCTIONS FOR THE MAIN AND AUXILIARY SYSTEMS

3.1. Analytical expressions for the transmissibility functions

Using Eqs (2)-(4) and the first of Eqs (6), calculate the ratio of the displacement amplitudes

y_1 and y_2 to the displacement amplitude y_0:

$$\begin{cases} \dfrac{y_1}{y_0} = \dfrac{(1 - i\,\eta_{p1})\ \left[1 - (\omega/\omega_{p2})^2 - i\,\eta_{p2}\right]}{\Psi(\omega)}, \\ \dfrac{y_2}{y_0} = \dfrac{(1 - i\,\eta_{p1})\ (1 - i\,\eta_{p2})}{\Psi(\omega)} \end{cases} \tag{22}$$

where the polynomial

$$\Psi(\omega) = [1 - (\omega/\Omega_1)^2 - i\,\eta_1]\ [1 - (\omega/\Omega_2)^2 - i\,\eta_2]. \tag{23}$$

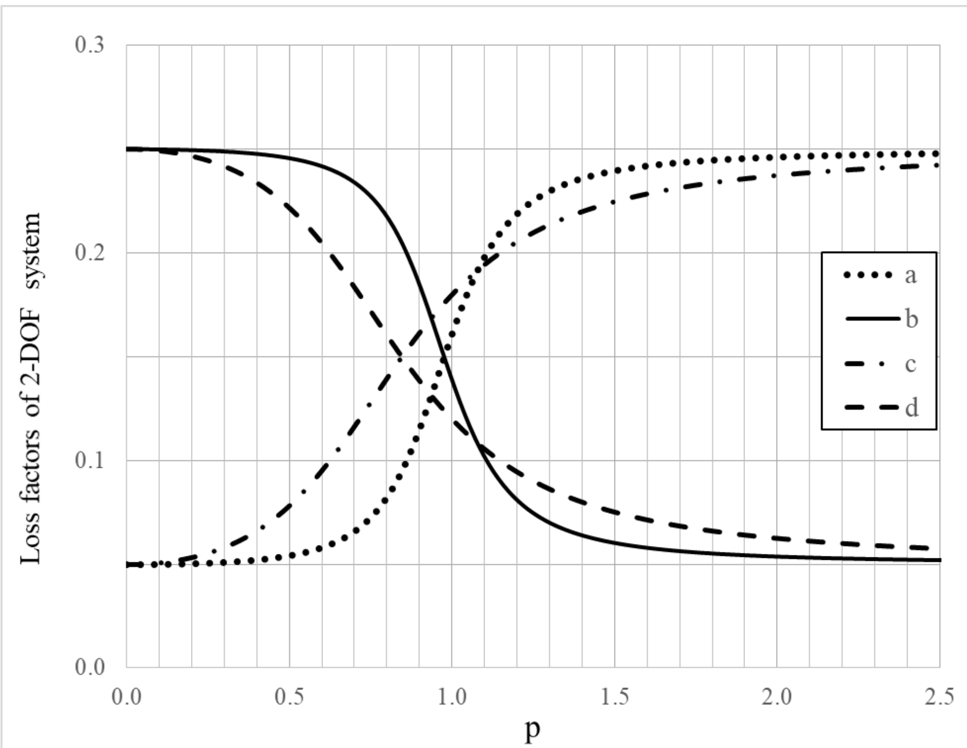

Fig. 3. Loss factors η_1 and η_2 of 2-DOF in-series system vs. $p = \omega_{p2}/\omega_{p1}$ if the partial loss factors are $\eta_{p1} = 0.05$ and $\eta_{p2} = 0.25$:

(a) η_1; $\mu = 0.05$, (b) η_2; $\mu = 0.05$, (c) η_1; $\mu = 0.4$, (d) η_2; $\mu = 0.4$.

It is noteworthy that Eqs (22) are valid too if the base does not move but the vibrating force $F = F_0 \exp(i \omega t)$ is applied to the first body and $y_0 = F_0 / K_1$.

Using Eqs (22) and (23), calculate the transmissibility functions for the first (main) structure

$$T_1(\omega) = \left| \frac{y_1}{y_0} \right| = \sqrt{\frac{\left(1 + \eta_{p1}^2\right) H(\omega)}{\Phi(\omega)}} \tag{24}$$

and for the second (auxiliary) structure

$$T_2(\omega) = \left| \frac{y_2}{y_0} \right| = \sqrt{\frac{(1 + \eta_{p1}^2)\ (1 + \eta_{p2}^2)}{\Phi(\omega)}}\ , \tag{25}$$

where the functions

$$\begin{cases} \Phi(\omega) = \{[1 - (\omega/\Omega_1)^2]^2 + \eta_1^2\}\ \{[1 - (\omega/\Omega_2)^2]^2 + \eta_2^2\}, \\ H(\omega) = [1 - (\omega/\omega_{p2})^2]^2 + \eta_{p2}^2\ . \end{cases} \tag{26}$$

To demonstrate the main trends, the results given by Eqs (24) and (25) could be compared with the transmissibility function of the main 1-DOF system alone

$$T_0(\omega) = \sqrt{\frac{(1 + \eta_{p1}^2)}{[1 - (\omega/\omega_1)^2]^2 + \eta_1^2}}\ , \tag{27}$$

which is derived from Eq. (24) by suggesting that $\Omega_1 \to \omega_1$, $\Omega_2 \to \infty$, and $\eta_2 = 0$.

