Discovering and Proving that π is Irrational, 2nd Edition

Timothy W. Jones

December 8, 2017

Abstract

Ivan Niven's proof of the irrationality of π is often cited because it is brief and uses only calculus. However it is not well motivated. Using the concept that a quadratic function with the same symmetric properties as sine should when multiplied by sine and integrated obey upper and lower bounds for the integral, a contradiction is generated for rational candidate values of π. This simplifying concept yields a more motivated proof of the irrationality of π and π^2.

Charles Hermite proved that e is transcendental in 1873 using a polynomial that is the sum of derivatives of another polynomial [8]. Ivan Niven in 1947 found a way to use Hermite's technique to prove that π is irrational [12]. Lambert in 1767 had proven this result in a twelve-page article using continued fractions [10]. Niven's half-page proof, using only algebra and calculus, is frequently cited and sometimes reproduced in textbooks [14]. Although his proof is brief and uses ostensibly simple mathematics, it begins by defining functions as in the technique of Hermite without any motivation. In this article a simplifying concept is used that provides a more motivated and straightforward proof than Niven's. Using this concept, we, as it were, discover that π might be irrational and then confirm that it is with a proof.

1 A MOTIVATED APPROACH.

We seek to combine a known falsity with a known truth and then to derive a contradiction from the combination. If π is assumed to be rational, $\pi = p/q$

with p and q natural numbers, then the maximum of $\sin x$ occurs at $p/(2q)$. The quadratic $-qx^2 + px = x(p - qx)$ will have its maximum at the same point, as will the product of the two functions. If we have a blender that allows inferences from this statement we might be able to derive a contradiction.

Such a blender exists in a definite integral. A definite integral allows for evaluations that might contradict upper or lower bounds. We have

$$0 < \int_0^{p/q} x(p - qx) \sin x \, dx \leq \frac{p^2}{4q} \cdot \frac{p}{q} = \frac{p^3}{4q^2}, \tag{1}$$

where the lower bound holds as the integrand is always positive,[1] and the upper bound is formed from the length of the interval of integration multiplied by the maximum value of the integrand [16, Property 8, p. 389].

For a polynomial $f(x)$, repeated integration by parts[2] gives the indefinite integral pattern

$$\int f(x) \sin x \, dx = f(x) \cos x - f'(x) \sin x + f''(x) \cos x - f'''(x) \sin x - \ldots.$$

For the function $f(x) = x(p - qx)$, as $f^{(k)}(x) = 0$ for $k \geq 3$, we have

$$\int_0^{p/q} f(x) \sin x \, dx = \{f(x) \cos x - f'(x) \sin x + f''(x) \cos x\}\Big|_0^{p/q}$$

and the odd term drops out ($\sin p/q = \sin 0 = 0$) leaving an alternating sum of even derivatives of $f(x)$ evaluated at the endpoints:

$$\int_0^{p/q} f(x) \sin x \, dx = f(p/q) + f(0) - f''(p/q) - f''(0). \tag{2}$$

The sum is $4q$. Combining (1) and (2), we have

$$0 < 4q \leq \frac{p^3}{4q^2}. \tag{3}$$

[1] To see that the inequality is strict, consider:

$$\int_0^{p/(4q)} x(p - qx) \sin x \, dx + \int_{p/(4q)}^{3p/(4q)} x(p - qx) \sin x \, dx + \int_{3p/(4q)}^{p/q} x(p - qx) \sin x \, dx.$$

[2] Tabular integration by parts (see [11, p. 532], [5] and Appendix A) is especially suited for integrals of the type given in (1).

2 DISCOVERING π IS IRRATIONAL.

2.1 Candidate π Values.

The inequalities in (3) show π does not equal 1 or 2. For $\pi = 7/2$, this $n = 1$ case of the general polynomial $x^n(p - qx)^n$ does not give a contradiction. We will try the $n = 2$ case and see if it works for this rational. This is possible as the same reasoning about $x(p-qx)$ applies to $x^n(p-qx)^n$; it is symmetric like $\sin x$ on $[0, p/q]$ and $x^n(p - qx)^n \sin x$ when integrated in that interval should have a value consistent with the integral's upper and lower bounds.

