Formulas and Spreadsheets for Simple, Composite, and Complex Rotations of Vectors and Bivectors in Geometric (Clifford) Algebra

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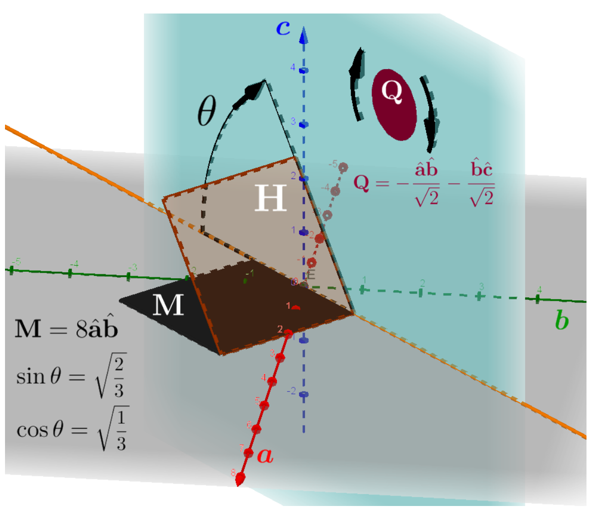
Abstract

We show how to express the representations of single, composite, and "rotated" rotations in GA terms that allow rotations to be calculated conveniently via spreadsheets. Worked examples include rotation of a single vector by a bivector angle; rotation of a vector about an axis; composite rotation of a vector; rotation of a bivector; and the "rotation of a rotation". Spreadsheets for doing the calculations are made available via live links.



"Rotation of the bivector $8\mathbf{ab}$ by the bivector angle $\mathbf{Q}\theta$ to give the new bivector, $\mathbf{H}$."

Contents

1 Introduction

References [1] (pp. 280-286) and [2] (pp. 89-91) derive and explain the following formula for finding the new vector, $\mathbf{w'}$, that results from the rotation of a vector $\mathbf{w}$ through the angle θ with respect to a plane that is represented by the unit bivector $\mathbf{Q}$ to which that plane is parallel:

We will see later (Sections 4 and 5) that this formula for rotating a vector extends to the rotation of bivectors, and to the rotation of any multivector $\mathcal{M}$.

$$\mathbf{w}' = \left[exp\left(-\mathbf{Q}\frac{\theta}{2}\right)\right][\mathbf{w}]\left[exp\left(\mathbf{Q}\frac{\theta}{2}\right)\right]. \tag{1.1}$$

That formula is convenient and efficient for manipulations of vectors that are represented abstractly as symbols, but what form does it take in a specific, concrete situation? For example, how do we use it when a client presents the vector $\mathbf{w}$ in terms of coordinates with respect to that client's chosen frame of reference, and wishes to know the coordinates of the vector that results when $\mathbf{w}$ is rotated through the angle θ about a given axis? What will we need to do to transform that problem into a form suitable for solution via Eq. (1.1), and what will the calculations "look like" as we work through them?

These are the sorts of questions that we will address in this document. Because geometric algebra (GA) rotates objects through bivector angles rather than around axes, Section 2.1 begins by deriving a formula for the rotation of a given vector through a given bivector angle. After introducing, briefly, the important subject of how GA "represents" rotations symbolically, we'll implement our formula in an Excel spreadsheet, which we'll then use to solve two example problems.

Having worked those examples, we'll show how we may derive a similar formula for rotating a vector about an axis, by transforming that rotation into one through a bivector angle. Again, a sample problem will be solved via a spreadsheet.

We'll then treat one of GA's strengths: its ability to formulate and calculate the result of sequence of rotations conveniently, using those rotations' representations. We'll derive formulas that will allow us to find, as an example problem, the single rotation that would have produced the same result as the combination of the rotations that were given in the two sample problems in Section 2.1.1 .

Vectors are not the only objects that we will want to rotate in GA; the rotation of bivectors is particularly useful. We'll take up that subject in a section that derives formulas that can be implemented in a spreadsheet to solve our sample problem.

Finally, we'll treat an interesting problem from Ref. [2]: the "rotation of a rotation". The derivation of a formula for that purpose makes use of our result for rotating a bivector. As in previous sections, we'll finish by solving a sample problem via a spreadsheet.

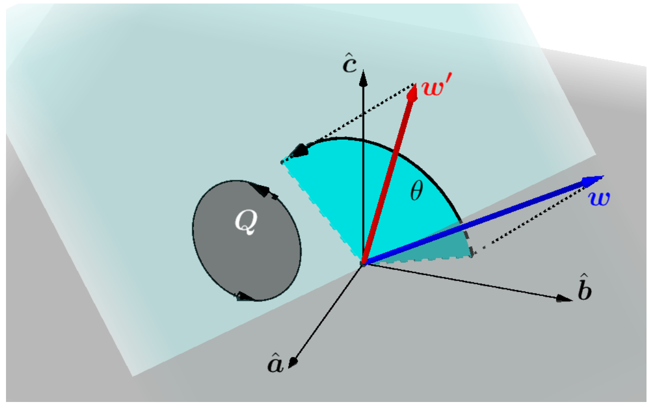


Figure 1: Rotation of the vector $\mathbf{w}$ through the bivector angle $\mathbf{Q}\theta$, to produce the vector $\mathbf{w'}$.

2 Rotation of a Given Vector

2.1 Rotation by a Bivector Angle

When describing an angle of rotation in GA, we are often well advised—for sake of clarity—to write it as the product of the angle's scalar measure (in radians) and the bivector of the plane of rotation. Following that practice, we would say that the rotation of a vector $\mathbf{w}$ through the angle θ (measured in radians) with respect to a plane that is parallel to the unit bivector $\mathbf{Q}$, is the rotation of $\mathbf{v}$ through the *bivector angle* $\mathbf{Q}\theta$. (For example, see Fig. 1.) References [1] (pp. 280-286) and [2] (pp. 89-91) derive and explain the following formula for finding the new vector, $\mathbf{w'}$, that results from that rotation :

Notation: $\mathsf{R}_{\mathbf{Q}\theta}(\mathbf{w})$ is the rotation of the vector $\mathbf{w}$ by the bivector angle $\mathbf{Q}\theta$.

$$\mathbf{w'} = \underbrace{\left[e^{-\mathbf{Q}\theta/2}\right][\mathbf{w}]\left[e^{\mathbf{Q}\theta/2}\right]}_{\text{Notation: } \mathsf{R}_{\mathbf{Q}\theta}(\mathbf{w})}. \tag{2.1}$$

For our convenience later in this document, we will follow Reference [2] (p. 89) in saying that the factor $e^{-\mathbf{Q}\theta/2}$ *represents* the rotation $\mathsf{R}_{\mathbf{Q}\theta}$. That factor is a quaternion, but in GA terms it is a multivector. We can see that it is a multivector from the following identity, which holds for any unit bivector $\mathbf{B}$ and any angle ϕ (measured in radians):

The *representation* of a rotation.

$$exp\,(\mathbf{B}\phi) \equiv \cos\phi + \mathbf{B}\sin\phi.$$

Thus,

$$e^{-\mathbf{Q}\theta/2} = \cos\frac{\theta}{2} - \mathbf{Q}\sin\frac{\theta}{2}. \tag{2.2}$$

In this document, we'll restrict our treatment of rotations to three-dimensional Geometric Algebra ($\mathbb{G}^3$). In that algebra, and using a right-handed reference system with orthonormal basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, we may express the unit bivector $\mathbf{Q}$ as a linear combination of the basis bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$:

$$\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}}q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}q_{ac},$$

in which q_{ab}, q_{bc}, and q_{ac} are scalars, and $q_{ab}^2 + q_{bc}^2 + q_{ac}^2 = 1$.

If we now write $\mathbf{w}$ as $\mathbf{w} = \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c$, Eq. (1.1) becomes

$$\begin{aligned}
\mathbf{w}' &= \left[\cos\frac{\theta}{2} - \mathbf{Q}\sin\frac{\theta}{2}\right][\mathbf{w}]\left[\cos\frac{\theta}{2} + \mathbf{Q}\sin\frac{\theta}{2}\right] \\
&= \left[\cos\frac{\theta}{2} - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}q_c + \hat{\mathbf{b}}\hat{\mathbf{c}}q_a - \hat{\mathbf{a}}\hat{\mathbf{c}}q_b\right)\sin\frac{\theta}{2}\right]\left[\hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right]\left[\cos\frac{\theta}{2} + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}q_c + \hat{\mathbf{b}}\hat{\mathbf{c}}q_a - \hat{\mathbf{a}}\hat{\mathbf{c}}q_b\right)\sin\frac{\theta}{2}\right]. \quad (2.3)
\end{aligned}$$

Expanding the right-hand side of that result, we'd obtain 48 (!) terms, some of which would simplify to scalar multiples of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and others of which will simplify to scalar multiples of the trivector $\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$. The latter terms would cancel, leaving an expression for $\mathbf{w}'$ in terms of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$.

The prospect of carrying out that expansion and simplification is fairly terrifying, so before we dive into that task, we might want to think a bit about which tools we'd use to carry out the calculation in practice. In the absence of specialized GA software, we might use Excel to calculate the coordinates of $\mathbf{w}'$ in terms of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$. With that end in mind, a reasonable step to take before expanding and simplifying the right-hand side of Eq. (2.3) is to define four scalar variables, which we'd use later in an Excel spreadsheet (Section 2.1.1):

- $f_o = \cos\frac{\theta}{2}$;
- $f_{ab} = q_{ab}\sin\frac{\theta}{2}$;
- $f_{bc} = q_{bc}\sin\frac{\theta}{2}$; and
- $f_{ac} = q_{ac}\sin\frac{\theta}{2}$.

Using these variables, Eq. (2.3) becomes

$$\mathbf{w}' = \left[f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right)\right]\left[\hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right]\left[f_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right)\right].$$

After expanding and simplifying the right-hand side, we obtain

$$\begin{aligned}
\mathbf{w}' = \hat{\mathbf{a}}&\left[w_a\left(f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2\right) + w_b\left({}^{-}2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_c\left({}^{-}2f_of_{ac} + 2f_{ab}f_{bc}\right)\right] \\
+ \hat{\mathbf{b}}&\left[w_a\left(2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_b\left(f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2\right) + w_c\left({}^{-}2f_of_{bc} - 2f_{ab}f_{ac}\right)\right] \quad (2.4) \\
+ \hat{\mathbf{c}}&\left[w_a\left(2f_of_{ac} + 2f_{ab}f_{bc}\right) + w_b\left(2f_of_{bc} - 2f_{ab}f_{ac}\right) + w_c\left(f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2\right)\right] .
\end{aligned}$$

Note that in terms of our four scalar variables f_o, f_{ab}, f_{bc}, and f_{ac}, the representation $\left(e^{-\mathbf{Q}\theta/2}\right)$ of the rotation is

$$e^{-\mathbf{Q}\theta/2} = f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right). \quad (2.5)$$

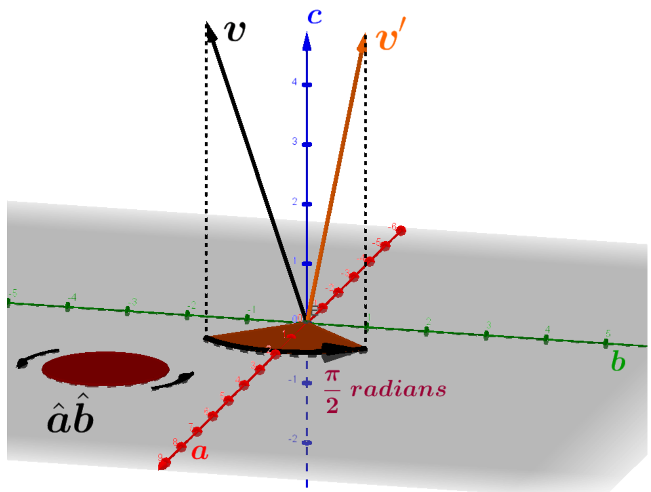


Figure 2: Rotation of the vector $\mathbf{v}$ through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$, to produce the vector $\mathbf{v}'$.

