On the general solution for the Troesch, sinh-Poisson and Poisson-Boltzmann equations

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Abstract

This paper shows, for the first time, that the explicit and exact solution to the Troesch nonlinear two-point boundary value problem may be computed in a direct and straightforward fashion from the general solution obtained by a generalized Sundman transformation for the related differential equation, which appeared with the sinh-Poisson and Poisson-Boltzmann equations to be special cases of a more general equation. As a result, various initial and boundary value problems for these equations may be solved explicitly and exactly.

Theory

The Troesch nonlinear two-point boundary value problem is well known to be of the form [1]

$$u''(x) + a\sinh au(x) = 0 \qquad (1)$$

where

$$u(0) = 0, \quad u(1) = 1 \qquad (2)$$

and a is a positive constant. The sinh-Poisson differential equation is of the form [2]

$$u''(x) + a^2 \sinh(u) = 0 \qquad (3)$$

and the Poisson-Boltzmann equation may be written [2]

$$u''(x) - b^2 \sinh(u) = 0 \qquad (4)$$

The purpose is now to establish that the equations (1), (3) and (4) are limiting cases of a more general equation.

1. Generalized equation

According to [3-5], the general class of equations for the application of the generalized Sundman linearization theory developed by Akande et al. [3] may be written as

$$u''(x) + a^2 e^{\varphi(u)} \int e^{\varphi(u)} du = 0 \qquad (5)$$

where $\gamma = 1$, under the conditions that

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$$y(\tau)=\int e^{\varphi(u)}du\,,\;\; d\tau=e^{\varphi(u)}dx \tag{6}$$

and $y(\tau)$ satisfies

$$\ddot{y}(\tau)+a^2 y(\tau)=0 \tag{7}$$

that is

$$y(\tau)=A_0\sin(a\tau+\alpha) \tag{8}$$

Inserting

$$\varphi(u)=\ln(\sinh qu) \tag{9}$$

into (5) yields

$$u''(x)+\frac{a^2}{2q}\sinh(2qu)=0 \tag{10}$$

where $q\neq 0$, is an arbitrary parameter. The equation (10) is the desired generalized differential equation. Substituting $2q=a$ into (10), leads to the differential equation (1) of the Troesch nonlinear two-point boundary value problem under the condition that $a\succ 0$. For $2q=1$, the generalized equation (10) gives the sinh-Poisson equation (3). The Poisson-Boltzmann equation (4) is obtained from (10) for $2q=1$, and $a^2=-b^2$ where $b^2\succ 0$. In such a situation the general solution to (1), (3), and (4) may be explicitly and exactly established from that of (10).

2. General solution for the generalized equation

The application of (6), taking into consideration the equations (8) and (9), may lead to

$$qA_0\sin(a\tau+\alpha)=\cosh(qu) \tag{11}$$

that is

$$u(x)=\frac{1}{q}\cosh^{-1}\left[qA_0\sin(a\tau+\alpha)\right] \tag{12}$$

such that

$$\frac{d\tau}{\sqrt{q^2A_0^2\sin^2(a\tau+\alpha)-1}}=dx \tag{13}$$

The integration of (13) yields after a few algebraic manipulations [6]

$$\cos(a\tau+\alpha)=p\,sn\left[aqA_0(x+C),p\right] \tag{14}$$

where $p=\sqrt{1-\left(\frac{1}{qA_0}\right)^2}$, and C is a constant of integration, so that the general solution (12) to the generalized equation (10) becomes [6,7]

$$u(x)=\frac{1}{q}\cosh^{-1}\left\{qA_0dn\left[aqA_0(x+C),p\right]\right\} \tag{15}$$

3. Exact general solution for the Troesch problem

Making $2q = a$, yields the general solution to the differential equation (1) of the Troesch nonlinear two-point boundary value problem as

$$u(x) = \frac{2}{a}\cosh^{-1}\left\{\frac{a}{2}A_0 dn\left[\frac{a^2}{2}A_0(x+C), p\right]\right\} \tag{16}$$

where $p = \sqrt{1-\left(\frac{2}{aA_0}\right)^2}$. Therefore, the exact solution to the Troesch nonlinear two-point boundary value problem may be computed by the determination of the two integration constants by applying the boundary conditions (2). In this perspective the application of (2) leads, for the constants of integration A_0 and C, from the general solution (15), to the two transcendental equations

$$dn(qaA_0C, p) = \frac{1}{aqA_0} \tag{17}$$

$$dn[aqA_0(1+C), p] = \frac{\cosh(q)}{aqA_0} \tag{18}$$

Therefore, the exact solution to the Troesch nonlinear two-point boundary value problem, for $q = \frac{a}{2}$, is given by the solution (16) under the conditions.

$$dn\left(\frac{a^2}{2}A_0C, p\right) = \frac{2}{a^2A_0} \tag{19}$$

$$dn\left[\frac{a^2}{2}A_0(1+C), p\right] = \frac{\cosh(\frac{a}{2})}{a^2A_0} \tag{20}$$

4. Exact general solution for the sinh-Poisson equation

For $2q = 1$, the exact general solution for (3), using (15) takes the form

$$u(x) = 2\cosh^{-1}\left\{\frac{A_0}{2}dn\left[\frac{aA_0}{2}(x+C), p\right]\right\} \tag{21}$$

where $p = \sqrt{1-\frac{4}{A_0^{\ 2}}}$.

5. Exact general solution for the Poisson-Boltzmann equation

The application of $2q=1$, and $a^2=-b^2$, where $b^2 \succ 0$ to (15), yields as general solution for (4), the expression

$$u(x)=2\cosh^{-1}\left\{\frac{A_0}{2}dn\left[\frac{i\varepsilon bA_0}{2}(x+C),p\right]\right\} \tag{22}$$

where $p^2=1-\frac{4}{A_0^{\ 2}}$. Using the identity

$$dn(iu,\ p)=dc(u,\ p')$$

where $p'=\sqrt{1-p^2}$, that is $p'=\frac{2}{A_0}$, the solution (22) takes definitively the form

$$u(x)=2\cosh^{-1}\left\{\frac{A_0}{2}dc\left[\frac{bA_0}{2}(x+C),\frac{2}{A_0}\right]\right\} \tag{23}$$

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