Systematic Synthetic Factoring

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December 7, 2017

Introduction

Various methods are used in elementary algebra to factor quadratics of the form $ax^2 + bx + c$ with a an integer not equal to one. Two methods are generally presented in current textbooks: trial and error and grouping [1, 2]. The goal in all factoring methods is to complete the task in a minimum amount of time; factoring, when possible with a few easy steps, is quicker than completing the square or using the quadratic formula.

However, the level of difficulty in factoring varies from problem to problem depending on the number of factors in the a and c coefficients. Trial and error is used when only a few divisors are present in these coefficients and grouping when several are present. In this article, we present a third method of factoring, synthetic systematic factoring (SSF) and argue that it provides a better pedagogical approach than that given in current textbooks. In particular, it enables many standards of mathematical practice (SMPs) as given by Common Core State Standards Initiative [3]. Students can explore and discover properties of factoring quadratics with a few leading questions.

Systematic synthetic factoring

Given the quadratic $ax^2 + bx + c$, the goal of factoring is to find d, f, g, and h such that

$$ax^2 + bx + c = (dx + f)(gx + h), \tag{1}$$

where all coefficients are natural numbers. From (1), we derive

$$(dx + f)(gx + h) = dgx^2 + (dh + fg)x + fh. \tag{2}$$

The left hand side of (2) demonstrates the FOIL acronym for multiplying the two binary forms $(dx + f)$ and $(gx + h)$. The first coefficients in each form are multiplied for the coefficient of x^2, then the outer, d and h, are multiplied and added to the inner, f and g multiplied, to form the coefficient of x, and, finally, the lasts, f and h, are multiplied to form the constant term. It follows from this that a and c from (1) must be split into pairs of divisors: (d, g) and (f, h). The goal then is to find pairs so that the outer plus the inner, $dh + fg$, equals the middle coefficient b.

We introduce a table that places all possible firsts (d, g)s, along the left-most column and all lasts (f, h)s on the top row; the interior cells give all outer dh and inner fg pairs: (dh, fg)s. These are summed and when this sum equals the b coefficient of the quadratic, the required firsts and lasts are directly to the left and above. These are transcribed from the table in the usual form of a factored quadratic: $(dx + f)(gx + h)$.

So, for example, given the quadratic $2x^2 + 11x + 12$, place all the ways of splitting up 2 and 12 into two natural numbers along the left column and along the top row, respectively, see Table 1.

	1 12	2 6	3 4	4 3	6 2	12 1
1 2	12 2	6 4	4 6	3 8	2 12	1 24
2 1	24 1	12 2	8 3	6 4	4 6	2 12

Table 1: Florida table for $2x^2 + 11x + 12$.

The tables are systematic in giving all possible combinations, given a and c, and synthetic by analogy with synthetic division, where the variable x does not appear, but just the coefficients. As these tables consist of firsts along the left and lasts on top, Florida tables (or FOIL tables) seems an appropriate name for them; FL is the abbreviation for the state of Florida.

By ordering pairs with the smallest of the two first and going up, as is done in Table 1, one has an assurance of getting all such possible pairs. The pairs are found, then, by testing divisibility by each natural number in turn. The interior table cells are formed by multiplying the outer and inner pairs of numbers from the top row and left column. So, for example, the interior cell 12 2 is a result of 1×12 (an outer) and 2×1 (an inner). When the sum equals the middle coefficient of the quadratic, in this example 11, the left column gives the firsts and top row gives the lasts. For this quadratic, we use 8 3 from the third row and fourth column; the numbers add to 11, the middle coefficient. The answer is then transcribed from the table: $(2x\ )(1x\ )$ for the firsts (from the left column) (2x 3)(1x 4) for the lasts (from the top row). For this form of a quadratic, $ax^2 + bx + c$ both factors use addition: the answer is $(2x + 3)(x + 4)$.

Here's an example that shows the use of Florida tables to determine that a quadratic is prime. The Florida table for the quadratic $10x^2 + 3x + 2$ is given in Table 2. One sees the systematic list of all two integer products giving 10, left column, and all such giving 2, top row. As the *outer and inner*'s, given by the interior cells for all these combinations, never yields a sum equal to 3, this quadratic is a prime.