Because of the convenience with which Eq. (2.4) can be implemented in a spreadsheet, the remainder of this document will express the representations of various rotations of interest in the form of Eq. (2.5).

2.1.1 Sample Calculations

Example 1 The vector $\mathbf{v} = \frac{4}{3}\hat{\mathbf{a}} - \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$ is rotated through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$ radians to produce a new vector, $\mathbf{v}'$. Calculate $\mathbf{v}'$.

The rotation is diagrammed in Fig. 2.

As shown in Fig. 3, $\mathbf{v}' = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$.

Example 2 The vector $\mathbf{v}'$ from **Example 1** is now rotated through the bivector angle $\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$ to produce vector $\mathbf{v}''$. Calculate $\mathbf{v}''$.

The rotation of $\mathbf{v}'$ by $\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$ is diagrammed in Fig. 4. Fig. 5 shows that $\mathbf{v}'' = \frac{4}{3}\hat{\mathbf{a}} + \frac{16}{3}\hat{\mathbf{b}} + \frac{4}{3}\hat{\mathbf{c}}$.

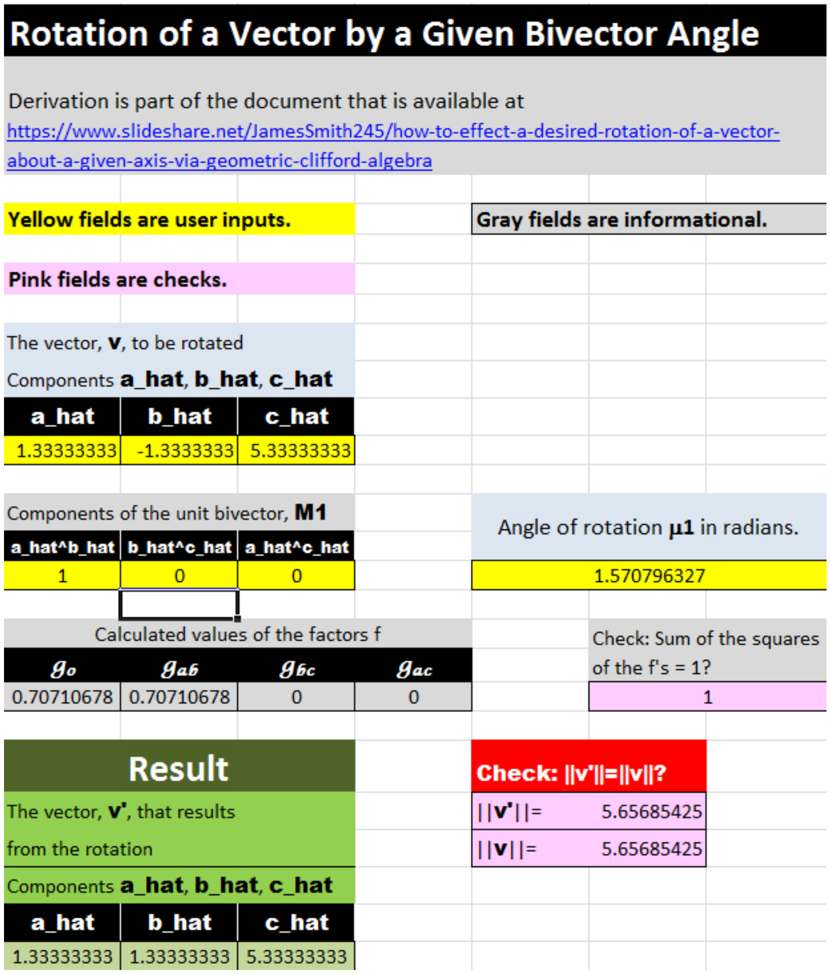


Figure 3: Screen shot of the Excel spreadsheet (Reference [3]) that uses Eq. (2.4) to calculate $\mathbf{v}'$ as the rotation of $\mathbf{v}$ through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$.

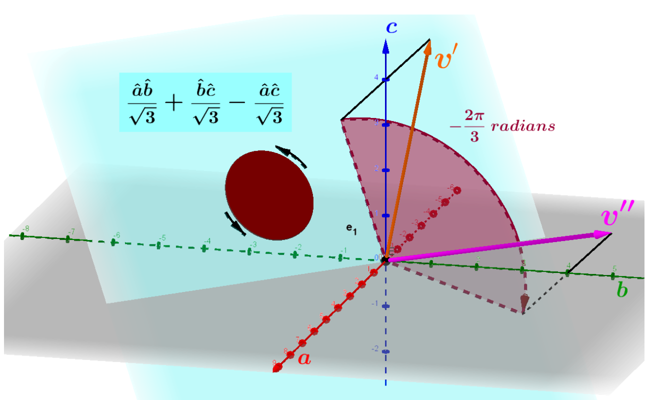


Figure 4: Rotation of $\mathbf{v}'$ to form $\mathbf{v}''$. Note the significance of the negative sign of the scalar angle: the direction in which $\mathbf{v}'$ is to be rotated is contrary to the rotation of the bivector.

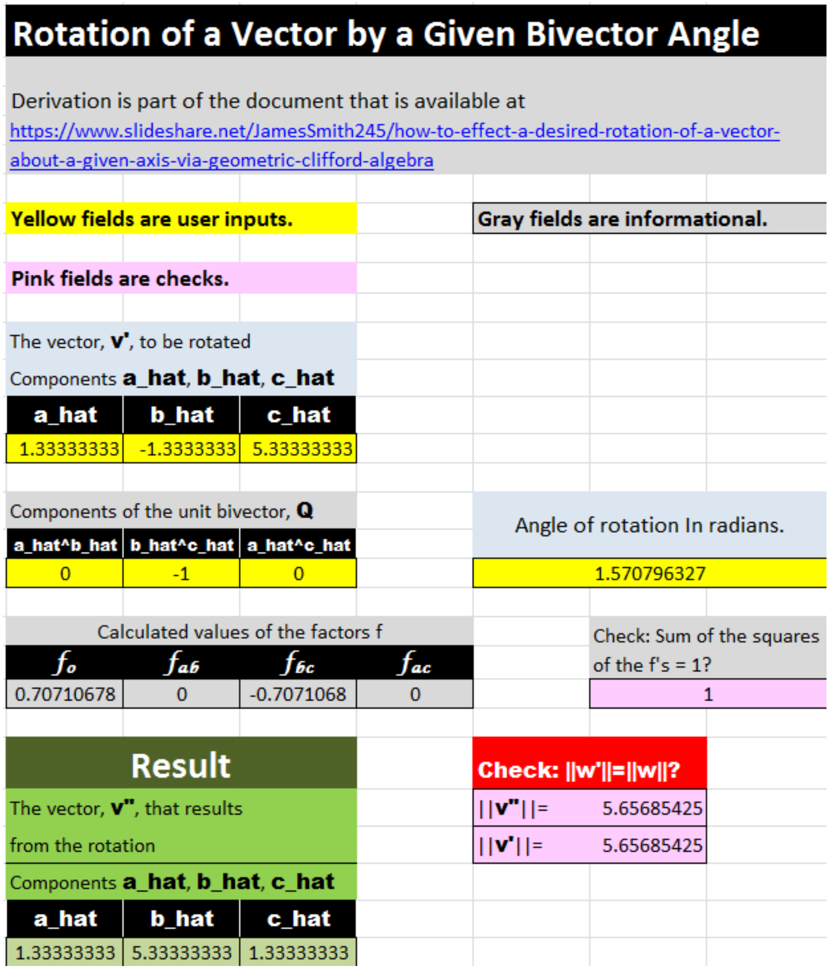


Figure 5: Screen shot of the Excel spreadsheet (Reference [3]) that uses Eq. (2.4) to calculate $\mathbf{v}''$ as the rotation of $\mathbf{v}'$. Compare the result to that shown in Fig. 14.

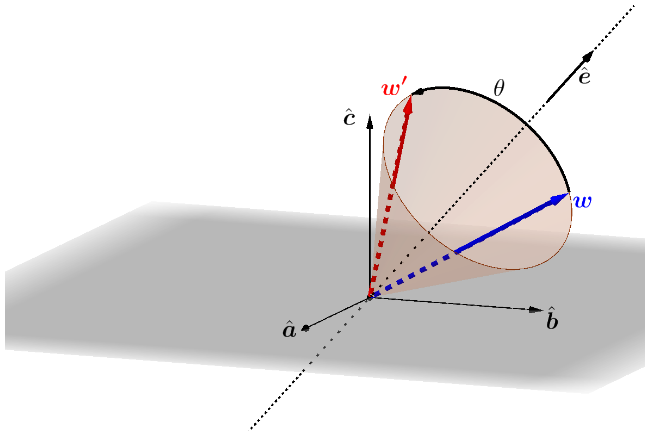


Figure 6: The new vector $\mathbf{w}'$ produced by the rotation of vector $\mathbf{w}$ through the angle θ, around an axis given by the vector $\hat{\mathbf{e}}$. The reference frame formed by vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ (taken in that order) is orthonormal and right-handed.

2.2 Rotation about a Given Axis

2.2.1 Statement and Transformation of the Problem

Statement of the Problem We state the problem as follows, with reference to Fig. 6:

> The vector $\mathbf{w}$ $\left(= \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right)$ is rotated through the angle θ, in the direction as indicated, about the unit vector $\hat{\mathbf{e}}$ $\left(= \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)$. Write the resulting vector $\mathbf{w}'$ in terms of the same basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$.

Note that the reference system is a right-handed orthonormal one (Reference [1], p. 53), and that the given rotation is a right-handed one about the unit vector $\hat{\mathbf{e}}$.

Transformation of the Problem

Why is a transformation necessary? As explained in Section 2.1, rotations in three-dimensional Geometric Algebra ($\mathbb{G}^3$) are effected with respect to planes, rather than axes. Or to put it more correctly, with respect to bivectors rather than vectors. We'd like to use Eq. (1.1) to solve our present problem, so we must first identify the unit bivector that corresponds to the given axis of rotation, $\hat{\mathbf{e}}$. What do we mean by "corresponds to"? We'll answer that question in the next section.