2.2 The $n = 2$ Case.

With $f(x) = x^2(p - qx)^2$, repeated integration by parts gives

$$\int_0^{p/q} f(x) \sin x \, dx = f^{(0)}(p/q, 0) - f^{(2)}(p/q, 0) + f^{(4)}(p/q, 0), \quad (4)$$

where $f^{(k)}(p/q, 0) = f^{(k)}(p/q) + f^{(k)}(0)$. Multiplying out $f(x)$, we have

$$f(x) = x^2(p - qx)^2 = q^2x^4 - 2pqx^3 + p^2x^2.$$

Derivatives for this function are easily computed. The values of these derivatives at the endpoints 0 and p/q are given in 1. Using Table 1, with the same logic used

k	$f^{(k)}(0)$	$f^{(k)}(p/q)$
0	0	0
1	0	0
2	$2! \cdot p^2$	$2! \cdot p^2$
3	$-4! \cdot pq$	$4! \cdot pq$
4	$4! \cdot q^2$	$4! \cdot q^2$

Table 1: Derivatives of $x^2(p - qx)^2$.

for the inequalities in (3), we form the inequality

$$0 < -4p^2 + 48q^2 \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^2$$

and letting $p = 7$ and $q = 2$, we get $-4p^2 + 48q^2 = -4$, a contradiction of the lower bound.

2.3 The $n = 3, 4$ Cases.

Similar calculations can be carried out for the $n = 3$ and $n = 4$ cases. The inequalities for each are

$$0 < -144p^2q + 1440q^3 \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^3 \tag{5}$$

and

$$0 < 48p^4 - 8640p^2q^2 + 80640q^4 \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^4, \tag{6}$$

respectively.[3]

For the $n = 3$ case, when p/q equals $3/1$, $13/4$, $16/5$, and $19/6$ the upper or lower bound of (5) is contradicted. We discover that $22/7$ is not π using (6), the $n = 4$ case.

3 PROVING π IS IRRATIONAL.

3.1 The General Case.

Referring to Table 1, it is likely that $f(x) = x^n(p - qx)^n$ will be such that the alternating sum of its even derivatives evaluated at the endpoints 0 and p/q will be divisible by $n!$. If the integral in

$$0 < \int_0^{p/q} x^n(p - qx)^n \sin x \, dx \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^n < p^{2n+1} \tag{7}$$

is divisible by $n!$ then the upper bound on (7) can be used to prove π is irrational. This follows as the integral is increasing with n factorially, but the upper bound has polynomial growth. We know factorial growth exceeds polynomial – see [16, Equation 10, p. 764]; [3, Example 2, p. 86] gives a direct proof of this result.

[3]Leibniz's formula [1, Problem 4, p. 222] gives a means of calculating nth derivatives of a product of two functions. In the case of the product of two polynomials, all derivatives can be calculated by placing the derivative of one polynomial along the top row of a table, the derivatives of the other polynomial along the left column, and forming a Pascal's triangle in the interior table. After forming products of these row and column entries with the binomial coefficients of Pascal's triangle, all derivatives are given by sums along interior diagonals, SW to NE, of the table. See Appendix B for details.

3.2 Proving the General Case.

The lower and upper bounds of (7) follow from the properties of the integrand. Repeated integration by parts establishes that

$$\int_0^{p/q} x^n (p - qx)^n \sin x \, dx = \sum_{k=0}^{n} (-1)^k f^{(2k)}(p/q, 0). \tag{8}$$

Consequently, we need only prove that the right-hand side of (8) is divisible by $n!$.

First, symmetry of $f(x)$ allows us to consider only the left endpoint in this sum. This follows as the equation $f(x) = f(p/q - x)$, differentiated repeatedly, gives $f'(x) = -f'(p/q - x)$, $f''(x) = f''(p/q - x)$, and, by induction, $f^{(k)}(x) = (-1)^k f^{(k)}(p/q - x)$. So $f^{(k)}(0) = (-1)^k f^{(k)}(p/q)$. For the even derivatives, with which we are concerned, we have $f^{(2k)}(0) = f^{(2k)}(p/q)$.

Next, $f(x)$ when expanded will have the form $a_n x^{2n} + \cdots + a_0 x^n$. For $k < n$, $f^{(k)}(0) = 0$, and for $k \geq n$, $f^{(k)}(0)$ is divisible by $k!$ and therefore $n!$. We have established that the sum in (8) is divisible by $n!$ and that π must be irrational.