	1 2	2 1
1 10	2 10	1 20
2 5	4 5	2 10
5 2	10 2	5 4
10 1	20 1	10 2

Table 2: Florida table for $10x^2 + 3x + 2$.

This approach to factoring trinomials stresses the Common Cores Standards for Mathematical Practice, in particular SMP 2: Reason abstractly and quantitatively [3, p. 7] and SMP 7: Look for and make use of structure [3, p. 8].

Other aspects of Florida tables

There are properties of these tables that provide good shortcuts to factoring and thus stress SMP 8: Look for and express regularity in repeated reasoning. First, only one of a and c needs to have all combinations of pairs given; the other just needs half. This can be observed in Tables 1 and 2. No new combinations are given by the additional three columns and two bottom rows, respectively; this follows from commutativity of multiplication and addition. In this regard, a nice classroom question is which should be given a complete list of pairs? The answer is, of course, the number with the fewest divisors. The second feature relates to the four cases of quadratics possible. Students can explore and find rules that determine which factor has an addition and which a subtraction for all the cases of quadratics. The details follow. In a classroom setting, guiding questions can be asked that foster perseverance in problem solving and thus stress SMP 1: Make sense of problems and persevere in solving them [3, p. 6].

There are four combinations possible in $ax^2 \pm bx \pm c$: + +, - +, + - , and - - with all coefficients natural numbers. When factoring for these cases it is possible to automatically find the signs to use in the factored form, $(cx \pm d)(rx \pm s)$. For the cases + + and - +, the sign combinations are $(cx + d)(rx + s)$ and $(cx - d)(rx - s)$, respectively. These are forced, meaning there is no decision to be made. For these two cases, the interior cells are always added and when the sum equals the absolute value of b, the corresponding cell in the left column gives the firsts and corresponding ones in the top row give the lasts. For the remaining two cases, - - and + -, the absolute value of the difference of the pairs in interior cells is used. When this absolute value of the difference equals the absolute value of b, once again, the corresponding left and top cells give the factors. Here are the rules for these cases that determine which factor gets the plus: with - -, the larger of the interior cell number points to the factor in the *last*'s, top row that gets the plus sign; with + -, the smaller. The other factor is always minus. The upper left corner of these tables is a perfect place to put the case of the quadratic being factored; its a good reminder; the second sign indicates whether interior cells are to be added or subtracted.

Here are examples of the various cases. Notice that left columns and top rows, as well as interior cells are generally not completed in these examples. These are intentional omissions to suggest that one does a few calculations with only partially completed left columns and top rows, and, factors not having been found, fills in more possible pairs on the top and left. Pairs in interior cells themselves can mentally be checked, generally, with only the correct combinations being written down and/or the corresponding top and left pairs circled.

Example 1. Factor $12x^2 - 11x - 15$. The Florida table for this example is given in Table 3. The factors are immediately generated by the table using the rule for this case. That is: the largest of the interior pair, 20, corresponds to the top row factor 3 that receives the plus sign. The answer is $(3x - 5)(4x + 3)$.

Example 2. Factor $4x^2 + 4x - 15$. The Florida table for this example is given in Table 4. We can immediately write (2x-3)(2x+5). The smaller interior

- -	1 15	3 5	5 3
1 12			
2 6			
3 4			9 20

Table 3: Florida table for $12x^2 - 11x - 15$ with use of the sign rule.

number, 6 (bolded), corresponds to the 5 (bolded) in the top row; 5 gets the plus for the factor (2x+5).

+ -	1 15	3 5
1 4	15 4	5 12
2 2	30 2	10 6
4 1		

Table 4: Florida table for $4x^2 + 4x - 15$.

Comparison with other methods

There are three factoring techniques that are generally presented in textbooks: simple factoring, trial and error factoring, and factoring by grouping (also known as the AC-method). We review each as it relates to SSF.

Simple factoring is used for monic quadratics, that is, quadratics with a leading coefficient of one. The challenge in factoring this type is to find two integers whose product is the last coefficient and whose sum is the middle. For example, given $x^2 - x - 12$, we consider the ways -12 can be factored into two numbers whose sum is -1.

- -	1 12	2 6	3 4
1 1			4 3

Table 5: Simple factoring: 4 points to 3 for $(x + 3)(x - 4)$.