From Eq. (27), the transmissibility of the main 1-DOF structure at its resonance frequency

$$T_0(\omega_{p1}) = \frac{\sqrt{1 + \eta_{p1}^2}}{\eta_{p1}} \approx \frac{1}{\eta_{p1}}\ . \tag{28}$$

3.2. Computation and graphical comparison of the typical transmissibility functions

The transmissibility functions given by Eqs (24) and (25) are computed and plotted in Figs 4-9 vs. the dimensionless variable $\xi = \omega/\omega_{p1}$ for the mass ratio values $\mu = 0.2$ or $\mu = 0.02$

which are typical for machinery and buildings, respectively. Here, the partial loss factor is $\eta_{p2} = 0.05$ or $\eta_{p2} = 0.25$ for the tuned mass damper and $\eta_{p1} = 0.05$ for the main structure, so as follows from Eq. (28), the peak transmissibility of the main structure alone is $T_0(\omega_{p1}) = 1/0.05 = 20$. To estimate the role of the ratio p, the results are plotted for three of its values: (a) $1/\sqrt{1+\mu}$ as given by Eq. (17b) for the "nearby" case, (b) $1/(1+\mu)$ as in the "classical" case [1, 15], and (c) $p = 1$ (the partial undamped natural frequencies coincide). It is noteworthy that $1/(1+\mu) < 1/\sqrt{1+\mu} < 1$.

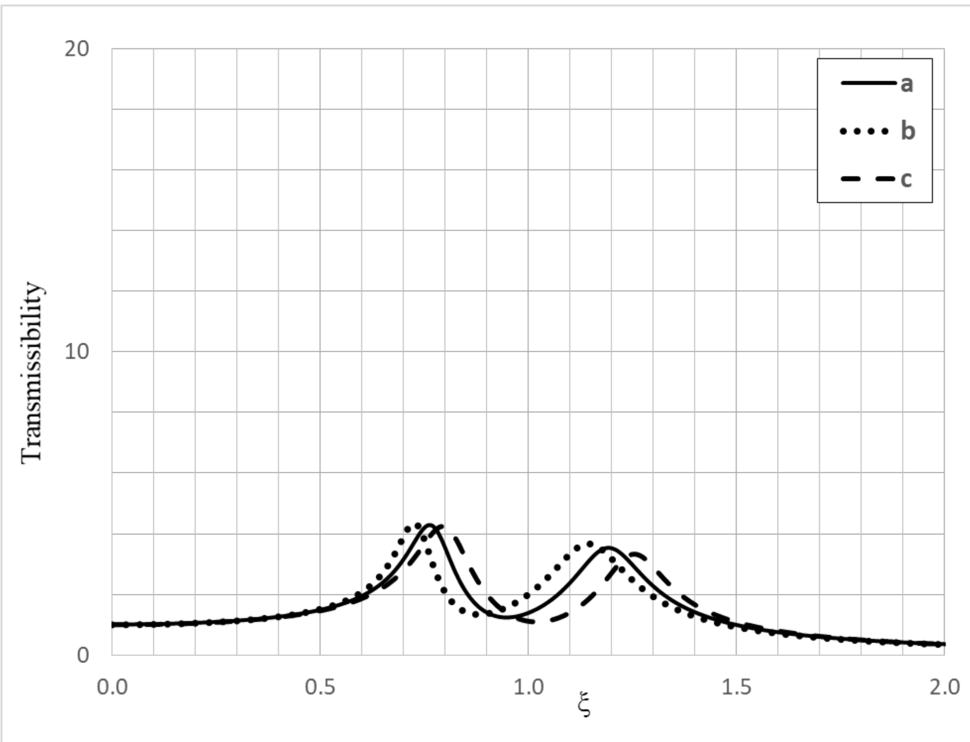

Fig. 4. Transmissibility vs. $\xi = \omega/\omega_{p1}$ for the main mass if

$\mu = 0.2$, $\eta_{p1} = 0.05, \eta_{p2} = 0.25$:

(a) $p = 1/\sqrt{1+\mu}$, (b) $p = 1/(1+\mu)$, (c) $p = 1$.

As seen from Figs (1)-(9), the results are not much affected by such a deviation of the parameter p above and below its "nearby" value. Hence, the effect of the mass ratio and loss factors on the transmissibility may be estimated just in the "nearby" case.