Identifying the unit bivector that "corresponds to" the given axis of rotation The rotation that we are asked to make is a right-handed one. Therefore —as may be inferred from a study of references [1] (p. (56, 63) and [2] (pp. 106-108) —the unit bivector $\mathbf{Q}$ that we seek is the one whose dual is $\hat{\mathbf{e}}$. That is, $\mathbf{Q}$ must satisfy the condition

$$\hat{\mathbf{e}} = \mathbf{Q} I_3^{-1};$$
$$\therefore \mathbf{Q} = \hat{\mathbf{e}} I_3. \tag{2.6}$$

Although we won't use that fact here, I_3^{-1} is I_3's negative: $I_3^{-1} = -\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$.

where I_3 is the right-handed pseudoscalar for $\mathbb{G}^3$. That pseudoscalar is the product, written in right-handed order, of our orthonormal reference frame's basis vectors: $I_3 = \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ (and is also $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}$). Therefore, proceeding from Eq. (2.6),

To make this simplification, we use the following facts:

- The product of two perpendicular vectors (such as $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$) is a bivector;
- Therefore, for any two perpendicular vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{qp} = -\mathbf{qp}$; and
- (Of course) for any unit vector $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}\hat{\mathbf{p}} = 1$.

$$\begin{aligned}\mathbf{Q} &= \hat{\mathbf{e}} I_3 \\ &= \left(\hat{\mathbf{a}} e_a + \hat{\mathbf{b}} e_b + \hat{\mathbf{c}} e_c\right) \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} \\ &= \hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} e_a + \hat{\mathbf{b}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} e_b + \hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} e_c \\ &= \hat{\mathbf{a}}\hat{\mathbf{b}} e_c + \hat{\mathbf{b}}\hat{\mathbf{c}} e_a - \hat{\mathbf{a}}\hat{\mathbf{c}} e_b. \end{aligned} \tag{2.7}$$

In writing that last result, we've followed [2]'s convention (p. 82) of using $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$ as our bivector basis. Examining Eq. (2.7) we can see that if we write $\mathbf{Q}$ in the form $\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}} q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}} q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}} q_{ac}$, then

$$q_{ab} = e_c, \quad q_{bc} = e_a, \quad q_{ac} = -e_c. \tag{2.8}$$

Two questions.

First, is $\mathbf{Q}$ a **unit** bivector, as Eq. (1.1) requires? Yes: for any bivector $\mathbf{B}$,

$$\|\mathbf{B}\| = \sqrt{\mathbf{B}\,(^{-}\mathbf{B})}.$$

If we calculate $\|\mathbf{Q}\|$ according to that formula, using the expression in Eq. (2.7), we find (after expansion and simplification) that

$$\|\mathbf{Q}\| = \sqrt{e_a^2 + e_b^2 + e_c^2}\ ,$$

which is equal to 1, because $\hat{\mathbf{e}}$ is a unit vector.

The second question is, "What would we have done if the required rotation had been a left-handed one around $\hat{\mathbf{e}}$, rather than a right-handed one?" There are two reasonable ways to handle such a case. We could either (1) make the rotation a right-handed one around the vector $^{-}\hat{\mathbf{e}}$; or (2) recognize that a left-handed rotation through an angle ψ is a right-handed rotation through the angle $^{-}\psi$. Therefore, using the latter idea, we'd use the given vector $\hat{\mathbf{e}}$, but use $^{-}\psi$ as our angle instead of ψ itself.

Now that we've identified the unit bivector $\mathbf{Q}$ for our problem, we can re-state our problem in terms that will enable us to use Eq. (1.1).

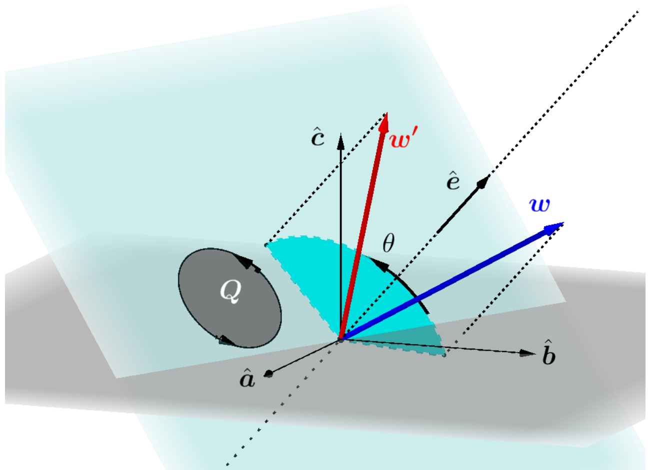


Figure 7: The same situation as in Fig. 6, translated into GA terms. The unit bivector $\mathbf{Q}$ $(= \hat{\mathbf{e}}I_3\ )$ is perpendicular to $\hat{\mathbf{e}}$. In GA, the angle of rotation would be the bivector $\mathbf{Q}\theta$ rather than the scalar θ.

2.2.2 Restatement of the Problem

We'll follow the practice of writing an angle of rotation as the product of the angle's scalar measure (in radians) and the bivector of the plane of rotation. In our present case, we would write that angle as $\mathbf{Q}\theta$. Therefore, using our expression for $\mathbf{Q}$ from Eq. (2.7), we restate our problem, with reference to Fig. 7, as

Recall that bivectors are "oriented areas": they do not possess the attribute "shape". Therefore, $\mathbf{Q}$ in Fig. 7 could have been drawn as any plane figure of unit area, and with the same orientation as shown.

> The vector $\mathbf{w}$ $\left(= \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right)$ is rotated through the bivector angle
> $\left(\hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b\right)\theta$. Write the resulting vector $\mathbf{w}'$ in terms of the same basis vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$.

We're ready, now, to employ Eq. (2.4) . All we need to do is make the appropriate substitutions in the list of f's that we developed in Section 2.1:

- $f_o = \cos\dfrac{\theta}{2}$;
- $f_{ab} = q_{ab}\sin\dfrac{\theta}{2} = e_c\sin\dfrac{\theta}{2}$;
- $f_{bc} = q_{bc}\sin\dfrac{\theta}{2} = e_a\sin\dfrac{\theta}{2}$; and
- $f_{ac} = q_{ac}\sin\dfrac{\theta}{2} = -e_b\sin\dfrac{\theta}{2}$.

Having made those substitutions, we simply use Eq. (2.4):

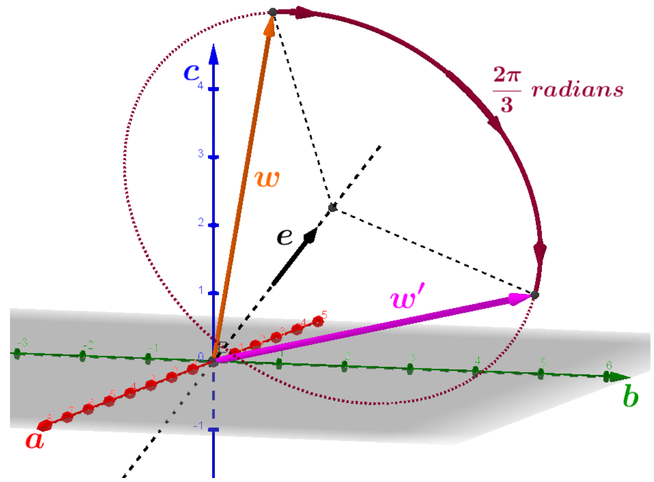


Figure 8: Vector $\mathbf{w} = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$ is rotated through $2\pi/3$ radians, in the direction shown, about an axis whose direction is given by the vector $\hat{\mathbf{e}} = \hat{\mathbf{a}}+\hat{\mathbf{b}}+\hat{\mathbf{c}}$. What is the new vector, $\mathbf{w}'$, that results? Note that the axes $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually perpendicular.

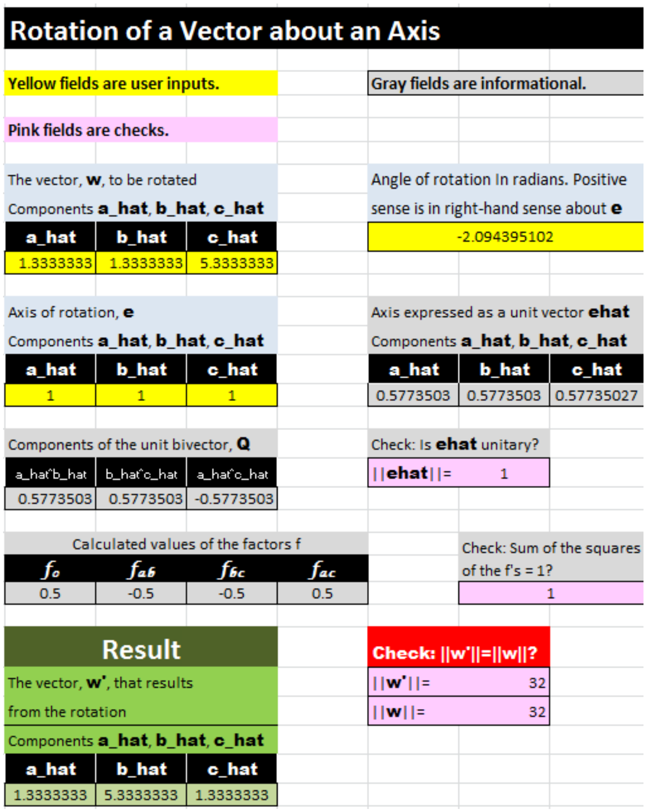
$$\begin{aligned}\mathbf{w}' = \hat{\mathbf{a}}&\left[w_a\left(f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2\right) + w_b\left({}^-2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_c\left({}^-2f_of_{ac} + 2f_{ab}f_{bc}\right)\right]\\ &+ \hat{\mathbf{b}}\left[w_a\left(2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_b\left(f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2\right) + w_c\left({}^-2f_of_{bc} - 2f_{ab}f_{ac}\right)\right]\\ &+ \hat{\mathbf{c}}\left[w_a\left(2f_of_{ac} + 2f_{ab}f_{bc}\right) + w_b\left(2f_of_{bc} - 2f_{ab}f_{ac}\right) + w_c\left(f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2\right)\right]\ .\end{aligned} \tag{2.9}$$

2.2.3 A Sample Calculation

We'll solve the following problem, with reference to Figs. 7 and 8, keeping our eyes open for data that will need to be transformed so that we may use Eq. (2.4) .

> The vector $\mathbf{w}$ is given by $\mathbf{w} = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$. It is rotated through $2\pi/3$ radians, in the direction shown, about an axis whose direction is given by the vector $\hat{\mathbf{e}} = \hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}$. What is the new vector, $\mathbf{w}'$, that results?

Examining Figs. 8 and 9, we see that the vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ (taken in that order) form an orthonormal reference frame, as required. However, the vector $\mathbf{e}$ is not a unit vector (its magnitude is $\sqrt{3}$), and the angle of rotation is in the direction defined as negative. Therefore, in our calculations we will use $\hat{\mathbf{e}}\ \left(= \frac{1}{\sqrt{3}}\hat{\mathbf{a}} + \frac{1}{\sqrt{3}}\hat{\mathbf{b}} + \frac{1}{\sqrt{3}}\hat{\mathbf{c}}\right)$ as our "axis" vector, and $-2\pi/3$ as θ. A screen shot of the Excel spreadsheet used for the calculation is shown in Fig. 9. From the perspective (Fig. 10) of someone who is looking at the origin along the direction



Rotation of a Vector about an Axis

Yellow fields are user inputs.

Gray fields are informational.

Pink fields are checks.

The vector, **w**, to be rotated
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.3333333	1.3333333	5.3333333

Angle of rotation In radians. Positive sense is in right-hand sense about **e**

-2.094395102

Axis of rotation, **e**
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1	1	1

Axis expressed as a unit vector **ehat**
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
0.5773503	0.5773503	0.57735027

Components of the unit bivector, **Q**

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
0.5773503	0.5773503	-0.5773503

Check: Is **ehat** unitary?
||**ehat**||= 1

Calculated values of the factors f

f_o	f_{ab}	f_{bc}	f_{ac}
0.5	-0.5	-0.5	0.5

Check: Sum of the squares of the f's = 1?
1

Result

The vector, **w'**, that results from the rotation
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.3333333	5.3333333	1.3333333

Check: ||w'||=||w||?
||**w'**||= 32
||**w**||= 32

Figure 9: Screen shot of the Excel spreadsheet (Reference [4]) used to calculate the coordinates for the vector $\mathbf{w}'$ that result form rotating $\mathbf{w}$ about the axis $\hat{\mathbf{e}} = \hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}$. Quantities shown in the spreadsheet are defined in the text.

