
DEMONSTRATION OF GOLDBACH CONJECTURE.

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Abstract : In this paper, we are going to give the proof of the Goldbach conjecture by introducing the lemma which implies Goldbach conjecture. first of all we are going to prove that the lemma implies Goldbach conjecture and in the following we are going to prove the validity of the lemma by using Chébotarev-Artin theorem's, Mertens formula and the Principle of inclusion - exclusion of Moivre.

1 Introduction

The Goldbach conjecture was introduced in 1742 and has never been proven though it has been verified by computers for all numbers up to 19 digits.

It states that all, even numbers above two are the sum of two prime numbers. All studies on Goldbach conjecture have failed. So we are going to give a complete proof of Goldbach conjecture.

1.1 Principle of the Demonstration

Let n an even integer such as above 20 and denote by \mathbb{C}_n the set of the composite integers of $[1, n-1]$ to what we add 1 and let f_n be the bijective mapping such that :

$$f_n : \mathbb{C}_n \rightarrow n - \mathbb{C}_n$$

$$m \mapsto n - m$$

Denote by G_n the subset of $n - \mathbb{C}_n$ consisting of prime numbers and G'_n that of composite numbers we have $n - \mathbb{C}_n = G_n \cup G'_n$. Let \mathcal{P}_n the set of prime numbers less than or equal to n . Let

$$\delta(n) = \text{card}(G_n), \alpha(n) = \text{card}(\mathcal{P}_n \setminus G_n), \Pi(n) = \text{card}(\mathcal{P}_n)$$

then $\Pi(n) = \delta(n) + \alpha(n)$, obviously $\alpha(n)$ represents the number of ways to write n as the sum of two primes

1.2 Lemma 1

$\forall n \in \mathbb{N}$, we have $\mathcal{P}_n \setminus G_n \neq \emptyset$

As we said we are going to give later the proof the lemma 1. Without loss of generality, suppose that the lemma 1 is true then we have :

1.3 Lemma 2

$\forall p \in \mathcal{P}_n \setminus G_n$, we have $n - p \in \mathcal{P}_n$

1.4 Proof of lemma 2

Let n be an even integer above 20, and suppose that $n-p$ is not prime, then

$$n - p \in \mathbb{C}_n$$

, as

$$p = n - (n - p)$$

hence

$$p \in G_n$$

.The lemma is thus proven .

Observe that each integer $m \in \mathbb{C}_n$ such that $m \geq 4$ has at least one prime divisor $p \leq \sqrt{n}$.

Let $\mathcal{P}_{\leq \sqrt{n}} = \{p_1, p_2, \dots, p_r\}$ where $p_1 = 2, p_2 = 3, \dots, p_r = \max(\mathcal{P}_{\leq \sqrt{n}})$.

Moreover, remembering that

$$\mathbb{C}_n = \bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}}, p \geq 2} A_{2p} \cup \{1\}$$

where

$$A_{2p} = \{2p, 3p, 4p, \dots, (\lfloor \frac{n}{p} \rfloor)p\}$$

. We notice that A_{2p} is an arithmetic sequence of first term $2p$ and reason p .

So

$$n - \mathbb{C}_n = f_n(\mathbb{C}_n) = \bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}}, p \geq 2} f_n(A_{2p}) \cup \{n - 1\}$$

As

$$f_n(A_{2p}) = \{n - 2p, n - 3p, n - 4p, \dots, n - \lfloor \frac{n}{p} \rfloor p\} = \{n - \lfloor \frac{n}{p} \rfloor p, n - (\lfloor \frac{n}{p} \rfloor - 1)p, \dots, n - 4p, n - 3p, n - 2p\}$$

Then $f_n(A_{2p})$ is an arithmetic sequence of first term $n - \lfloor \frac{n}{p} \rfloor p$ and reason p .

We will evaluate the quantity of prime numbers in $\bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}}, p \geq 2} f_n(A_{2p})$

by applying the principle -exclusion of Moivre and Chébotarev -Artin theorem in each $f_n(A_{2p})$ in the case where $p \nmid n$

2 Chebotarev-Artin 's Theorem

Let $a, b > 0$ such that $\gcd(a, b) = 1, \Pi(X, a, b) = \text{card}(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that $\Pi(X, a, b) = \frac{L_i(X)}{\phi(b)} + \mathcal{O}(cXe^{-\sqrt{\ln X}})$