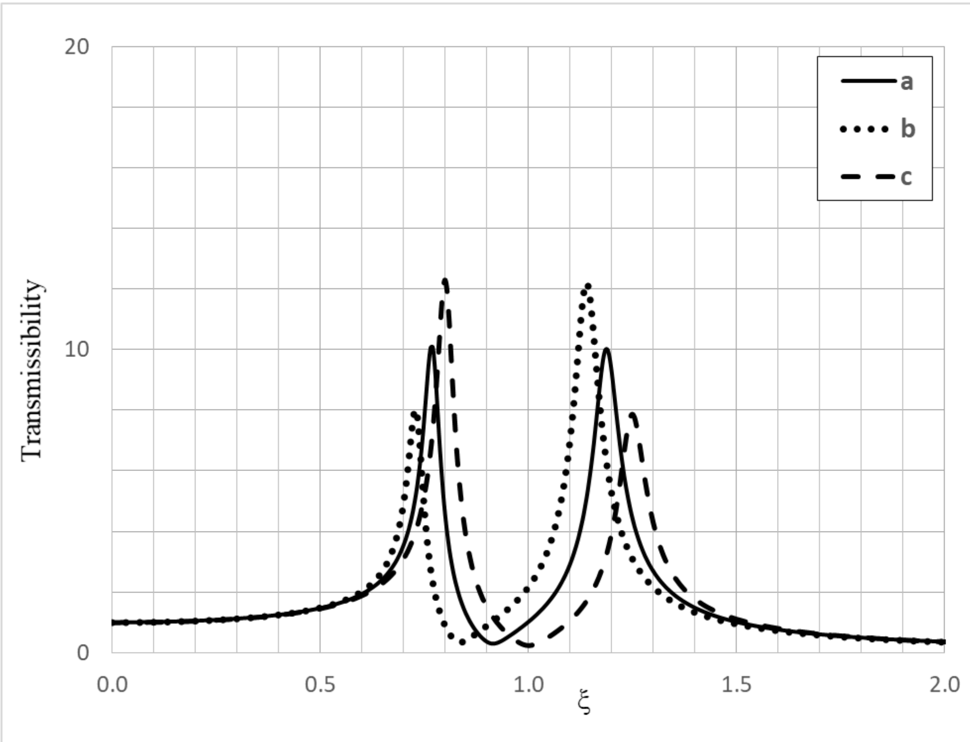

Fig. 5. Transmissibility vs. $\xi = \omega/\omega_{p1}$ for the main mass if $\mu = 0.2$, $\eta_{p1} = \eta_{p2} = 0.05$:

(a) $p = 1/\sqrt{1+\mu}$, (b) $p = 1/(1+\mu)$, c) $p = 1$.

As seen in Fig. 4, the effect of TMD with a relatively high mass ratio and partial loss factor ($\mu = 0.2$, $\eta_{p2} = 0.25$) is quite positive: the transmissibility of the main structure at both resonance frequencies is about 4 (much below $T_0(\omega_{p1}) = 20$). But if the TMD partial loss factor is low ($\eta_{p2} = 0.05$) for the same mass ratio $\mu = 0.2$, the transmissibility of the main structure at both resonance frequencies grows up to 10 (Fig. 5). On the other hand, if the TMD partial loss factor is relatively high ($\eta_{p2} = 0.25$) but the mass ratio is as low as $\mu = 0.02$ (Fig. 6), the transmissibility at the first resonance frequency is about 7 (almost twice that for $\mu = 0.2$) and there is no second resonance peak. Such a partially degenerate case for the auxiliary system was described in paper [14] with the following conclusion: the second resonance peak disappears if $\eta > [1-(\Omega_1/\Omega_2)^2]/2^{3/2}$ where the average loss factor η is given by Eq. (21). A similar effect can also occur in the main structure but at a higher loss factor.

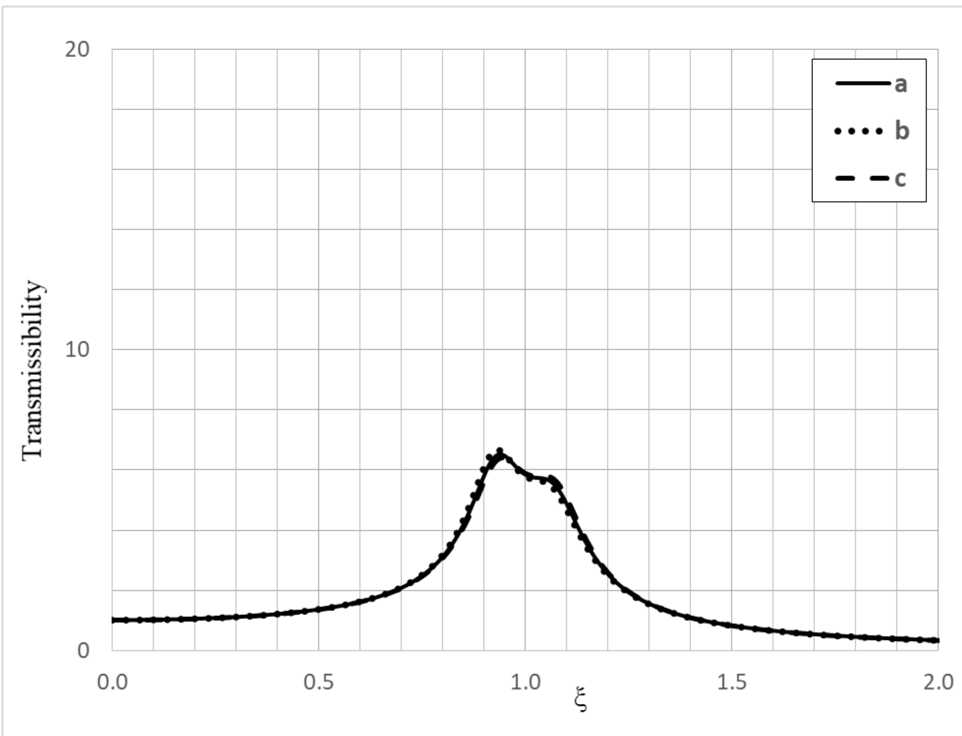

Fig. 6. Transmissibility vs. $\xi = \omega/\omega_{p1}$ for the main mass if

$\mu = 0.02$, $\eta_{p1} = 0.05, \eta_{p2} = 0.25$:

(a) $p = 1/\sqrt{1+\mu}$, (b) $p = 1/(1+\mu)$, c) $p = 1$.