4 CONCLUSION.

Niven gives two proofs of the irrationality of π. One has been cited in the introduction. The other occurs in his book on irrational numbers [13]; there he shows the irrationality of π^2. We will re-examine these proofs.[4]

Looking at Hermite's transcendence of e proof [7, p. 152], one sees definitions of two functions $f(x)$ and $F(x)$ with the derivatives of $f(x)$ being used in the definition of $F(x)$. An integral is then used with the integrand having e^{-x} in it. In Niven's π and π^2 proofs he defines one function as the sum of the derivatives of the other, as Hermite does. The manipulations Niven performs are to obtain forms like Hermite's. In both articles the integral of one function equals an expression involving the other. To someone un-steeped in Hermite's technique the motivation for the proof must be unclear.

In this note a concept motivates the introduction of the polynomial Niven defines. The concept is that if π is rational then the evaluation of a definite integral comprised of the product of two functions symmetric about $x = \pi/2$ should be consistent with bounds for the integral. This being shown not to be the case, a

[4]See Appendix C for a more detailed analysis of Niven's proofs.

contradiction occurs and π is proven irrational. The graphs of $\sin x$, $x(p-qx)$, and their product give the concept – visually.

The same logic used for π can be applied to π^2. Assume $\pi^2 = a/b$. We have

$$0 < \int_0^{a/b} x^n(a-bx)^n \sin\frac{x}{\sqrt{a/b}}\,dx \le \frac{a}{b}\left(\frac{a^2}{4b}\right)^n, \tag{9}$$

with the same reasoning as before: the integrand by assumption is a symmetric function with its maximum at $x = a/(2b)$. The integral, using repeated integration by parts, evaluates to

$$\sum_{k=0}^{n}(-1)^k(\sqrt{a/b})^{2k+1}(f^{(2k)}(a/b) + f^{(2k)}(0)),$$

where $f(x) = x^n(a-bx)^n$. With some factoring, this sum is

$$\frac{\pi}{b^n}\sum_{k=0}^{n}(-1)^k b^{n-k}a^k(f^{(2k)}(a/b) + f^{(2k)}(0)).$$

With a multiplication by b^n/π to clear π/b^n from this sum, we have

$$0 < \frac{b^n}{\pi}\int_0^{a/b} x^n(a-bx)^n \sin\frac{x}{\sqrt{a/b}}\,dx = n!R_n \le \frac{b^n}{\pi}\frac{a}{b}\left(\frac{a^2}{4b}\right)^n < a^{3n+1},$$

which gives a contradiction.

Note: reproductions of older articles by Hermite [8] and others can be found in [2].

ACKNOWLEDGMENTS. I would like to thank E. F. for helping me to believe that one could spell π without an e. Thanks also go to Richard Foote of the University of Vermont for his patience with me over the years.

References

[1] T. Apostol, *Calculus*, vol. 1, 2nd ed., John Wiley, New York, 1967.

[2] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[3] G. Chrystal, *Algebra: An Elementary Textbook*, vol. 2, 7th ed., American Mathematical Society, Providence, RI, 1964.

[4] P. Eymard and J.-P. Lafon, *The Number* π, American Mathematical Society, Providence, RI, 2004.

[5] K. W. Folley, Integration by parts, *Amer. Math. Monthly*, **54** (1947) 542–543.

[6] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[7] C. Hermite, Sur la fonction exponientielle, *Compt. Rend. Acad. Sci. Paris* **77** (1873) 18–24, 74–79, 226–233, 285–293.

[8] ———, *Oeuvres Complétes*, vol. 3, Hermann, Paris, 1912.

[9] I. N. Herstein, *Topics in Algebra*, 2nd ed., John Wiley, New York, 1975.

[10] J. Lambert, Mémoire sur quelques propriétés remarquables des quantitiés transcendentes circulaires et logarithmiques, *Historie de l'Académie Royale des Sciences et des Belles-Lettres der Berlin* **17** (1761) 265–276.

[11] B. Larson and B. H. Edwards, *Calculus*, 9th ed., Brooks/Cole, Belmont, CA, 2010.

[12] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947) 509.

[13] ———, *Irrational Numbers*, Carus Mathematical Monographs, no. 11, Mathematical Association of America, Washington, DC, 1985.

[14] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.

[15] G. F. Simmons, *Calculus Gems: Brief Lives and Memorable Mathematics*, Mathematical Association of America, Washington, DC, 2007.