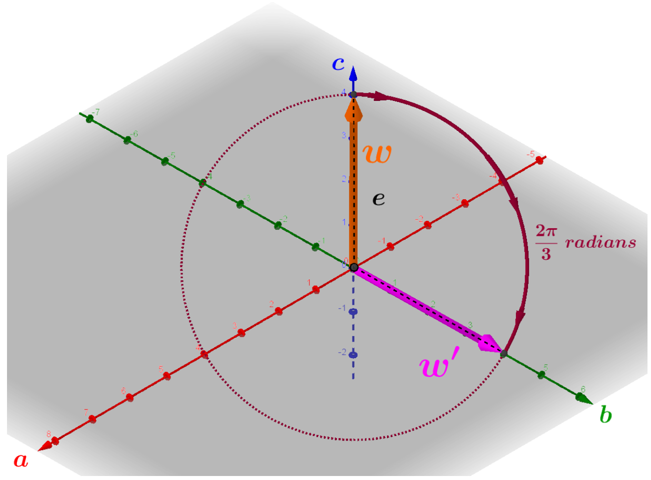


Figure 10: The same situation as in Fig. 8, but looking toward the origin along the direction $-\left(\hat{\mathbf{a}}+\hat{\mathbf{b}}+\hat{\mathbf{c}}\right)$.

$-\left(\hat{\mathbf{a}}+\hat{\mathbf{b}}+\hat{\mathbf{c}}\right)$, the rotation by $-2\pi/3$ brings $\mathbf{w}$ into alignment with the $\mathbf{b}$ axis. In the spreadsheet, that result is indicated by the fact that $\mathbf{w}'$'s $\hat{\mathbf{b}}$ coordinate is $\mathbf{w}$'s $\hat{\mathbf{c}}$ coordinate, and $\mathbf{w}'$'s $\hat{\mathbf{a}}$ coordinate is $\mathbf{w}$'s $\hat{\mathbf{c}}$ coordinate.

2.2.4 Summary of Rotating a Vector about a Given Axis

We have seen how to transform a "rotate a vector around a given axis" problem into one that may be solved via GA, which rotates objects with respect to bivectors. We have also seen how to calculate the result conveniently via an Excel spreadsheet. Two important cautions are (1) the axis must be expressed as a unit vector; and (2) the sign of the angle of rotation must be determined correctly.

3 Composite Rotations of Vectors

Suppose that we rotate some vector $\mathbf{v}$ through the bivector angle $\mathbf{M}_1\mu_1$ to produce the vector that we shall call $\mathbf{v}'$ (Fig. 11), and that we then rotate $\mathbf{v}'$ through the bivector angle $\mathbf{M}_2\mu_2$ to produce the vector that we shall call $\mathbf{v}''$ (Fig. 12). That sequence of rotations is called the *composition* of the two rotations.

In this section, we will derive an expression for the representation of a composition of two rotations. We'll write that representation in the form of Eq. (2.5), so that we may then use Eq. (2.4) to calculate the resulting vector, $\mathbf{v}''$. We'll also calculate the bivector angle that produces the same rotation in

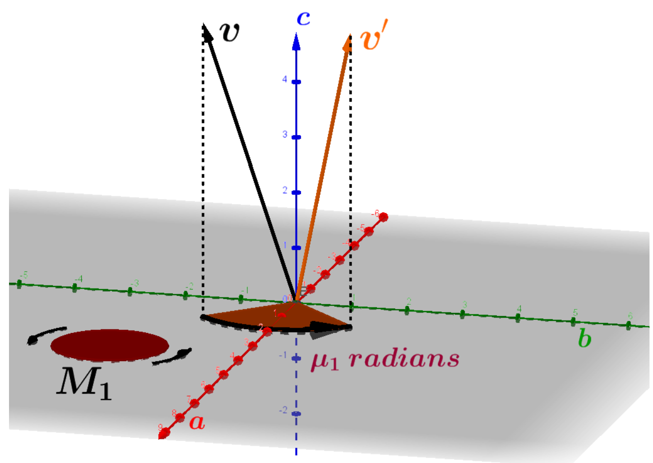


Figure 11: Rotation of the vector $\mathbf{v}$ through the bivector angle $\mathbf{M}_1\mu_1$, to produce the vector $\mathbf{v}'$.

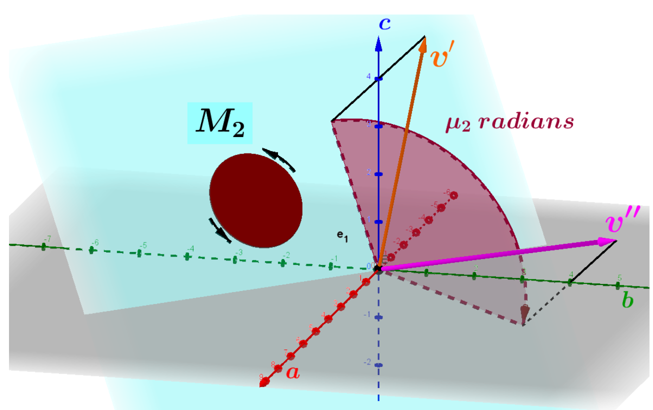


Figure 12: Rotation of the vector $\mathbf{v}'$ through the bivector angle $\mathbf{M}_2\mu_2$, to produce the vector $\mathbf{v}''$.

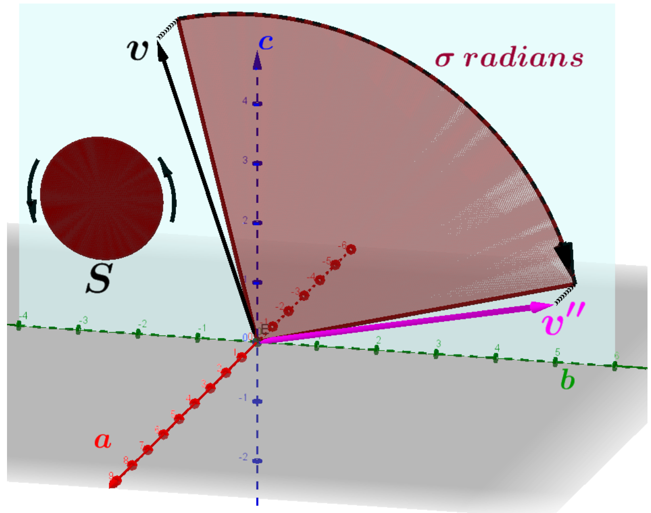


Figure 13: Rotation of $\mathbf{v}$ through the bivector angle $\mathbf{S}\sigma$, to produce the vector $\mathbf{v}''$ in a single operation.

a single operation. (The existence of that bivector angle is proved in [2], pp. 89-91.) We'll then work a sample problem in which we'll calculate the results of successive rotations of a vector.

3.1 Identifying the "Representation" of a Composite Rotation

Let's begin by defining two unit bivectors, $\mathbf{M}_1$ and $\mathbf{M}_2$:

$$\mathbf{M}_1 = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{1ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{1bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{1ac};$$
$$\mathbf{M}_2 = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{2ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{2bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{2ac}.$$

Now, write the rotation of a vector $\mathbf{v}$ by the bivector angle $\mathbf{M}_1\mu_1$ to produce the vector $\mathbf{v}'$:

$$\mathbf{v}' = \left[e^{-\mathbf{M}_1\mu_1/2}\right][\mathbf{v}]\left[e^{\mathbf{M}_1\mu_1/2}\right].$$

Next, we will rotate $\mathbf{v}'$ by the bivector angle $\mathbf{M}_2\mu_2$ to produce the vector $\mathbf{v}''$:

$$\mathbf{v}'' = \left[e^{-\mathbf{M}_2\mu_2/2}\right][\mathbf{v}']\left[e^{\mathbf{M}_2\mu_2/2}\right].$$

Combining those two equations,

$$\mathbf{v}'' = \left[e^{-\mathbf{M}_2\mu_2/2}\right]\left\{\left[e^{-\mathbf{M}_1\mu_1/2}\right][\mathbf{v}]\left[e^{\mathbf{M}_1\mu_1/2}\right]\right\}\left[e^{\mathbf{M}_2\mu_2/2}\right].$$

The vector $\mathbf{v}''$ was produced from $\mathbf{v}$ via the composition of the rotations through the bivector angles $\mathbf{M}_1\mu_1$ and $\mathbf{M}_2\mu_1$. The representation of that composition is the product $\left[e^{-\mathbf{M}_2\mu_1/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right]$. We'll rewrite the previous equation to make that idea clearer:

$$\mathbf{v}'' = \underbrace{\left\{\left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right]\right\}}_{\substack{\textit{Representation}\\ \textit{of the composition}}}[\mathbf{v}]\left\{\left[e^{\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]\right\}.$$

There exists an identifiable bivector angle —we'll call it $\mathbf{S}\sigma$—through which $\mathbf{v}$ could have been rotated to produce $\mathbf{v}''$ in a single operation rather than through the composition of rotations through $\mathbf{M}_1\mu_1$ and $\mathbf{M}_2\mu_2$. (See Section 3.2.) But instead of going that route, we'll write $e^{-\mathbf{M}_1\mu_1/2}$ and $e^{-\mathbf{M}_2\mu_2/2}$ in a way that will enable us to use Eq. (2.5):

$$\begin{aligned}
e^{-\mathbf{M}_1\mu_1/2} &= g_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}g_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}g_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}g_{ac}\right), \text{and}\\
e^{-\mathbf{M}_2\mu_2/2} &= h_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}h_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}h_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}h_{ac}\right),
\end{aligned}$$

where $g_o = \cos\frac{\mu_1}{2}$; $g_{ab} = m_{1ab}\sin\frac{\mu_1}{2}$; $g_{bc} = m_{1bc}\sin\frac{\mu_1}{2}$; and $g_{ac} = m_{1ac}\sin\frac{\mu_1}{2}$, and $h_o = \cos\frac{\mu_2}{2}$; $h_{ab} = m_{2ab}\sin\frac{\mu_2}{2}$; $h_{bc} = m_{2bc}\sin\frac{\mu_2}{2}$; and $h_{ac} = m_{2ac}\sin\frac{\mu_2}{2}$. Now, we write the representation of the the composition as

$$\underbrace{\left[h_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}h_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}h_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}h_{ac}\right)\right]}_{e^{-\mathbf{M}_2\mu_2/2}}\underbrace{\left[g_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}g_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}g_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}g_{ac}\right)\right]}_{e^{-\mathbf{M}_1\mu_1/2}}.$$

After expanding that product and grouping like terms, the representation of the composite rotation can be written in a form identical to Eq. (2.5):

$$\mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right), \tag{3.1}$$

with

$$\begin{aligned}
\mathcal{F}_o &= \langle e^{-\mathbf{M}_2\mu_2/2}e^{-\mathbf{M}_1\mu_1/2}\rangle_0\\
&= h_og_o - h_{ab}g_{ab} - h_{bc}g_{bc} - h_{ac}g_{ac}\ ,\\
\mathcal{F}_{ab} &= h_og_{ab} + h_{ab}g_o - h_{bc}g_{ac} + h_{ac}g_{bc}\ ,\\
\mathcal{F}_{bc} &= h_og_{bc} + h_{ab}g_{ac} + h_{bc}g_o - h_{ac}g_{ab}\ , \text{ and}\\
\mathcal{F}_{ac} &= h_og_{ac} - h_{ab}g_{bc} + h_{bc}g_{ab} + h_{ac}g_o\ .
\end{aligned} \tag{3.2}$$