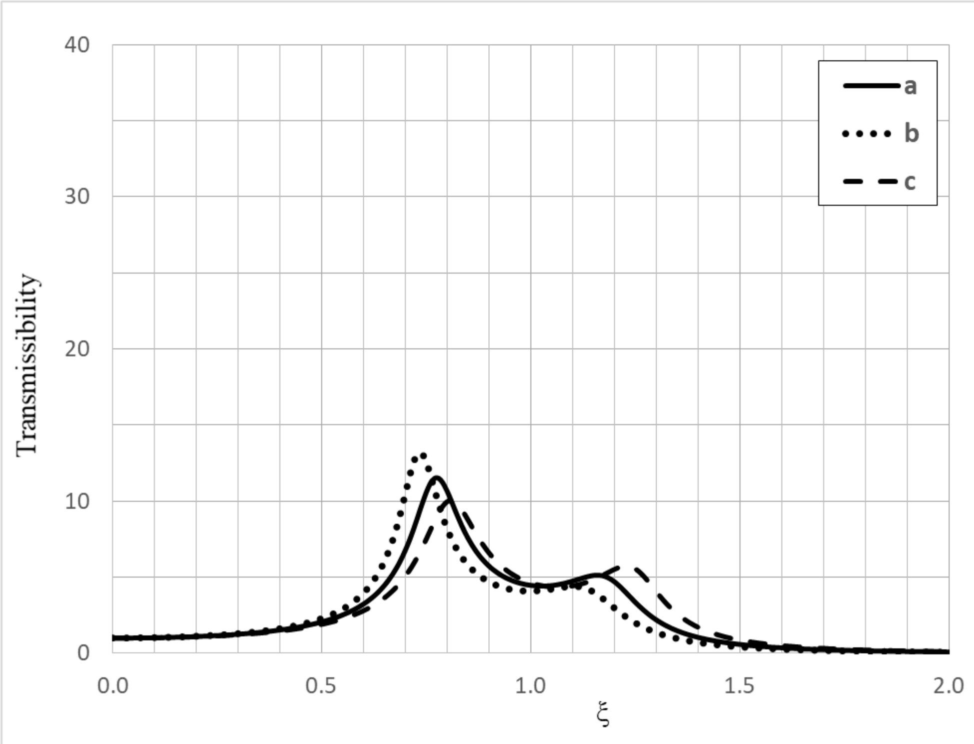

Fig. 7. Transmissibility vs. $\xi = \omega/\omega_{p1}$ for the auxiliary mass if

$\mu = 0.2$, $\eta_{p1} = 0.05, \eta_{p2} = 0.25$:

(a) $p = 1/\sqrt{1+\mu}$, (b) $p = 1/(1+\mu)$, (c) $p = 1$.

The transmissibility of the auxiliary mass for $\mu = 0.2$ and $\eta_{p2} = 0.25$ (Fig. 7) is 12 and 5 at the first and second resonance frequency, respectively. But if the TMD partial loss factor is low ($\eta_{p2} = 0.05$) for the same mass ratio $\mu = 0.2$, the transmissibility of the auxiliary structure at its resonance frequencies increases trice as much (Fig. 8). If the TMD partial loss factor is relatively high ($\eta_{p2} = 0.25$) but the mass ratio is as low as $\mu = 0.02$ (Fig. 9), the resonance peaks of the auxiliary structure merge together with a transmissibility of 26 (Fig. 9).

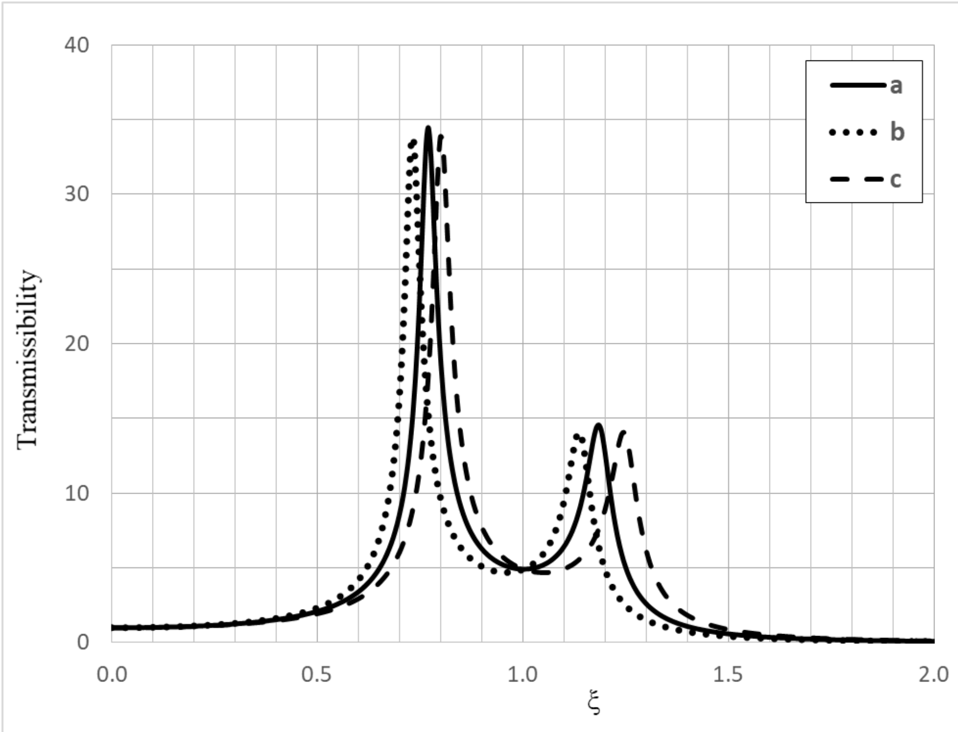

Fig. 8. Transmissibility vs. $\xi = \omega/\omega_{p1}$ for the auxiliary mass if

$\mu = 0.2$, $\eta_{p1} = \eta_{p2} = 0.05$: (a) $p = 1/\sqrt{1+\mu}$, (b) $p = 1/(1+\mu)$, c) $p = 1$.