The prime number theorem states that $L_i(X) \sim_{+\infty} \Pi(X)$ so

$$\Pi(X, a, b) = \frac{\Pi(X)}{\phi(b)} + \mathcal{O}(cXe^{-\sqrt{\ln X}})$$

3 corollary

Let $a, b > 0$ such that $\gcd(a, b) = 1, \Pi(X, a, b) = \text{card}(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that

$$\frac{\Pi(X, a, b)}{\Pi(X)} = \frac{1}{\phi(b)} + \mathcal{O}(c \ln X e^{-\sqrt{\ln X}})$$

From probabilistic point of view, the probability of prime numbers less than or equal to X in an arithmetic progression of reason b and of the first term has such that $\gcd(a, b) = 1$ is worth $\frac{1}{\phi(b)} + O(c \ln X e^{-\sqrt{\ln X}})$ for X large enough. In the following we will justify the application of Chebotain-Artin's theorem for sets $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^k f_n(A_{2p_{i_j}})$ for $1 \leq i_1 < i_2 < \dots < i_k$

3.1 Remarks

It is obvious to note that for $k > 2$, $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^k f_n(A_{2p_{i_j}})$ is the set of multiples of $\prod_{j=1}^k p_{i_j}$ which allows us to write

$$\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^k f_n(A_{2p_{i_j}}) = \{n - m \prod_{j=2}^k p_{i_j} \mid 1 \leq m \leq \lfloor \frac{n}{\prod_{j=2}^k p_{i_j}} \rfloor\}$$

This set is an arithmetic sequence of reason $\prod_{j=2}^k p_{i_j}$ and first term $n - \lfloor \frac{n}{\prod_{j=2}^k p_{i_j}} \rfloor \prod_{j=2}^k p_{i_j}$.

The hypothesis of application of Chebotarev-Artin's theorem will be justified if and only if $\gcd(2 \prod_{j=2}^k p_{i_j}, \prod_{j=2}^k p_{i_j} + n) = 1$ which is the case if $\prod_{j=2}^k p_{i_j} \nmid n$

4 Demonstration of Goldbach's conjecture

4.1 Theorem

Let n an even integer be arbitrarily large ,

$$\alpha(n) = \text{card}(\mathcal{P}_n \setminus G_n)$$

the numbers of way to write n in sum of two prime numbers ,

$$\beta_n = \prod_{p=3}^{\sqrt{n}} \frac{p(p-2)}{(p-1)^2} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

$\exists n_0$ such that $\forall n \geq n_0$

$$\alpha(n) \geq \frac{\beta_n \Pi(n)}{2 \ln n}$$

4.2 Useful Lemma

Let a_1, a_2, \dots, a_r be r numbers then

$$1 - \sum_{i=1}^r \frac{1}{a_i} + \sum_{1 \leq i < j \leq r} \frac{1}{a_i a_j} + \dots + \frac{(-1)^r}{a_1 a_2 \dots a_r} = \prod_{i=1}^r \frac{a_i - 1}{a_i}$$

4.3 Proof

Let us consider the polynomial : $P(X) = \prod_{i=1}^r (X - \frac{1}{a_i})$ from the coefficient-root relations

$$P(X) = X^r + \sum_{k=1}^r \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq r} \frac{(-1)^k X^{r-k}}{\prod_{j=1}^k a_{i_j}}$$

taking $X = 1$, the lemma is thus proved.

4.4 Proof of Theorem

According to the principle of inclusion -exclusion of Moivre we have :

$$\varrho\left(\bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}, p \geq 3, p \nmid n}} f_n(A_{2p})\right) = \sum_{k=2}^r (-1)^k \sum_{2 \leq i_2 < i_3 < \dots < i_k \leq r} \varrho\left(\bigcap_{j=2, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}, p_{i_j} \nmid n}}^k f_n(A_{2p_{i_j}})\right)$$

où ϱ represents the probability of prime numbers so

$$\varrho(n - \mathbb{C}_n \setminus n - 1) = \varrho\left(\bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}, p \geq 3, p \nmid n}} f_n(A_{2p})\right) = \frac{\delta(n) - \psi_{n-1}}{\Pi(n)}$$

. where $\psi_{n-1} = 1, 0$ according to $n-1$ is prime or not According to Chebotarev's theorem -Artin :
 $\forall k \geq 2$

$$\varrho\left(\bigcap_{j=2, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}, p_{i_j} \nmid n}}^k f_n(A_{2p_{i_j}})\right) = \frac{1}{\phi(\prod_{j=2}^k p_{i_j})} + h(n)$$