Therefore, with these definitions of $\mathcal{F}_o$, $\mathcal{F}_{ab}$, $\mathcal{F}_{bc}$, and $\mathcal{F}_{ac}$, $\mathbf{v}''$ can be calculated from $\mathbf{v}$ (written as $\hat{\mathbf{a}}v_a + \hat{\mathbf{b}}v_b + \hat{\mathbf{c}}v_c$) via an equation that is analogous, term for term, with Eq. (2.4):

$$\begin{aligned}
\mathbf{v}'' = \hat{\mathbf{a}}&\left[v_a\left(\mathcal{F}_o^2 - \mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 - \mathcal{F}_{ac}^2\right) + v_b\left(-2\mathcal{F}_o\mathcal{F}_{ab} - 2\mathcal{F}_{bc}\mathcal{F}_{ac}\right) + v_c\left(-2\mathcal{F}_o\mathcal{F}_{ac} + 2\mathcal{F}_{ab}\mathcal{F}_{bc}\right)\right]\\
+ \hat{\mathbf{b}}&\left[v_a\left(2\mathcal{F}_o\mathcal{F}_{ab} - 2\mathcal{F}_{bc}\mathcal{F}_{ac}\right) + v_b\left(\mathcal{F}_o^2 - \mathcal{F}_{ab}^2 - \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2\right) + v_c\left(-2\mathcal{F}_o\mathcal{F}_{bc} - 2\mathcal{F}_{ab}\mathcal{F}_{ac}\right)\right]\\
+ \hat{\mathbf{c}}&\left[v_a\left(2\mathcal{F}_o\mathcal{F}_{ac} + 2\mathcal{F}_{ab}\mathcal{F}_{bc}\right) + v_b\left(2\mathcal{F}_o\mathcal{F}_{bc} - 2\mathcal{F}_{ab}\mathcal{F}_{ac}\right) + v_c\left(\mathcal{F}_o^2 + \mathcal{F}_{ab}^2 - \mathcal{F}_{bc}^2 - \mathcal{F}_{ac}^2\right)\right].
\end{aligned} \tag{3.3}$$

At this point, you may (and should) be objecting that I've gotten ahead of myself. Please recall that Eq. (2.4) was derived starting from the "rotation" equation (Eq. (1.1))

$$\mathbf{w}' = \left[e^{-\mathbf{Q}\theta/2}\right][\mathbf{w}]\left[e^{\mathbf{Q}\theta/2}\right].$$

The quantities f_o, f_o, f_{ab}, f_{bc}, and f_{ac} in Eq. (2.4), for which

$$e^{-\mathbf{Q}\theta/2} = f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right), \tag{3.4}$$

also meet the condition that

$$e^{\mathbf{Q}\theta/2} = f_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right). \tag{3.5}$$

We are not justified in using $\mathcal{F}_o$, $\mathcal{F}_{ab}$, $\mathcal{F}_{bc}$, and $\mathcal{F}_{ac}$ in Eq. (2.4) unless we first prove that these composite-rotation "$\mathcal{F}$'s", for which

$$\mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right) = e^{-\mathbf{M}_2\mu_2/2}e^{-\mathbf{M}_1\mu_1/2}, \tag{3.6}$$

also meet the condition that

$$\mathcal{F}_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right) = e^{\mathbf{M}_1\mu_1/2}e^{\mathbf{M}_2\mu_2/2}. \tag{3.7}$$

Although more-elegant proofs may well exist, "brute force and ignorance" gets the job done. We begin by writing $e^{\mathbf{M}_1\mu_1/2}e^{\mathbf{M}_2\mu_2/2}$ in a way that is analogous to that which was presented in the text that preceded Eq. (3.1):

$$\underbrace{\left[g_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}g_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}g_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}g_{ac}\right)\right]}_{e^{\mathbf{M}_1\mu_1/2}}\underbrace{\left[h_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}h_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}h_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}h_{ac}\right)\right]}_{e^{\mathbf{M}_2\mu_2/2}}.$$

Expanding, simplifying, and regrouping, we fine that $e^{\mathbf{M}_1\mu_1/2}e^{\mathbf{M}_2\mu_2/2}$ is indeed equal to $\mathcal{F}_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right)$, as required.

3.2 Identifying the Bivector Angle $\mathbf{S}\sigma$ through which the Vector $\mathbf{v}$ Can be Rotated to Produce $\mathbf{v}''$ in a Single Operation

Let $\mathbf{v}$ be an arbitrary vector. We want to identify the bivector angle $\mathbf{S}\sigma$ through which the initial vector, $\mathbf{v}$, can be rotated to produce the same vector $\mathbf{v}''$ that results from the rotation of $\mathbf{v}$ through the composite rotation by $\mathbf{M}_1\mu_1$, then by $\mathbf{M}_2\mu_2$:

$$\left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right][\mathbf{v}]\left[e^{\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right] = \mathbf{v}'' = \left[e^{-\mathbf{S}\sigma/2}\right][\mathbf{v}]\left[e^{\mathbf{S}\sigma/2}\right]. \tag{3.8}$$

We want Eq. (3.8) to be true for all vectors $\mathbf{v}$. Therefore, $e^{\mathbf{S}\sigma/2}$ must be equal to $\left[e^{\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]$, and $e^{-\mathbf{S}\sigma/2}$ must be equal to $\left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right]$. The second of those conditions is the same as saying that the representations of the $\mathbf{S}\sigma$ rotation and the composite rotation must be equal. We'll write that condition using the $\mathcal{F}_o$'s defined in Eq. (3.2), with $\mathbf{S}$ expressed in terms of the unit bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$:

$$\cos\frac{\sigma}{2} - \underbrace{\left(\hat{\mathbf{a}}\hat{\mathbf{b}}S_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}S_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}S_{ac}\right)}_{\mathbf{S}}\sin\frac{\sigma}{2} = \mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right).$$

Now, we want to identify σ and the coefficients of $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$. First, we note that both sides of the previous equation are multivectors. According to the postulates of GA, two multivectors $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal if and only if for every grade k, $\langle \mathcal{A}_1 \rangle_k = \langle \mathcal{A}_2 \rangle_k$. Equating the scalar parts, we see that $\cos\frac{\sigma}{2} = \mathcal{F}_o$. Equating the bivector parts gives $\left(\hat{\mathbf{a}}\hat{\mathbf{b}}S_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}S_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}S_{ac}\right)\sin\frac{\sigma}{2} = \hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}$. Comparing like terms, $S_{ab} = \mathcal{F}_{ab}/\sin\frac{\sigma}{2}$, $S_{bc} = \mathcal{F}_{bc}/\sin\frac{\sigma}{2}$, and $S_{ac} = \mathcal{F}_{ac}/\sin\frac{\sigma}{2}$.

Why is it correct to identify the $\mathcal{S}$'s by comparing like terms? In simple terms, because the unit bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$ $\hat{\mathbf{a}}\hat{\mathbf{b}}$ are orthogonal. Two linear combinations of those bivectors are equal if and only if the coefficients match, term for term.

Next, we need to find $\sin\frac{\sigma}{2}$. Although we could do so via $\sin\frac{\sigma}{2} = \sqrt{1-\cos^2\frac{\sigma}{2}}$, for the purposes of this discussion we will use the fact that $\mathbf{S}$ is, by definition, a unit bivector. Therefore, $||\mathbf{S}|| = 1$, leading to

$$\begin{aligned}\|\sin\frac{\sigma}{2}\| &= \|\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\| \\ &= \sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}\,.\end{aligned}$$

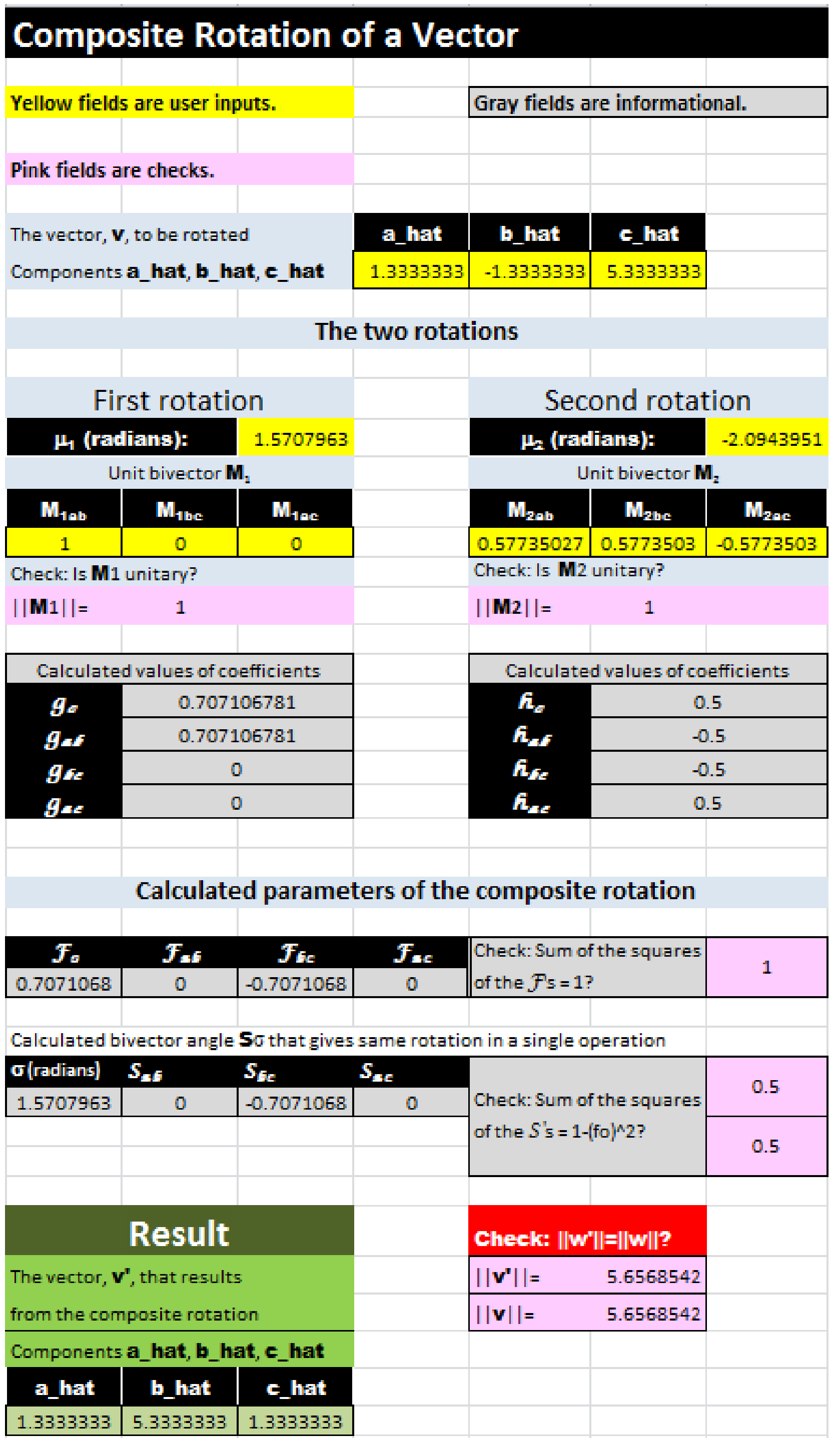
Now, the question is whether we want to use $\sin\frac{\sigma}{2} = +\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$, or $\sin\frac{\sigma}{2} = -\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$. The truth is that we can use either: if we use $-\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$ instead of $+\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$, then the sign of $\mathbf{S}$ changes as well, leaving the product $\mathbf{S}\sin\frac{\sigma}{2}$ unaltered.