4. CLOSED-FORM SIMPLE EQUATIONS FOR TRANSMISSIBILITY AT THE RESONANCE FREQUENCIES

4.1. General relationships for transmissibility at the resonance peak amplitudes

For simplicity suppose that the resonance frequencies of 2-DOF in-series system coincide with the appropriate undamped natural frequencies. This is true for 1-DOF systems with hysteretic friction [12] but if the loss factors are relatively small, this suggestion is also reasonable for 2-DOF in-series systems with hysteretic friction, in particular for the frequency response of the main structure. For the auxiliary structure, the second resonance frequency can notably shift down from the appropriate undamped natural frequency if the average loss factor is high [14].

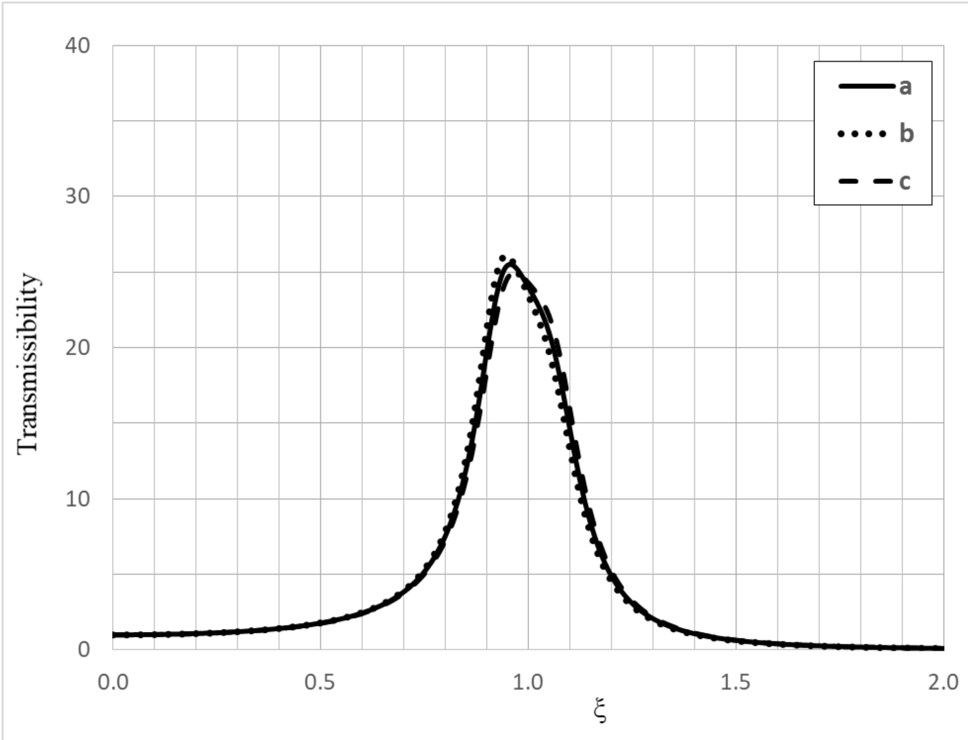

Fig. 9. Transmissibility vs. $\xi = \omega/\omega_{p1}$ for the auxiliary mass if

$\mu = 0.02$, $\eta_{p1} = 0.05, \eta_{p2} = 0.25$:

(a) $p = 1/\sqrt{1+\mu}$, (b) $p = 1/(1+\mu)$, (c) $p = 1$.

Using Eqs (19) and (17b), derive four helpful intermediate relationships for the nearby case:

$$\left(\frac{\Omega_1}{\Omega_2}\right)^2 = \frac{\sqrt{1+\mu} - \sqrt{\mu}}{\sqrt{1+\mu} + \sqrt{\mu}} = \left(\sqrt{1+\mu} - \sqrt{\mu}\right)^2 = 1 - 2\sqrt{\mu}\left(\sqrt{1+\mu} - \sqrt{\mu}\right), \tag{29}$$

$$\left(\frac{\Omega_2}{\Omega_1}\right)^2 = \frac{\sqrt{1+\mu} + \sqrt{\mu}}{\sqrt{1+\mu} - \sqrt{\mu}} = \left(\sqrt{1+\mu} + \sqrt{\mu}\right)^2 = 1 + 2\sqrt{\mu}\left(\sqrt{1+\mu} + \sqrt{\mu}\right), \tag{30}$$

$$\left(\frac{\Omega_1}{\omega_{p2}}\right)^2 = \left(\frac{\Omega_1}{\omega_{p1}}\frac{\omega_{p1}}{\omega_{p2}}\right)^2 = \frac{\sqrt{1+\mu} - \sqrt{\mu}}{\sqrt{1+\mu}}(1+\mu) = \sqrt{1+\mu}\left(\sqrt{1+\mu} - \sqrt{\mu}\right), \tag{31}$$

$$\left(\frac{\Omega_2}{\omega_2}\right)^2 = \left(\frac{\Omega_2}{\omega_{p1}}\frac{\omega_{p1}}{\omega_{p2}}\right)^2 = \frac{\sqrt{1+\mu}+\sqrt{\mu}}{\sqrt{1+\mu}}(1+\mu) = \sqrt{1+\mu}\left(\sqrt{1+\mu}+\sqrt{\mu}\right). \quad (32)$$