$\forall i \geq 2$

$$\varrho(f_n(A_{2p_i, p_i \nmid n})) = \frac{1}{\phi(p_i)} - \frac{\psi_{n-p_i}}{\Pi(n)} + h(n)$$

where $\psi_{n-p_i} = 1, 0$ depending on whether $n - p_i$ is a prime number or not
 with $h(n) = \mathcal{O}(c \ln(n) e^{-\sqrt{\ln(n)}})$

Thus

$$\frac{\delta(n) - \psi_{n-1}}{\Pi(n)} = h(n) - \sum_{k=2}^r \frac{\psi_{n-p_k}}{\Pi(n)} + \sum_{k=2}^r \sum_{2 \leq i_2 < i_3 < \dots < i_k \leq r} \frac{(-1)^k}{\prod_{j=2}^k (p_{i_j} - 1), p_{i_j} \nmid n}$$

.Noting that $\sum_{k=2}^r \psi_{n-p_k} = \alpha(r)$ and applying the useful lemma, we have :

$$\frac{\delta(n) - \psi_{n-1}}{\Pi(n)} = h(n) - \frac{\alpha(r)}{\Pi(n)} + \left(1 - \prod_{i=2, p_i \nmid n}^r \frac{p_i - 2}{p_i - 1}\right)$$

.As $\delta(n) = \Pi(n) - \alpha(n)$ et $r = \max\{i | p_i \leq \sqrt{n}\}$ hence

$$\alpha(n) - \alpha(\sqrt{n}) = \Pi(n) \prod_{p=3, p \nmid n}^{\sqrt{n}} \frac{p-2}{p-1} - \Pi(n)h(n) - \psi_{n-1}$$

.In the following we will apply the Mertens' theorem to evaluate $c_n = \prod_{p=3, p \nmid n}^{\sqrt{n}} \frac{p-2}{p-1}$.
 As

$$\prod_{p=3}^{\sqrt{n}} \frac{p-2}{p-1} = \prod_{p=3, p \nmid n}^{\sqrt{n}} \frac{p-2}{p-1} \prod_{p=3, p | n}^{\sqrt{n}} \frac{p-2}{p-1}$$

hence

$$c_n = \prod_{p=3, p | n}^{\sqrt{n}} \frac{p-2}{p-1} \prod_{p=3}^{\sqrt{n}} \frac{p-1}{p-2}$$

Mertens formula is written

$$\prod_{p \leq n} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\ln n} \left(1 + \mathcal{O}\left(\frac{1}{\ln n}\right)\right)$$

therefore

$$\prod_{p \leq \sqrt{n}} \left(1 - \frac{1}{p}\right) = \frac{2e^{-\gamma}}{\ln n} \left(1 + \mathcal{O}\left(\frac{1}{\ln n}\right)\right)$$

Let's put

$$c_2(n) = \prod_{p=3}^{\sqrt{n}} \frac{p(p-2)}{(p-1)^2} = \prod_{p=3}^{\sqrt{n}} \frac{p}{p-1} \prod_{p=3}^{\sqrt{n}} \frac{p-2}{p-1}$$

hence

$$c_n = 2c_2(n) \prod_{p=2}^{\sqrt{n}} \left(1 - \frac{1}{p}\right) \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

From the previous part

$$c_n(n) = \frac{4c_2(n)e^{-\gamma}}{\ln n} \left(1 + \mathcal{O}\left(\frac{1}{\ln n}\right)\right) \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

$$\alpha(n) - \alpha(\sqrt{n}) = \Pi(n) \left[\frac{4c_2(n)e^{-\gamma}}{\ln n} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2} - h(n) + \mathcal{O}\left(\prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2} \frac{1}{\ln^2 n}\right) \right]$$

Let

$$\beta_n = c_2(n) \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

and noting that $\left|\frac{h(n)\ln(n)}{\beta_n}\right|_{n \rightarrow +\infty} \mapsto 0$
then $\exists n_0 \forall n \geq n_0$

$$h(n) \leq \frac{c_2(n)(4e^{-\gamma} - \frac{1}{2})}{\ln n} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

$\forall n \geq n_0$

$$\alpha(n) \geq \alpha(n) - \alpha(\sqrt{n}) \geq \frac{c_2(n)\Pi(n)}{2 \ln n} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

4.5 proof of lemma 1

According to the theorem the lemma is proven asymptotically .by computing we can conclude and for an even integer given the pairwise of Goldbach prime is $(p, n - p)$ where $p \in \mathcal{P}_n \setminus G_n$

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