The choice having been made, we can find the scalar angle σ from the values of $\sin\frac{\sigma}{2}$ and $\cos\frac{\sigma}{2}$, thereby determining the bivector angle $\mathbf{S}\sigma$.

3.3 A Sample Calculation

In Section 2.1.1, we solved a problem in which the vector $\mathbf{v} = \frac{4}{3}\hat{\mathbf{a}} - \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$ was rotated through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$ radians to produce a new vector, $\mathbf{v}'$, which was then rotated through the bivector angle $\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$ to produce vector $\mathbf{v}''$. Here, we'll calculate $\mathbf{v}''$ directly from $\mathbf{v}$ using Eqs. (3.1), (3.2) and 3.3, We'll also calculate the bivector angle $\mathbf{S}\sigma$ through which $\mathbf{v}$ could have been rotated to produce $\mathbf{v}''$ in a single operation.

As we can see by comparing from Figs. 14 and 5, the result $\left(\mathbf{v}'' = \frac{4}{3}\hat{\mathbf{a}} + \frac{16}{3}\hat{\mathbf{b}} + \frac{4}{3}\hat{\mathbf{c}}\right)$ obtained via the composite-rotation formula agrees with that which was obtained by calculating $\mathbf{v}''$ in two steps. Fig. 14 also shows that the bivector angle $\mathbf{S}\sigma$ is $\hat{\mathbf{b}}\hat{\mathbf{c}}\,(-\pi/2)$, which we can also write as $\hat{\mathbf{c}}\hat{\mathbf{b}}\,(\pi/2)$. That rotation is diagrammed in Fig. 15.



Composite Rotation of a Vector

Yellow fields are user inputs.		Gray fields are informational.
Pink fields are checks.		

	a_hat	b_hat	c_hat
The vector, **v**, to be rotated Components **a_hat**, **b_hat**, **c_hat**	1.3333333	-1.3333333	5.3333333

The two rotations

First rotation

μ_1 (radians):	1.5707963	
Unit bivector $\mathbf{M}_1$		
$\mathbf{M}_{1ab}$	$\mathbf{M}_{1bc}$	$\mathbf{M}_{1ac}$
1	0	0
Check: Is **M**1 unitary?		
||**M**1||=	1	

Calculated values of coefficients	
g_o	0.707106781
g_{ab}	0.707106781
g_{bc}	0
g_{ac}	0

Second rotation

μ_2 (radians):	-2.0943951	
Unit bivector $\mathbf{M}_2$		
$\mathbf{M}_{2ab}$	$\mathbf{M}_{2bc}$	$\mathbf{M}_{2ac}$
0.57735027	0.5773503	-0.5773503
Check: Is **M**2 unitary?		
||**M**2||=	1	

Calculated values of coefficients	
h_o	0.5
h_{ab}	-0.5
h_{bc}	-0.5
h_{ac}	0.5

Calculated parameters of the composite rotation

f_o	f_{ab}	f_{bc}	f_{ac}	Check: Sum of the squares of the f's = 1?	
0.7071068	0	-0.7071068	0		1

Calculated bivector angle **S**σ that gives same rotation in a single operation

σ (radians)	S_{ab}	S_{bc}	S_{ac}	Check: Sum of the squares of the S's = 1-(fo)^2?	
1.5707963	0	-0.7071068	0		0.5
					0.5

Result

The vector, **v'**, that results from the composite rotation

Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.3333333	5.3333333	1.3333333

Check: ||w'||=||w||?	
||**v'**||=	5.6568542
||**v**||=	5.6568542

Figure 14: Screen shot of the Excel spreadsheet (Reference [5]) that uses Eq. 3.3 to calculate $\mathbf{v}''$ via the composite rotation of $\mathbf{v}$.

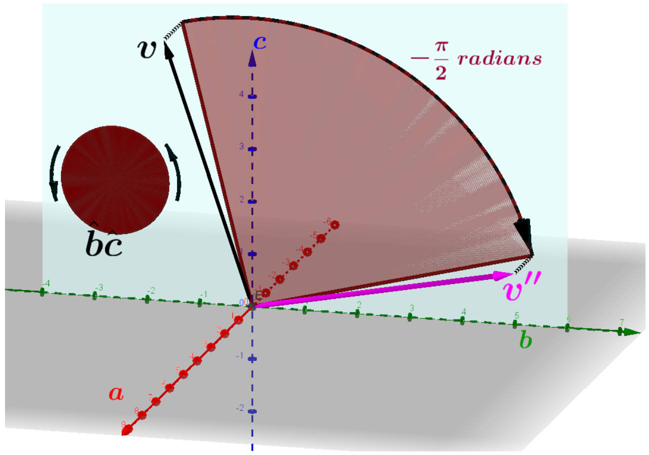


Figure 15: Rotation of $\mathbf{v}$ by $\mathbf{S}\sigma$ to produce $\mathbf{v}''$ in a single operation. Note the significance of the negative sign of the scalar angle: the direction in which $\mathbf{v}'$ rotated is contrary to the orientation of the bivector $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and contrary also to the direction of the rotation from $\hat{\mathbf{b}}$ to $\hat{\mathbf{c}}$.

4 Rotation of a Bivector

4.1 Derivation of a Formula for Rotation of a Bivector

In his Theorem 7.5, Macdonald ([2], p. 125) states that if a blade $\mathbf{M}$ is rotated by the bivector angle $\mathbf{Q}\theta$, the result will be the blade

$$\mathsf{R}_{\mathbf{Q}\theta}\left(\mathbf{M}\right) = \left[e^{-\mathbf{Q}\theta/2}\right]\left[\mathbf{M}\right]\left[e^{\mathbf{Q}\theta/2}\right]. \tag{4.1}$$

To express the result as a linear combination of the unit bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$, we begin by writing the unit bivector $\mathbf{Q}$ as $\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}}q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}q_{ac}$, so that we may write the representation of the rotation in exactly the same way as we did for the rotation of a vector:

$$e^{-\mathbf{Q}\theta/2} = f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right),$$

with $f_o = \cos\frac{\theta}{2}$; $f_{ab} = q_{ab}\sin\frac{\theta}{2}$; $f_{bc} = q_{bc}\sin\frac{\theta}{2}$; $f_{ac} = q_{ac}\sin\frac{\theta}{2}$.

Next, we write $\mathbf{M}$ as $\mathbf{M} = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{ac}$. Making these substitutions in Eq. (4.1), then expanding and simplifying, we obtain

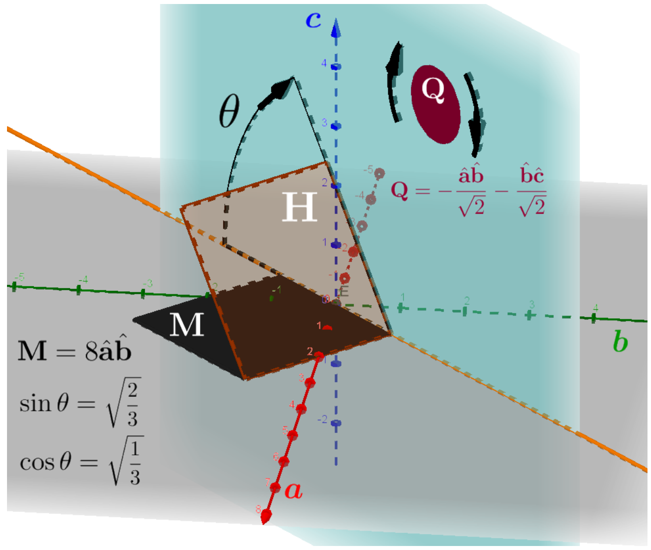


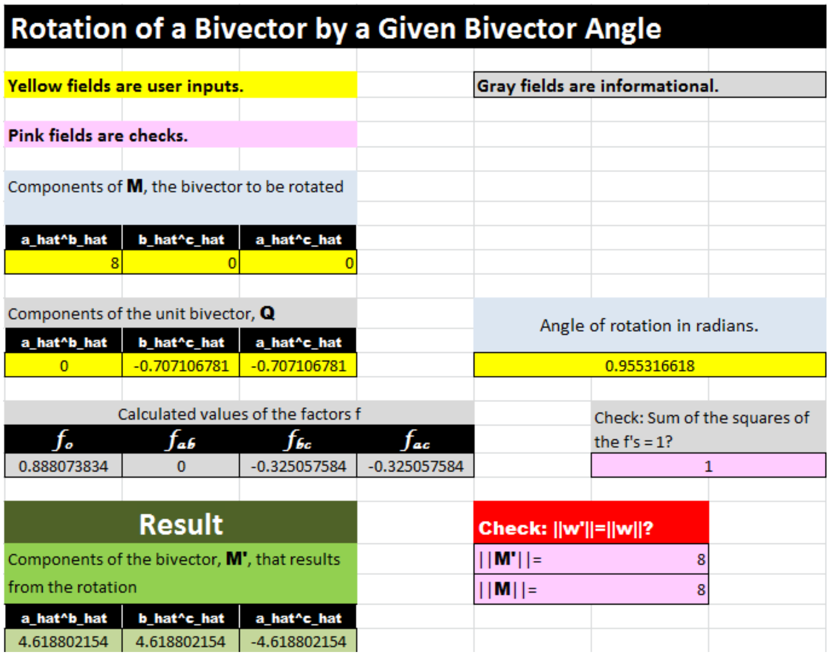
Figure 16: Rotation of the bivector $8\mathbf{ab}$ by the bivector angle $\mathbf{Q}\theta$ to give the new bivector, $\mathbf{H}$.

$$\begin{aligned}
\mathsf{R}_{\mathbf{Q}\theta}\left(\mathbf{M}\right) = \hat{\mathbf{a}}\hat{\mathbf{b}}\,&\left\{m_{ab}\left(1-2f_{bc}^2-2f_{ac}^2\right)\right.\\
&\left.+2\left[f_{ab}\left(f_{bc}m_{bc}+f_{ac}m_{ac}\right)+f_o\left(f_{ac}m_{bc}-f_{bc}m_{ac}\right)\right]\right\}\\
+\,\hat{\mathbf{b}}\hat{\mathbf{c}}\,&\left\{m_{bc}\left(1-2f_{ab}^2-2f_{ac}^2\right)\right.\\
&\left.+2\left[f_{bc}\left(f_{ab}m_{ab}+f_{ac}m_{ac}\right)+f_o\left(f_{ab}m_{ac}-f_{ac}m_{ab}\right)\right]\right\}\\
+\,\hat{\mathbf{a}}\hat{\mathbf{c}}\,&\left\{m_{ac}\left(1-2f_{ab}^2-2f_{bc}^2\right)\right.\\
&\left.+2\left[f_{ac}\left(f_{ab}m_{ab}+f_{bc}m_{bc}\right)+f_o\left(f_{bc}m_{ab}-f_{ab}m_{bc}\right)\right]\right\}.
\end{aligned} \tag{4.2}$$

4.2 A Sample Calculation

In Fig. 16, $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{1}{3}}$. The bivector $\mathbf{M} = 8\hat{\mathbf{a}}\hat{\mathbf{b}}$ is rotated by the bivector angle $\mathbf{Q}\theta$, with $\mathbf{Q} = -\frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{2}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{2}}$, to give the new bivector, $\mathbf{H}$. Calculate $\mathbf{H}$.