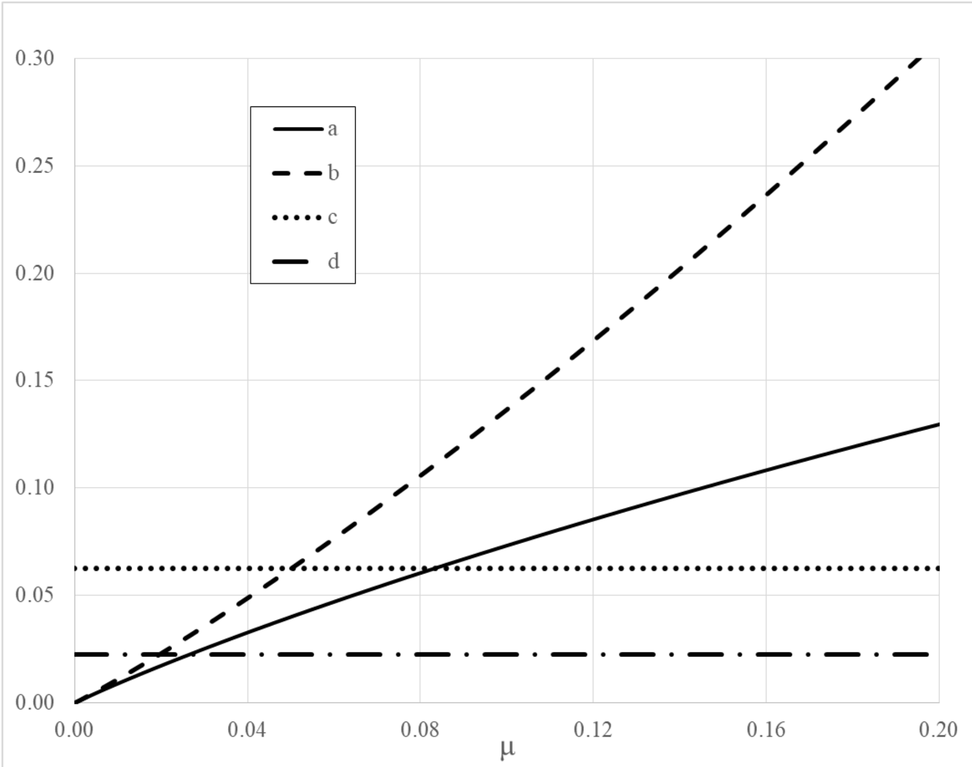

Fig. 10. Functions (a) $\varphi_1(\mu)$ and (b) $\varphi_2(\mu)$ in comparison with

(c) $\eta_{p2}^2 = 0.25^2 = 0.0625$ and (d) $\eta^2 = 0.15^2 = 0.0225.$

Substituting Eqs (29)-(32) into Eqs (24) and (25), calculate the transmissibility at the resonance frequencies of the main structure

$$T_1(\Omega_1) = \frac{\sqrt{1+\eta_{p1}^2}}{2\eta}\sqrt{\frac{\varphi_1+\eta_{p2}^2}{\varphi_1+(\eta/2)^2}} \approx \frac{1}{2\eta}\sqrt{\frac{\varphi_1+\eta_{p2}^2}{\varphi_1+(\eta/2)^2}}, \quad (33)$$

$$T_1(\Omega_2) = \frac{\sqrt{1+\eta_{p1}^2}}{2\eta}\sqrt{\frac{\varphi_2+\eta_{p2}^2}{\varphi_2+(\eta/2)^2}} \approx \frac{1}{2\eta}\sqrt{\frac{\varphi_2+\eta_{p2}^2}{\varphi_2+(\eta/2)^2}}, \quad (34)$$

and at the first resonance frequency of the auxiliary structure (TMD)

$$\mathrm{T}_2(\Omega_1) = \frac{\sqrt{1+\eta_{p1}^2}}{2\,\eta}\;\sqrt{\frac{1+\eta_{p2}^2}{\varphi_1 + (\eta/2)^2}} \approx \frac{1}{2\,\eta}\;\frac{1}{\sqrt{\varphi_1 + (\eta/2)^2}}\,. \tag{35}$$

where the functions

$$\begin{cases} \varphi_1(\mu) = \mu\left(\sqrt{1+\mu} - \sqrt{\mu}\right)^2 = \dfrac{\mu}{\left(\sqrt{1+\mu} + \sqrt{\mu}\right)^2} \\ \varphi_2(\mu) = \mu\left(\sqrt{1+\mu} + \sqrt{\mu}\right)^2 \end{cases}, \tag{36}$$

are plotted in Fig. 10, and the average loss factor η is given by Eq. (21). The transmissibility at the resonance frequencies, given by Eqs (33) – (35), depends on three parameters: the mass ratio μ, partial loss factor η_{p2}, and average partial loss factor η. These relationships are plotted vs. the mass ratio μ for $\eta_{p1} = 0.05$ and $\eta_{p2} = 0.25$ (Fig. 11) or $\eta_{p2} = 0.05$ (Fig. 12). Generally, Eqs (33)-(35) are in good agreement with the Eqs (24) and (25) at the resonance frequencies. They are simple, and can be further simplified for relatively low or high mass ratios.

4.2. Approximate expressions for relatively high mass ratios

From Eqs (36), $\varphi_1(\mu) \to \frac{1}{4}$, $\varphi_2(\mu) \to \infty$ if $\mu \to \infty$. In this asymptotic case, $\varphi_{1,2}(\mu) >> \eta_2^2 \ge \eta_1^2$, and Eqs (33)-(35) become very simple and straightforward:

$$\mathrm{T}_1(\Omega_1) \approx \mathrm{T}_1(\Omega_2) \approx \frac{1}{\eta_{p1} + \eta_{p2}} = \frac{1}{2\,\eta}\,, \tag{37}$$

$$\mathrm{T}_2(\Omega_1) \approx \frac{2}{\eta_{p1} + \eta_{p2}} = \frac{1}{\eta}\;. \tag{38}$$