[16] J. Stewart, *Calculus: Early Transcendentals*, 5th ed., Thomson Brooks/Cole, Belmont, CA 2003

A Tabular Integration

Tabular integration is based on integration by parts. Integration by parts is, in turn, based on the product rule. Consider

$$(uv)' = u'v + uv'$$

implies that

$$\int (uv)' = \int u'v + \int uv'$$

and so

$$\int uv' = uv - \int u'v. \tag{10}$$

Using placement as an organizing principal, (10) translates into

$$\int [1][2] = [same][up] - \int [down][up], \tag{11}$$

where $[down]$ means a derivative is taken and $[up]$ an integral. We now start with the integral on the right of (11) and apply the pattern again:

$$\int [down][up] = [down][upup] - \int [downdown][upup]$$

to get

$$\int [1][2] = [same][up] - ([down][upup] - \int [downdown][upup]).$$

Using exponential notation in an obvious way, the formula for repeated integration by parts is

$$\int [1][2] = su - du^2 + d^2u^3 - d^3u^4 + \cdots + (-1)^n \int d^n u^n. \tag{12}$$

For polynomials that only have a finite number of non-zero derivatives eventually, if d is the polynomial all zero terms will be reached. Also, a table is suggested as it is generally easy to repeat taking progressive derivatives and integrals – just use the one above for guidance. Table 2 gives the paradigm and Table 3 gives an easy first example: $x \sin x$. Reading the non-zero rows, we arrive at $\int x \sin x = -x \cos x + \sin x$ and, taking derivatives of the right-hand side, we confirm that it is correct.

A table to evaluate the definite integral in (8) is given in Table 4.

	d/dx	$\int$
		[2]
$+$	[1]	

Table 2: Paradigm for tabular integration.

	d/dx	$\int$	
		$\sin x$	
$+$	x	$-\cos x$	su
$-$	1	$-\sin x$	$-du^2$
$+$	0	$\cos x$	d^2u^3

Table 3: Tabular integration shows $\int x \sin x = -x \cos x + \sin x$.

	d/dx	$\int$	$\Big\vert_0^\pi$	$\sum$
		$\sin x$		
$+$	$f(x)$	$-\cos x$	$-\cos \pi f(\pi) - (-\cos 0) f(0)$	$f(\pi) + f(0)$
$-$	$f'(x)$	$-\sin x$	0	0
$+$	$f''(x)$	$\cos x$	$\cos \pi f(\pi) - \cos 0 f(0)$	$-f''(\pi) - f''(0)$
$-$	$f^{(3)}(x)$	$\sin x$	0	0
$+$	$f^{(4)}(x)$	$-\cos x$	$-\cos \pi f^{(4)}(\pi) - (-\cos 0) f^{(4)}(0)$	$f^{(4)}(\pi) + f^{(4)}(0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4: Tabular integration used to derive (8).

B Leibniz Tables

Leibniz tables are an application of Leibniz's formula. Here is Leibniz's formula:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

This formula should immediately strike one as similar to the binomial theorem, also a formula:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^{(k)}.$$

Both formulas are proved using essentially the same induction argument.

Pascal's triangle is a computing device used to find the binomial coefficients that occur in both formulas. Strangely, no one, before now, has thought of making a similar device for finding derivatives of all orders for products of two functions. Table 5 gives the idea.

	A	B	C	D	E	F	G
1		f	f'	f''	$f^{(3)}$	$f^{(4)}$	$\dots$
2	g	1	1	1	1	1	$\dots$
3	g'	1	2	3	4	5	$\dots$
4	g''	1	3	6	10	15	$\dots$
5	$g^{(3)}$	1	4	10	20	35	$\dots$
6	$g^{(4)}$	1	5	15	35	70	$\dots$
7	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 5: A Leibniz table for general functions f and g.

We can quickly find the value of the right hand side of (4). In Table 6, we have used $f(x) = x^2$ and $g(x) = (p - qx)^2$ in a Leibniz table. Tables 7 and 8 evaluate this table at $x = 0$ and $x = p/q$, respectively. The tables show $f^{(0)}(p/q, 0) - f^{(2)}(p/q, 0) + f^{(4)}(p/q, 0) = -4p^2 + 48q^2$. With a little practice it is unnecessary to make separate tables for evaluation purposes.

	x^2	$2x$	$2!$
$(p - qx)^2$	1,0	1, 1	1,2
$2(p - qx)(-q)$	1, 1	2,2	3,3
$2q^2$	1,2	3,3	6,4

Table 6: Leibniz table for $x^2(p - qx)^2$ with binomial coefficients and order of derivatives: $(coefficient, order)$.

