Beal's Conjecture as Univariate Polynomial Identity Derived from Algebraic Expansion of Powers of Binomials: Analysis and Proof

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Abstract: The general equation of Beal's conjecture $x^a + y^b = z^c$, at points where its variables are numerically equal, is identified as a univariate polynomial identity derived from algebraic expansion of powers of binomials which upon expansion and reduction to two terms produces $\alpha x^l + \beta x^l \equiv \delta x^l$, where the general polynomial equation has integer solution at the intersection with the line $x - y = 0$ as a special case and satisfies Beal's condition of perfect power terms; $\alpha, \beta, \delta,\ l$ are positive integers, $l > 2$ and $\alpha + \beta = \delta$. This algebraic identity can be represented by the addition of two vectors in the vector space of the set of all polynomials in the form $p(x) = a\, x^l$ for $a \in \mathbb{Q}$ as a subspace of the infinite vector space over $\mathbb{Q}$ of all polynomials with basis $1, x, x^2, ...$ with the ordinary addition of polynomials and multiplication by a scalar from $\mathbb{Q}$, where l is particular to any solution to the equation. Here we look for elements in the $\mathbb{Q}$ field where the rational number can be converted to a number in exponential form that successfully combines with the basis-element x^l to produce perfect power terms. Accordingly, it is shown that all three monomials of the identity equation numerically produce terms of perfect powers by following the rules of exponentiation which produces integer solutions to Beal's general equation and can be obtained by expanding the corresponding binomial identity.

Key words: Beal's conjecture, Binomial identity, Diophantine equations, Univariate polynomial

1. Introduction and conclusion

Beal's conjecture states that if $x^a + y^b = z^c$, where *a*, *b*, *c*, *x*, *y* and *z* are positive integers and *a*, *b*, c > 2, then *x*, *y*, and *z* have a common prime factor. The conjecture was made by math enthusiast Daniel Andrew Beal in 1997 [1]. It is a generalization of Fermat's Last Theorem (FLT) which has been considered extensively in the literature [2-7] and was proved by Andrew Wiles [8]. Similar problems to Beal's conjecture have been suggested as early as the year 1914 [9] and the conjecture maybe referred to by different names in the literature [10-11]. So far a proof to the conjecture has been a challenge to the public as well as to mathematicians and no counterexample has been successfully presented to disprove it, i.e. Peter Norvig reported having conducted a series of numerical searches for counterexamples to Beal's conjecture. Among his results, he excluded all possible solutions having each of $a, b, c \leq 7$ and each of $x, y, z \leq 250{,}000$, as well as possible solutions having each of $a, b, c \leq 100$ and each of $x, y, z \leq 10{,}000$ [12].

Even though many partial results of the problem have been proved [13-16] and physical elementary approaches have been attempted [17], but no solid approach to solve the general problem has been presented in the literature and therefore any serious contribution to it is considered an important advancement in algebraic number theory especially because it implies FLT. In this paper the conjecture is presented by a univariate polynomial identity that produces solutions with terms that must share the indeterminate (bound variable) to fulfill the identity property, and hence share a prime factor.

The search for solutions to Diophantine Beal's type equations by designating each term a different variable complicates finding one tremendously; i.e. the proof of Fermat's type Diophantine equations of powers 2 is made difficult by seeking a proof via modular arithmetic of elliptic curves. It has been the trend to consider such algebraic equations as equations in several variables. Such approach is extensively handled by algebraic geometry of systems of polynomial equations in several variables, Diophantine geometry, and algebraic number theory, which studies the existence of solutions to Diophantine equations of univariate nature. For higher number of addends, the solutions or proofs become progressively difficult. Unlike polynomial equations in several variables as handled by the literature, here all equations that satisfy Beal's conjecture are treated as polynomial equations in single variable that follow the proposed identity and simplify to Beal's type numerical equations.