As shown in Fig. 17, $\mathbf{H} = \frac{8}{\sqrt{3}}\hat{\mathbf{a}}\hat{\mathbf{b}} + \frac{8}{\sqrt{3}}\hat{\mathbf{b}}\hat{\mathbf{c}} - \frac{8}{\sqrt{3}}\hat{\mathbf{a}}\hat{\mathbf{c}}$.



Rotation of a Bivector by a Given Bivector Angle

Yellow fields are user inputs.

Gray fields are informational.

Pink fields are checks.

Components of **M**, the bivector to be rotated

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
8	0	0

Components of the unit bivector, **Q**

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
0	-0.707106781	-0.707106781

Angle of rotation in radians.

0.955316618

Calculated values of the factors f

f_o	f_{ab}	f_{bc}	f_{ac}
0.888073834	0	-0.325057584	-0.325057584

Check: Sum of the squares of the f's = 1?

1

Result

Components of the bivector, **M'**, that results from the rotation

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
4.618802154	4.618802154	-4.618802154

Check: ||w'||=||w||?

||**M'**||=	8
||**M**||=	8

Figure 17: Screen shot of the Excel spreadsheet (Reference [6]) used to calculate the rotation of the bivector $8\hat{\mathbf{a}}\hat{\mathbf{b}}$. (See text.)

5 Rotation of a Rotation

References [1] and [2] discuss, in detail, how to rotate vectors and planes via Geometric Algebra (GA). Here, we solve a problem from Ref. [2] , p. 127, which reads (paraphrasing),

> (Rotating rotations). Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be unit bivectors. Consider a rotation by the bivector angle $\mathbf{M}_1\mu_1$. Now, "rotate the rotation": that is, rotate $\mathbf{M}_1\mu_1$ by the bivector angle $\mathbf{M}_2\mu_2$ to obtain the new bivector rotation angle $e^{-\mathbf{M}_2\mu_2/2}\left[\mathbf{M}_1\mu_1\right]e^{\mathbf{M}_2\mu_2/2}$. Show that this rotated rotation is represented by $Z'' = Z'ZZ'^{-1}$, where $Z = e^{-\mathbf{M}_1\mu_1/2}$ represents the original rotation, $Z' = e^{-\mathbf{M}_2\mu_2/2}$ represents its rotation, and $Z'^{-1} = e^{\mathbf{M}_2\mu_2/2}$.
>
> *Hint:* The unit bivector $\mathbf{M}_1$ is the product $\mathbf{ef}$ of orthonormal vectors. Thus
>
> $$e^{-\mathbf{M}_2\mu_2/2}\left[\mathbf{M}_1\right]e^{\mathbf{M}_2\mu_2/2} = \left[e^{-\mathbf{M}_2\mu_2/2}\left(\mathbf{e}\right)e^{\mathbf{M}_2\mu_2/2}\right]\left[e^{-\mathbf{M}_2\mu_2/2}\left(\mathbf{f}\right)e^{\mathbf{M}_2\mu_2/2}\right]$$
>
> is also a unit bivector.

Proof

Let's use the symbol $\mathbf{M}_3$ to represent the rotated bivector $e^{-\mathbf{M}_2\mu_2/2}\left[\mathbf{M}_1\right]e^{\mathbf{M}_2\mu_2/2}$. Then, the new bivector rotation angle $e^{-\mathbf{M}_2\mu_2/2}\left[\mathbf{M}_1\mu_1\right]e^{\mathbf{M}_2\mu_2/2}$ is $\mathbf{M}_3\mu_1$. Because $\mathbf{M}_3$ is a unit bivector (per the "hint"), the representation of the rotated rotation is

$$\begin{aligned} Z'' &= e^{-\mathbf{M}_3\mu_1/2} \\ &= \cos\frac{\mu_1}{2} - \mathbf{M}_3 \sin\frac{\mu_1}{2}. \end{aligned}$$

Now, we'll note that $\cos\frac{\mu_1}{2} = \left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[\cos\frac{\mu_1}{2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]$. Therefore,

$$\begin{aligned} Z'' &= \left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[\cos\frac{\mu_1}{2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right] - \underbrace{\left\{e^{-\mathbf{M}_2\mu_2/2}\left[\mathbf{M}_1\right]e^{\mathbf{M}_2\mu_2/2}\right\}}_{\mathbf{M}_3}\sin\frac{\mu_1}{2} \\ &= \left[e^{-\mathbf{M}_2\mu_2/2}\right]\underbrace{\left[\cos\frac{\mu_1}{2} - \mathbf{M}_1\sin\frac{\mu_1}{2}\right]}_{=e^{-\mathbf{M}_1\mu_1/2}}\left[e^{\mathbf{M}_2\mu_2/2}\right] \qquad (5.1) \\ &= Z'ZZ'^{-1}. \ \square \end{aligned}$$

5.1 Formulas for Components of the Representation of a Rotation of a Rotation

In this section, the variables $\mathbf{M}_1$, μ_1, $\mathbf{M}_2$, and μ_2 have the same significance as in the previous. Macdonald's Theorem 7.6 ([2], p. 126) states that for any multivectors $\mathcal{N}$ and $\mathcal{P}$,

$$\mathsf{R}_{\mathbf{i}\theta}\left(\mathcal{N}+\mathcal{P}\right) = \mathsf{R}_{\mathbf{i}\theta}\left(\mathcal{N}\right) + \mathsf{R}_{\mathbf{i}\theta}\left(\mathcal{P}\right).$$

The representation of the rotation by $\mathbf{M}_1\mu_1$ is the bivector $\cos\frac{\mu_1}{2} - \mathbf{M}_1\sin\frac{\mu_1}{2}$. Therefore, Eq. (5.1) becomes

$$\begin{aligned} Z'' &= \left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[\cos\frac{\mu_1}{2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right] - \left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[\mathbf{M}_1\sin\frac{\mu_1}{2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right] \\ &= \cos\frac{\mu_1}{2} - \left\{\left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[\mathbf{M}_1\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]\right\}\sin\frac{\mu_1}{2}. \end{aligned} \qquad (5.2)$$

From Section 4, we recognize the second term on the last line of that result as the product of $\sin\frac{\mu_1}{2}$ and the rotation of $\mathbf{M}_1$ by the bivector angle $\mathbf{M}_2\mu_2$. Thus, so that we we may use (4.2), we write $\mathbf{M}_2$ as $\mathbf{M}_2 = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{2ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{2bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{2ac}$. Having done so, we may write $e^{-\mathbf{M}_2\mu_2/2}$ as

$$e^{-\mathbf{M}_2\mu_2/2} = f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right),$$

with $f_o = \cos\frac{\mu_2}{2}$; $f_{ab} = m_{2ab}\sin\frac{\mu_2}{2}$; $f_{bc} = m_{2bc}\sin\frac{\mu_2}{2}$; $f_{ac} = m_{2ac}\sin\frac{\mu_2}{2}$.

Next, we write $\mathbf{M}_1$ as $\mathbf{M}_1 = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{1ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m1_{1bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{1ac}$. Making these substitutions in Eq. (5.2), then expanding and simplifying, we obtain

$$
\begin{aligned}
Z'' = \cos\frac{\mu_1}{2} & \\
&+ \hat{\mathbf{a}}\hat{\mathbf{b}}\left\{m_{ab}\left(1-2f_{bc}^2-2f_{ac}^2\right)\right. \\
&\qquad \left.+2\left[f_{ab}\left(f_{bc}m_{bc}+f_{ac}m_{ac}\right)+f_o\left(f_{ac}m_{bc}-f_{bc}m_{ac}\right)\right]\right\}\sin\frac{\mu_1}{2} \\
&+ \hat{\mathbf{b}}\hat{\mathbf{c}}\left\{m_{bc}\left(1-2f_{ab}^2-2f_{ac}^2\right)\right. \\
&\qquad \left.+2\left[f_{bc}\left(f_{ab}m_{ab}+f_{ac}m_{ac}\right)+f_o\left(f_{ab}m_{ac}-f_{ac}m_{ab}\right)\right]\right\}\sin\frac{\mu_1}{2} \\
&+ \hat{\mathbf{a}}\hat{\mathbf{c}}\left\{m_{ac}\left(1-2f_{ab}^2-2f_{bc}^2\right)\right. \\
&\qquad \left.+2\left[f_{ac}\left(f_{ab}m_{ab}+f_{bc}m_{bc}\right)+f_o\left(f_{bc}m_{ab}-f_{ab}m_{bc}\right)\right]\right\}\sin\frac{\mu_1}{2}.
\end{aligned}
\tag{5.3}
$$

5.2 A Sample Calculation

From the result of the sample problem in Section 4.2, we can deduce that the rotation of $\hat{\mathbf{a}}\hat{\mathbf{b}}$ through the bivector angle $\mathbf{M}_2\mu_2$, with $\mu_2 = \arcsin\left(\sqrt{\frac{2}{3}}\right) = 0.95532$ radians and $\mathbf{M}_2 = -\frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{2}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{2}}$, will yield the bivector $\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}$. In addition, we know from Example 2 in Section 2.1.1 that rotating the vector $\mathbf{v} = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$ by the bivector angle

$$\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$$

produces the vector $\frac{4}{3}\hat{\mathbf{a}} + \frac{16}{3}\hat{\mathbf{b}} + \frac{4}{3}\hat{\mathbf{c}}$. Therefore, if we rotate $\mathbf{v}$ by the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\left(-\frac{2\pi}{3}\right)$, then "rotate that rotation" by the $\mathbf{M}_2\mu_2$ described at the beginning of this paragraph, the result should be $\frac{4}{3}\hat{\mathbf{a}} + \frac{16}{3}\hat{\mathbf{b}} + \frac{4}{3}\hat{\mathbf{c}}$. Fig. 18 confirms that result.

6 Summary

This document has shown (1) how to effect simple rotations of vectors and bivectors via GA, and (2) how to calculate the results of composite and "rotated" rotations expeditiously by using the concept of a rotation's 'representation". The formulas that we derived for those rotations are presented in the Appendix.

"Rotation of a Rotation" of a Vector

Yellow fields are user inputs.	Pink fields are checks.	Gray fields are informational.