Tables 9 and 11 confirm the results referenced in (5) and (6), respectively. The tables allow for efficient calculations: one can read which left and top entries yield 0 at $x = 0$ and $x = p/q$. Also, only even derivatives need be calculated. In both cases the calculations are easily done with a calculator. One can of course implement Table 9 in a spreadsheet and then enter x values and have the spreadsheet crunch away.

To verify that the candidate rationals contradict these $n = 3$ and $n = 4$ cases, use a spreadsheet.

	0	0	2
$(p)^2$	0	0	$2p^2$
$2(p)(-q)$	0	0	$-12pq$
$2q^2$	0	0	$24q^2$

Table 7: Leibniz table for $x^2(p-qx)^2$ evaluated at $x=0$.

	$(p/q)^2$	$2(p/q)$	$2!$
0	0	0	0
0	0	0	0
$2q^2$	$2p^2$	$12pq$	$24q^2$

Table 8: Leibniz table for $x^2(p-qx)^2$ evaluated at $x=p/q$.

	x^3	$3x^2$	$3!x$	$3!$
$(p-qx)^3$	1	1	1	1
$3(p-qx)^2(-q)$	1	2	3	4
$3!(p-qx)(-q)^2$	1	3	6	10
$-3!q^3$	1	4	10	20

Table 9: Leibniz table for $x^3(p-qx)^3$. Only the circled entries need to be evaluated.

	x^4	$4x^3$	$12x^2$	$4!x$	$4!$
$(p-qx)^4$	1	1	1	1	1
$4(p-qx)^3(-q)$	1	2	3	4	5
$12(p-qx)^2(-q)^2$	1	3	6	10	15
$4!(p-qx)(-q)^3$	1	4	10	20	35
$4!q^4$	1	5	15	35	70

Table 10: Leibniz table for $x^4(p-qx)^4$. Only the circled entries need to be evaluated.

Leibniz tables can also be used for theoretical calculations. In Section 3.2 we proved a result about an integral we can observe in a Leibniz table. As the integral in that section is strictly positive for all n, we can conclude, with the help of Table 11, that the integral is a multiple of $n!$.

	x^n	nx^{n-1}	$\dots$	$n!x$	$n!$
$(p-qx)^n$					
$n(p-qx)^{n-1}(-q)$					
$\vdots$					
$n!(p-qx)(-q)^{n-1}$					
$n!q^n$					

Table 11: Leibniz table for $x^n(p-qx)^n$. Only the $n!$ factors are relevant.

C Analysis of Niven's proofs

Arguably, the proofs given in this article are even shorter than Niven's famously short π is irrational proof of 1946. Here goes. Assume $\pi = p/q$. Evaluating with tabular integration the integral in

$$0 < \int_0^{p/q} x^n(p-qx)^n \sin x \, dx \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^n < p^{2n+1}$$

gives, via a Leibniz table evaluation, a contradiction of the upper bound. Most of the work is done for a proof of π^2's irrationality. Assume $\pi^2 = a/b$. Evaluating with tabular integration the integral in

$$0 < \int_0^{a/b} x^n(a-bx)^n \sin\frac{x}{\sqrt{a/b}} \, dx \leq \frac{a}{b}\left(\frac{a^2}{4b}\right)^n$$

gives, via a Leibniz table evaluation, a contradiction of the upper bound. Note: the denominator in the argument of sin in the integral just makes it shaped like $x^n(a-bx)^n$ with the right zero values.

Niven effectively defines with a function what his (and our) integral evaluates to [2]. Why not just start with the integral?

The whole point of the math is to get $F(x)$, large factorial growth, to be expressed with $f(x)$, small polynomial growth. In Hermite's original transcendence proof for e, and subsequent variations of it, the mean value theorem is used for this trick. Consider that the derivative of $e^{-x}F(x)$ is $-e^{-x}F(x) + e^{-x}F'(x) = -e^{-x}(F(x) - F'(x))$. As $F(x) - F'(x) = f(x)$, this reduction is accomplished: $(f(x) + f'(x) + \ldots) - (f'(x) + \ldots)$ telescopes to $f(x)$. Niven almost accomplishes this same feat with

$$\frac{d}{dx}F(x)\sin x = F'(x)\sin x + F(x)\cos x,$$

but not quite. We need the difference of $F(x) - F'(x)$, so let's try

$$\frac{d}{dx}\left(F'(x)\sin x - F(x)\cos x\right).$$

And this equals $(F''(x) + F(x))\sin x$ and if we define our F function as the alternating sum of even derivatives, this reduces to $f(x)\sin x$. We also have an easy time of integration, as we are starting with a derivative.