It is claimed here that any integer solution to the general equation of Beal's conjecture is a special case over the rational field where the solution of the equation lies on the line $x - y = 0$ renders the equation single variable polynomial identity, e.g., binomial equations whose solutions lie on the line $x - y = 0$ and whose terms are perfect powers. Perhaps the simplest example of a polynomial equation in several variables that can be treated as polynomial in one variable with a solution on the line $x - y = 0$ is the equation of the circle centered at the origin $x^2 + y^2 = z^c$. This can be viewed as due to symmetry at 45^0 angle. Here, Beal's numerical solution is where the circle intersects the line $x - y = 0$. This leads to $x = y$ and the polynomial becomes $2x^2 = z^c$. As an identity, the bound variable x must take the value 2 and the algebraic sum z^c must then be 2^3. An integer solution to the Diophantine equation $x^2 + y^2 = z^c$ can be uniquely then identified from all rational solutions by recognizing it as a univariate identity and utilizing the exponential identities. It simply means I am adding two same squared basis-elements of 2 on the 2-dimensional Euclidean space. This Diophantine equation $2^2 + 2^2 = 2^3$ is not classified as Beal's but it is a simple example of a single polynomial identity that is expressed in perfect power terms as Beal's conjecture requires. Obviously, the equation has a common prime factor 2. For powers > 2, the similar equation $2^3 + 2^3 = 2^4$, generated by the general equation of $x^3 + y^3 = z^c$ and a solution lies on $x - y = 0$, fits Beal's requirements.

The origin of Beal's equation is a binomial identity which upon expansion and by Pascal's rule produces terms of the same monomial degree if we convert the binomial identity into univariate equation by equating the two variables as suggested in this paper. We can proceed to prove by elementary algebra that Beal's equation is in fact algebraic identity. Let's recall that a binomial identity describes the expansion of powers of a binomial to produce terms of the same power as the degree of the binomial if we replace y with x. If the terms on both sides of the expanded binomial identity,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(\lambda x + \delta y)^n = \lambda^n x^n + \cdots + \delta^n y^n \tag{1}$$

where the coefficients λ, δ and the power n are positive integers, are converted into monomials with the same variable as a special case as suggested in this paper, the terms can be added algebraically to produce a univariate polynomial identity describing the LHS of Beal's equation while the RHS is the algebraic sum of its monomials producing an identity on its own. It is to be noticed that replacing y with x on the LHS of the identity $(\lambda+\delta)^n x^n$ generates the RHS of Beal's equation upon expanding and reducing the number of terms to 2, which is doable since all of the terms have the same power. This is clear since after we replace y with x we can expand the polynomial $(x+x)^n$ into a sum of terms of the form $\alpha x^b x^c$, where the coefficient α is governed by Pascal's rule and the exponents b and c are positive integers with $b+c=n$.

Starting with binomial of power 1 we simply add two terms and choosing the terms carefully we get,

$$(2^3x^3+x^3)^1=3^2x^3$$

Recognizing x must equal 3 and simplifying by following exponential rules we get,

$$6^3+3^3=3^5$$

We obtain Beal's equation of the form $a,b,c=(3,3,5)$ in this case. By carefully choosing our terms we can obtain different orders of (a,b,c).

For binomial of power 2, $(x+y)^2$ produces Beal's-like equation of the form $a,b,c=(2,3,2)$. Expanding the binomial identity we get,

$$(x+x)^2=x^2+2x^2+x^2$$

Reducing to two terms and simplifying with the proper choice of the variable x, as $x=3$, to comply with the exponential rules we get,

$$(2x)^2=x^2+3x^2$$

$$3^2+3^3=6^2$$

This is not Beal's equation because of the powers of 2. The terms in the equation are not coprime and therefore they are not considered to be of Fermat-Catalan form as well. We can generate other similar equations by varying the coefficients λ and δ in the binomial equation (1), e.g. the binomial $(2x+y)^2$ produces another non-Beal, non-Fermat-Catalan equation,

$$10^2+5^3=15^2$$

For $n\geq 3$, the identity produces Beal's equations of powers $(a,b,c)>2$. For $n=3$, the binomial $(\lambda x+\delta y)^3$ produces Beal's equations of the form $a,b,c=(3,4,3)$, e.g. the equation,

$$(x+x)^3=x^3+3x^2x+3x^2x+x^3$$

Simplifying we get,

$$(2x)^3=x^3+7x^3$$

Choosing $x=7$ we get Beal's equation,

$$7^3+7^4=14^3$$

Another example of binomial of power 3 is $(2x + y)^3$. Expanding and following the same steps as above, we get,

$$(2x + x)^3 = 2^3x^3 + 3 \cdot 4x^2x + 3x^2 \cdot 2x + x^3$$

Simplifying and taking the term 2^3x^3 as the leading term; since it produces perfect power on its own, to produce only two terms on the RHS we get,

$$(3x)^3 = 2^3x^3 + 19x^3$$

This identity requires $x = 19$ for the equation to have single power terms as required by Beal's conjecture and produces Beal's equation,

$$38^3 + 19^4 = 57^3$$

If we take x^3 to be the leading term as it also makes perfect power, and proceed as above, we get Beal's equation,

$$26^3 + 26^4 = 78^3$$

We can see that for binomials of powers $n > 2$ the identity produces Beal's equations of the order $a, b, c = (n, n + 1, n)$. Therefore, for any power of $n \geq 3$, there is an infinite number of Beal's triples of this form. For a list of first orders see Appendix A.