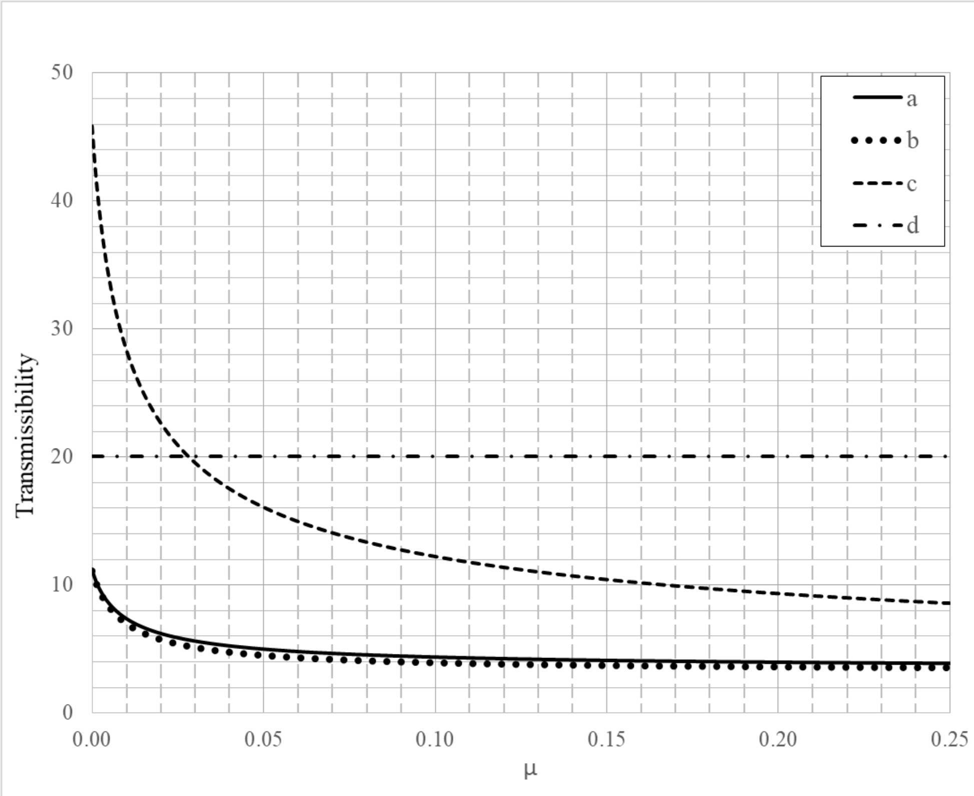

Fig. 11. Transmissibility at the resonance frequencies vs. the mass ratio μ if

$\eta_{p1} = 0.05, \eta_{p2} = 0.25$:

(a) $T_1\,(\Omega_1)$, (b) $T_1\,(\Omega_2)$, (c) $T_2\,(\Omega_1)$, (d) $T_0\,(\omega_{p1}) = 20$.

According to Eq (37), the transmissibility at both resonance frequencies for the main structure of 2-DOF in-series system with a relatively high mass ratio is similar to the peak transmissibility of a 1-DOF system where the loss factor equals the sum of the partial loss factors. In particular if $\eta_{p2} = \eta_{p1}$, the transmissibility $T_1(\Omega_1) \approx T_1(\Omega_2) \approx \dfrac{1}{2\,\eta_{p1}} = \dfrac{T_0(\omega_1)}{2}$, so the original resonance peak is reduced by half. To produce a much higher attenuation, the second partial loss factor should notably exceed the first partial loss factor ($\eta_{p2} >> \eta_{p1}$). One more important conclusion comes from Eq. (37) if $\eta_{p2} << \eta_{p1}$: no real vibration attenuation is accomplished in this uncommon case, because $T_1(\Omega_1) \approx T_1(\Omega_2) \approx \dfrac{1}{\eta_{p1}} = T_0(\omega_1)$.

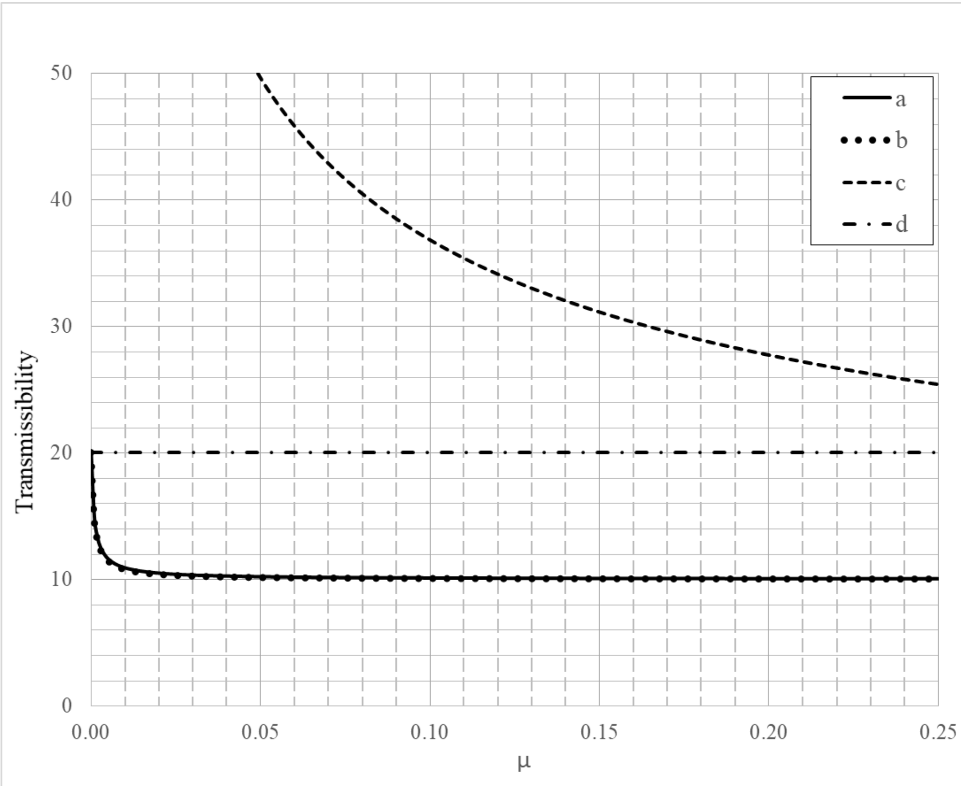

Fig. 12. Transmissibility at the resonance frequencies vs. the mass ratio μ

if $\eta_{p1} = \eta_{p2} = 0.05$:

(a) $T_1\ (\Omega_1)$, (b) $T_1\ (\Omega_2)$, (c) $T_2\ (\Omega_1)$, (d) $T_0\ (\omega_{p1}) = 20$.

To estimate a lower bound of the application range for the simplified Eq. (37), compare it numerically with Eqs (33) and (34) if $\eta_{p1} = 0.05, \eta_{p2} = 0.25$, and $\eta_{p1} = \eta_{p2} = 0.05$. In such typical cases, Eq. (37) calculates the transmissibility of 3.3 and 10 which is in good agreement with the plots in Figs 10 and 11 if $\mu \geq 0.04$ and $\mu \geq 0.01$, respectively. This makes Eq. (37) convenient for fast engineering estimations. It should be noted that the asymptotic value of transmissibility, expressed by Eq. (38) for the first resonance frequency of the auxiliary structure, is achieved at a much higher mass ratio.

4.3. Approximate expressions for relatively low mass ratios

If $\mu \to 0$, Eqs (36) are reduced to $\varphi_1(\mu) \approx \varphi_2(\mu) \approx \mu \to 0,$ and the mass ratios can be so small that

$$\varphi_{1,2}(\mu) << \eta_1^2 \leq \eta_2^2 . \tag{39}$$

Using the condition (39), simplify Eqs (33)-(35) as

$$T_1(\Omega_1) \approx T_1(\Omega_2) \approx \frac{4\,\eta_{p2}}{\left(\eta_{p1} + \eta_{p2}\right)^2} = \frac{\eta_{p2}}{\eta^2} , \tag{40}$$

$$T_2(\Omega_1) \approx \frac{4}{\left(\eta_{p1} + \eta_{p2}\right)^2} = \frac{1}{\eta^2} . \tag{41}$$

At first glance, the transmissibility expressed by Eq. (40) can be very low in the uncommon case $\eta_{p2} \to 0$ but such a formal approximation may not be valid since it can be in contradiction with Eq. (39). If $\eta_{p2} \geq \eta_{p1}$, the transmissibility values calculated by Eqs (40) and (41) are notably over those resulted from Eqs (37) and (38). In particular if $\eta_{p2} = \eta_{p1}$, Eq. (40) calculates $T_1(\Omega_1) \approx T_1(\Omega_2) \approx \dfrac{1}{\eta_{p1}} = T_0(\omega_1)$ which is the peak transmissibility of the main structure alone. To provide a reasonable attenuation, the TMD partial loss factor must notably exceed that for the main structure. This requirement is important for the building applications where the effective mass factor can be rather low: about 0.01 or less.

5. CONCLUSIONS

The forced vibration of 2-DOF in-series system with hysteretic friction is analyzed in case where the first and second partial 1-DOF systems play the roles of main structure and tuned (auxiliary) mass damper (TMD), respectively. The assumption of hysteretic damping is reasonable if the loss factor remains about stable at least in the frequency range containing the resonance peaks. Generally, the partial loss factor of a tuned mass damper should be much over that for the main structure but sometimes the role of TMD is played by a system with a similar loss factor: for instance, in automotive vehicles where an auxiliary cooling module attached to the main radiator via vibration isolators may serve as TMD.

The closed-form and simple relationships for a transmissibility at the resonance frequencies, given by Eqs (33)-(35), were derived in the “nearby” case (described by Eq. (17b)) where (1) the natural undamped frequencies of 2-DOF in-series system are most close to each other and (2) both loss factors of the 2-DOF system get equal to the arithmetic average of the partial loss factors. The transmissibility peak magnitudes in cases, where the ratio of natural undamped frequencies of 2-DOF system are moderately lower or higher than in the “nearby case”, prove to be about similar to those for the “nearby” case.

The independent parameters are the mass ratio and both partial loss factors.

Two limit cases are analyzed: for the relatively high and low mass ratios. If the mass ratio is relatively high (over 0.01-0.04 for most practical cases), the relationships given by Eq. (37) for the main structure and by Eq. (38) for the auxiliary structure become quite simple. In particular by Eq. (37), the transmissibility at both resonance frequencies for the main structure of 2-DOF in-series system is similar to the peak transmissibility of a 1-DOF system where the loss factor equals the sum of both partial loss factors. In this case, a notable positive effect is achieved even if the TMD (second) loss factor is similar to the partial loss factor of the main structure: the transmissibility of the main structure is reduced by half. Certainly, the higher the TMD loss factor, the lower the transmissibility at both resonance frequencies.

If the mass ratio is relatively low (about 0.01 or less), the transmissibility is given by Eq. (39) for the main structure and by Eq. (40) for the auxiliary structure. In this case, no positive effect is achieved if the TMD (second) loss factor is similar to the partial loss factor of the main structure. Here, for real improvement the TMD loss factor should notably exceed the partial loss of the main structure; this condition is important for the building applications where the effective mass factor used to be rather low.

The closed-form and simple equations developed in this paper are easy for physical interpretation and can be helpful for the noise and vibration control in machinery and buildings.

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