A Florida table organizes and systematizes the search for such a pair, see Table 5. It also abstracts just the essentials of the problem. The only factors for a=1 in ax^2+bx+c are 1 1, so only half the combinations need to be given in the top row for the c coefficient. The rules for the four cases given for non-monic quadratics apply. Simple factoring is the base case for SSF; students can build on their knowledge of simple factoring using SSF [5]. The trial and error method, as given in text books, is used for non-monic quadratics when the a and c coefficients have relatively few factors. The method is to apply an ad hoc way of trying factors until the correct ones are found. SSF provides a way to perform this method in an organized and systematic way. It removes the guess work involved.

The AC-method makes the method for non-monic quadratics similar to the method of simple factoring for monic quadratics. Given $ax^2 \pm bx \pm c$, multiply a and c and then look for factors of this product that when added or subtracted give b. With simple factoring, there is no multiplication (other than by 1); one just uses factors of c that add to b. This method is frequently called the grouping method as this is the next step in the factoring procedure. Well give an example using this method.

- -	1 24	2 12	3 8
1 1			8 3

Table 6: Florida table used for the key step in the AC-method.

Example 3. Factor $2x^2 - 5x - 12$ using the AC-method. Look for the factors of 2(-12) = -24 such that when added give -5. We can do this with a Florida table, see Table 6. Technically, we would have to consider all plus and minus combinations in using the AC-method, but $38 = -5$ is apparent from the table. Next, the quadratic is re-expressed using the found numbers: $2x^2 - 5x - 12 = 2x^2 + 3x - 8x - 12$. The first and last two terms are then grouped and recombined for the answer:

$$2x^2 + 3x - 8x - 12 = x(2x+3)4(2x+3) = (x-4)(2x+3). \tag{3}$$

Note the rules for the plus sign determination in the - - case apply in the sense that the larger of the interior cell pair, 8 3, points to the 3 which is added in $+3x - 8x$ of the grouping step in (3). We also get 8 3 as an interior cell value using SSF without the non-sensical multiplication before factoring of the AC-method.

A single edit of Table 1, a change from + + to - - as the case type, allows a fast factoring of this quadratic using SSF. Multiplication of a and c and the grouping step are not necessary. It is a faster, more direct way to factor than the AC-method. The time savings is pronounced when the product of ac results in a relatively large number. Factoring $8x^2+30x-27$, for example, with both methods demonstrates this.

Conclusion

Unlike the three methods currently used in textbooks, SSF gives a way to systematically and in an organized manner find all pertinent factors when factoring a quadratic. In the case of the grouping method, it provides the key pair necessary, but without the need for the grouping step. This suggests that, if speed and ease are the criteria for comparing factoring methods, SSF is more efficient than the grouping method.

Additionally, SSF allows for a natural pedagogical development in the textbook treatments of factoring trinomials. Simple factoring can be taught using Florida tables. The case mechanisms for determining the signs inside the factors apply to simple factoring, so these rules can be developed with the simplest monic quadratics first. Factoring non-monic quadratics is then a natural extension of simple factoring. Thus with SSF all factorable quadratics are given a unified method that fosters systematic thinking and abstraction of essential elements in solving problems. Students, with some guiding questions, can evolve their sophistication and understanding of factoring from the simplest monic quadratics with two additions to general non-monic quadratics with all the plus/minus combinations. These goals conform to the essence of modern pedagogical goals: enable students to explore and discover mathematics on their own [4, 5].

References

[1] R. C. Angel and D. C. Runde, *Intermediate Algebra for College Students*, 8th ed., Prentice Hall, New York, 2011.

[2] R. Blitzer, *Algebra and Trigonometry*, 4th ed., Pearson, Upper Saddle River, NJ, 2010.

[3] CCSSI, *Common Core State Standards Initiative: Common Core State Standards for Mathematics*, National Governors Association Center for Best Practices and the Council of Chief State School Officers. available at

`http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf`

[4] NCTM, *Principles and Standards for School Mathematics*, Reston, VA, NCTM, 2000.

[5] ______, *Focus in High School Mathematics: Reasoning and Sense Making*, Reston, VA, NCTM, 2009.