On the general solution to the mathematical pendulum and generalized mathematical pendulum equations

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Abstract

This paper shows, for the first time, that the mathematical pendulum and generalized mathematical pendulum initial and boundary value problems may be computed from the explicit and exact general solution to the corresponding differential equation in a straightforward fashion by a direct method.

Theory

The mathematical pendulum equation is well known in the literature as [1]

$$u''(x)+\sin u(x)=0 \tag{1}$$

The equation (1) has been intensively investigated as a boundary or initial value problem for a long time without being able to determine its explicit and exact general solution in a straightforward fashion by a direct method. The aim of this work is to remedy this drawback that persists since its establishment.

1. Generalized pendulum equation

The problem to be solved here is to show that the equation (1) is a special case of a more general equation that belongs to the general class of quadratic Liénard type equation highlighted by Akande et al. [2]. So consider the class of equations [2, 3]

$$u''(x)+a^2 e^{\gamma\varphi(u)}\int e^{\gamma\varphi(u)}du=0 \tag{2}$$

For $\varphi(u)=\ln(f(u))$, with $f(u)\succ 0$, (2) becomes

$$u''(x)+a^2 f(u)^{\gamma}\int f(u)^{\gamma}du=0 \tag{3}$$

Putting $f(u)=\cos(qu)$, yields

$$u''(x)+\frac{a^2}{2q}\sin(2qu)=0 \tag{4}$$

for $\gamma=1$. The equation (4) is the desired generalized mathematical pendulum equation. The equation (4) reduces to the well-known mathematical pendulum equation (1) for $2q=1$, and $a^2=1$.

2. General solutions

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Knowing that [2, 3]

$$y(\tau) = A_0 \sin(a\tau + \alpha) \quad (5)$$

is the solution to the linear harmonic oscillator equation

$$\ddot{y}(\tau) + a^2 y(\tau) = 0 \quad (6)$$

the generalized Sundman transformation [2, 3]

$$y(\tau) = \int f(u)du\,, \quad \frac{d\tau}{dx} = f(u) \tag{7}$$

leads to

$$\sin(qu) = qA_0 \sin(a\tau + \alpha)\,, \qquad d\tau = \cos(qu)dx \tag{8}$$

so that

$$\frac{d\tau}{\sqrt{1 - q^2 A_0{}^2 \sin^2(a\tau + \alpha)}} = \varepsilon dx \tag{9}$$

Setting $p = qA_0$, and $\theta = a\tau + \alpha$, with $p^2 \succ 1$, the integration of the left hand side of (9) gives

$$J = \frac{1}{ap} F(\delta, \frac{1}{p}) \tag{10}$$

where $\delta = \arcsin(p \sin\theta)$, that is

$$F(\delta, \frac{1}{p}) = ap\varepsilon(x + K) \tag{11}$$

with $\varepsilon = \pm 1$, and K designing a constant. In this way

$$\sin\theta = \frac{1}{p} sn\left[ap\varepsilon(x + K), \frac{1}{p} \right] \tag{12}$$

such that the desired general solution to the generalized mathematical pendulum equation(4) takes the form

$$u(x) = \frac{1}{q} \arcsin\left\{ sn\left[aqA_0\varepsilon(x + K), \frac{1}{qA_0} \right] \right\} \tag{13}$$

that is

$$u(x) = \frac{\varepsilon}{q} \arcsin\left\{ sn\left[aqA_0(x + K), \frac{1}{qA_0} \right] \right\} \tag{14}$$

For $q=\frac{1}{2}$, and $a=1$, the equation (14) becomes the desired explicit and exact general solution to the mathematical pendulum equation (1), that is

$$u(x)=2\varepsilon\arcsin\left\{sn\left[\frac{A_0}{2}(x+K),\frac{2}{A_0}\right]\right\} \tag{15}$$

In this perspective, the integration constants A_0 and K may be computed following the initial or boundary conditions to which the equation (1) or (4) is subjected.

References

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