Components of **v**, the vector to be rotated

a_hat	b_hat	c_hat
1.333333333	1.333333333	5.333333333

Parameters of the initial rotation (i.e., that by the bivector angle $M_1\mu_1$

Components of **M**, the bivector to be rotated

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
1	0	0

Angle μ_1 in radians.
-2.094395102

Parameters of the rotation of $M_1\mu_1$ by $M_2\mu_2$

Components of the unit bivector $\mathbf{M_2}$

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
0	-0.707106781	-0.707106781

Angle μ_2 in radians.
0.955316618

Components of the rotation by $M_2\mu_2$

Calculated values of the factors f

f_o	f_{ab}	f_{bc}	f_{ac}
0.888073834	0	-0.325057584	-0.325057584

Check: Sum of the squares of the factors f = 1?
1

Components of the Representation of the "Rotation of the Rotation"

Calculated values of the factors $\mathcal{F}$

$\mathcal{F}_o$	$\mathcal{F}_{ab}$	$\mathcal{F}_{bc}$	$\mathcal{F}_{ac}$
0.5	-0.5	-0.5	0.5

Check: Sum of the squares of the factors $\mathcal{F}$ = 1?
1

Results

Components of the bivector, **M'**, that results from the rotation of M_1 by $M_2\mu_2$

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
0.577350269	0.577350269	-0.577350269

Components of the vector, **v'**, that results from the rotation of the rotation

a_hat	b_hat	c_hat
1.333333333	5.333333333	1.333333333

Check: ||w'||=||w||?	
||**M'**||=	1
||**M**||=	1

Check: ||w'||=||w||?	
||**v'**||=	5.656854249
||**v**||=	5.656854249

Figure 18: Screen shot of the Excel spreadsheet (Reference [7]) used to calculate the result of the "rotation of a rotation" of a vector. Please see text for explanation.

References

[1] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[2] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[3] J. A. Smith, 2017a, "Rotation of a Vector by a Given Bivector Angle" (an Excel spreadsheet), https://drive.google.com/file/d/0B2C4TqxB32RRX2JfcDd5NjZiZ00/view?usp=sharing .

[4] J. A. Smith, 2017b, "Rotation of a Vector about an Axis" (an Excel spreadsheet), https://drive.google.com/file/d/0B2C4TqxB32RRNHBHV2tpSUhRTUk/view?usp=sharing.

[5] J. A. Smith, 2017c, "Composite rotation in GA" (an Excel spreadsheet), https://drive.google.com/file/d/0B2C4TqxB32RRaktDZktjcExPeUE/view?usp=sharing .

[6] J. A. Smith, 2017d, "Rotation of a Bivector by a Given Bivector Angle", https://drive.google.com/file/d/1pF_EhK57L4Eojgub0Ckc3TPcojVJ0N4w/view?usp=sharing.

[7] J. A. Smith, 2017, "Rotation of a Rotation" (an Excel spreadsheet), https://drive.google.com/file/d/1w5YAf8XjHf3NHTt0hhAVqdG2KKUB1xg0/view?usp=sharing .

Appendix: List of the Formulas Derived in this Document, and the Spreadsheets that Implement Them

All formulas presented in this Appendix are for three-dimensional Geometric Algebra ($\mathbb{G}^3$), using a right-handed reference system with orthonormal basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, with unit bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$.

Rotation of a given vector w by the bivector angle Qθ
See Section 2.1. Write the vector $\mathbf{w}$ as

$$\mathbf{w} = \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c,$$

and the unit bivector $\mathbf{Q}$ as

$$\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}}q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}q_{ac}.$$

Now, define

- $f_o = \cos\dfrac{\theta}{2}$;
- $f_{ab} = q_{ab}\sin\dfrac{\theta}{2}$;

- $f_{bc} = q_{bc} \sin \dfrac{\theta}{2}$; and
- $f_{ac} = q_{ac} \sin \dfrac{\theta}{2}$.

Then, the result $\mathbf{w}'$ of the rotation of $\mathbf{w}$ is

$$\begin{aligned}
\mathbf{w}' &= \left[e^{-\mathbf{Q}\theta/2}\right][\mathbf{w}]\left[e^{\mathbf{Q}\theta/2}\right] \\
&= \hat{\mathbf{a}}\left[w_a\left(f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2\right) + w_b\left(-2f_o f_{ab} - 2f_{bc}f_{ac}\right) + w_c\left(-2f_o f_{ac} + 2f_{ab}f_{bc}\right)\right] \\
&\quad + \hat{\mathbf{b}}\left[w_a\left(2f_o f_{ab} - 2f_{bc}f_{ac}\right) + w_b\left(f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2\right) + w_c\left(-2f_o f_{bc} - 2f_{ab}f_{ac}\right)\right] \\
&\quad + \hat{\mathbf{c}}\left[w_a\left(2f_o f_{ac} + 2f_{ab}f_{bc}\right) + w_b\left(2f_o f_{bc} - 2f_{ab}f_{ac}\right) + w_c\left(f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2\right)\right] .
\end{aligned}$$

The spreadsheet that implements this formula is Reference [3].

Rotation of a given vector $\mathbf{w}$ by θ radians about the axis given by the unit vector $\hat{\mathbf{e}}$

See Section 2.2. Write the vector $\mathbf{w}$ as $\mathbf{w} = \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c$, and the unit vector $\hat{\mathbf{e}}$ as $= \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$. Define

- $f_o = \cos \dfrac{\theta}{2}$;
- $f_{ab} = e_c \sin \dfrac{\theta}{2}$;
- $f_{bc} = e_a \sin \dfrac{\theta}{2}$; and
- $f_{ac} = -e_b \sin \dfrac{\theta}{2}$.

Then, the vector $\mathbf{w}'$ that results from the rotation is

$$\begin{aligned}
\mathbf{w}' &= \hat{\mathbf{a}}\left[w_a\left(f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2\right) + w_b\left(-2f_o f_{ab} - 2f_{bc}f_{ac}\right) + w_c\left(-2f_o f_{ac} + 2f_{ab}f_{bc}\right)\right] \\
&\quad + \hat{\mathbf{b}}\left[w_a\left(2f_o f_{ab} - 2f_{bc}f_{ac}\right) + w_b\left(f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2\right) + w_c\left(-2f_o f_{bc} - 2f_{ab}f_{ac}\right)\right] \\
&\quad + \hat{\mathbf{c}}\left[w_a\left(2f_o f_{ac} + 2f_{ab}f_{bc}\right) + w_b\left(2f_o f_{bc} - 2f_{ab}f_{ac}\right) + w_c\left(f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2\right)\right] .
\end{aligned}$$

The spreadsheet that implements this formula is Reference [4].

Composition of two rotations of a vector or bivector

See Section 3. Write the vector $\mathbf{v}$ as $\hat{\mathbf{a}}v_a + \hat{\mathbf{b}}v_b + \hat{\mathbf{c}}v_c$. Let the bivector angle of the first rotation be $\mathbf{M}_1\mu_1$, and the bivector angle of the second be $\mathbf{M}_2\mu_2$. Write the two bivectors as

$$\begin{aligned}
\mathbf{M}_1 &= \hat{\mathbf{a}}\hat{\mathbf{b}}m_{1ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{1bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{1ac}; \\
\mathbf{M}_2 &= \hat{\mathbf{a}}\hat{\mathbf{b}}m_{2ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{2bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{2ac}.
\end{aligned}$$

The representation of the composite rotation can be written as

$$\mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right), \tag{6.1}$$

with

$$\begin{aligned}
\mathcal{F}_o &= \langle e^{-\mathbf{M}_2\mu_2/2} e^{-\mathbf{M}_1\mu_1/2} \rangle_0 \\
&= h_o g_o - h_{ab} g_{ab} - h_{bc} g_{bc} - h_{ac} g_{ac} \ , \\
\mathcal{F}_{ab} &= h_o g_{ab} + h_{ab} g_o - h_{bc} g_{ac} + h_{ac} g_{bc} \ , \\
\mathcal{F}_{bc} &= h_o g_{bc} + h_{ab} g_{ac} + h_{bc} g_o - h_{ac} g_{ab} \ , \text{ and} \\
\mathcal{F}_{ac} &= h_o g_{ac} - h_{ab} g_{bc} + h_{bc} g_{ab} + h_{ac} g_o \ .
\end{aligned}$$

The vector $\mathbf{v}''$ that results from the composite rotation is

$$\begin{aligned}
\mathbf{v}'' = \hat{\mathbf{a}}&\left[v_a\left(\mathcal{F}_o^2 - \mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 - \mathcal{F}_{ac}^2\right) + v_b\left({}^{-}2\mathcal{F}_o\mathcal{F}_{ab} - 2\mathcal{F}_{bc}\mathcal{F}_{ac}\right) + v_c\left({}^{-}2\mathcal{F}_o\mathcal{F}_{ac} + 2\mathcal{F}_{ab}\mathcal{F}_{bc}\right)\right] \\
+ \hat{\mathbf{b}}&\left[v_a\left(2\mathcal{F}_o\mathcal{F}_{ab} - 2\mathcal{F}_{bc}\mathcal{F}_{ac}\right) + v_b\left(\mathcal{F}_o^2 - \mathcal{F}_{ab}^2 - \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2\right) + v_c\left({}^{-}2\mathcal{F}_o\mathcal{F}_{bc} - 2\mathcal{F}_{ab}\mathcal{F}_{ac}\right)\right], \\
+ \hat{\mathbf{c}}&\left[v_a\left(2\mathcal{F}_o\mathcal{F}_{ac} + 2\mathcal{F}_{ab}\mathcal{F}_{bc}\right) + v_b\left(2\mathcal{F}_o\mathcal{F}_{bc} - 2\mathcal{F}_{ab}\mathcal{F}_{ac}\right) + v_c\left(\mathcal{F}_o^2 + \mathcal{F}_{ab}^2 - \mathcal{F}_{bc}^2 - \mathcal{F}_{ac}^2\right)\right] .
\end{aligned}$$

See Section 3.2 regarding calculation of the bivector angle $\mathbf{S}\sigma$ that would give the same rotation in a single operation.

The spreadsheet that implements this formula is Reference [5].

Note that that same $\mathcal{F}$'s can be used in place of their respective f's to effect a composite rotation of a bivector, via the formula that is given next.

Rotation of a bivector M by the bivector angle Qθ

See Section 4. Write $\mathbf{M}$ as $\mathbf{M} = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{ac}$, and $\mathbf{Q}$ as $\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}}q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}q_{ac}$. Then the bivector that results from the rotation

$$\begin{aligned}
\mathsf{R}_{\mathbf{Q}\theta}(\mathbf{M}) = \hat{\mathbf{a}}\hat{\mathbf{b}}&\left\{m_{ab}\left(1 - 2f_{bc}^2 - 2f_{ac}^2\right)\right. \\
&\left. +2\left[f_{ab}\left(f_{bc}m_{bc} + f_{ac}m_{ac}\right) + f_o\left(f_{ac}m_{bc} - f_{bc}m_{ac}\right)\right]\right\} \\
+ \hat{\mathbf{b}}\hat{\mathbf{c}}&\left\{m_{bc}\left(1 - 2f_{ab}^2 - 2f_{ac}^2\right)\right. \\
&\left. +2\left[f_{bc}\left(f_{ab}m_{ab} + f_{ac}m_{ac}\right) + f_o\left(f_{ab}m_{ac} - f_{ac}m_{ab}\right)\right]\right\} \\
+ \hat{\mathbf{a}}\hat{\mathbf{c}}&\left\{m_{ac}\left(1 - 2f_{ab}^2 - 2f_{bc}^2\right)\right. \\
&\left. +2\left[f_{ac}\left(f_{ab}m_{ab} + f_{bc}m_{bc}\right) + f_o\left(f_{bc}m_{ab} - f_{ab}m_{bc}\right)\right]\right\},
\end{aligned}$$

with $f_o = \cos\frac{\theta}{2}$; $f_{ab} = q_{ab}\sin\frac{\theta}{2}$; $f_{bc} = q_{bc}\sin\frac{\theta}{2}$; $f_{ac} = q_{ac}\sin\frac{\theta}{2}$.

The spreadsheet that implements this formula is Reference [6].

Rotation of a Rotation

Because the details of this operation are complex, the reader is referred to Section 5. The spreadsheet that implements the result is Reference [7].