It is important to emphasize here that algebraic theorems involving numerical operations from arithmetic are generalized to cover non-numerical objects such as polynomials. In this regard, we identify Beal's equation $x^a + y^b = z^c$ as a particular numerical solution to the general polynomial identity $\alpha x^l + \beta x^l \equiv \delta x^l$, where α, β, δ, l are positive integers, $l > 2$ and x is the indeterminate whose value must combine with the coefficient of each term to produce a perfect power term following the rules of exponentiation, and $(\alpha + \beta) = \delta$. The LHS of the univariate identity represents the sum of two vectors in a polynomial vector space. The variable x is a bound variable since we identified the equation as algebraic identity.

To clarify the connection between the general equation of Beal's conjecture (Beal's equation) and the proposed binomial univariate identity, let's emphasize that the two terms on the LHS of Beal's equation can be combined by factorizing a CF and using the power rules to yield the RHS. In this sense, the LHS of Beal's equation can be treated as the sum of two single same-variable monomials of the same degree that necessarily must produce the monomial on the RHS of the equation. A greatest common divider (GCD) of the two monomials must exist then which allows for the process of combining the two LHS expressions into one by exponential rules.

Suppose a solution to Beal's equation produces $c^z = 3^5$, which can in turn be split to $3^2 \cdot 3^3$ and the coefficient term 3^2 expands to $(1 + 2^3)$ producing the equation,

$$3^3 + 6^3 = 3^5$$

The path to proving Beal's conjecture is seeking a polynomial identity such as the one that solves the Diophantine equation $x^a + y^b = z^c$ at the intersection points with the line $x - y = 0$ which satisfies Beal's condition of perfect power terms. The idea is to consider z^c as an element in a polynomial vector space over the rational numbers $\mathbb{Q}[x]$, i.e. the numerical value of $z^c = 3^5$ is obtained from the rational polynomial function ax^3, *where x is allowed to assume values in $\mathbb{Z}$,* of indeterminate $x = 3$ and coefficient 3^2. For later comparison of adding the two LHS terms to addition of fractions, the coefficient in exponential form as well as improper fractional form is introduced, e.g. $3^2 = \frac{27}{3}$. Therefore an integer solution to Beal's equation represents selective numbers in $\mathbb{Q}$ that meet the requirement of single integer power terms. In other words, the equation $3^3 + 6^3 = 3^5$ is an integer solution of the general single polynomial identity $\alpha x^l + \beta x^l \equiv \delta x^l$. Beal's equation then may be represented by the sum of two same degree monomials representing the addition of two vectors in the vector space of all polynomials $p(x) = ax^l$ over $\mathbb{Q}$, being a subspace to the infinite-dimensional vector space of all polynomials over $\mathbb{Q}$ with basis $1, x, x^2$ …., where l is positive integer indicating a particular solution to Beal's equation. In other words, we are adding two same degree single-variable monomials with coefficients from $\mathbb{Q}$ that produce a sum of same degree monomial with the condition that the numerical valuation leads to perfect power terms. In the above example, the monomial-equation is,

$$2^3x^3 + x^3 = 3^2x^3$$

The restriction introduced by Beal's equation of perfect power terms requires the proper choice of the subspace as well as the proper choice of the coefficients in the rational numbers. The coefficients in the above equation are particular numbers in the rational field, specifically, 2^3 is the improper fraction $24/3$ and 3^2 is $27/3$. A solution to this numerical identity equation requires the indeterminate to be 3 and the specific subspace to be $p(x) = ax^3$. For this particular example, the point on the line $x - y = 0$ that satisfies Beal's requirement of perfect power is $x = y = 3$ when employing the Diophantine equation $2^3x^3 + y^3 = 3^5$.

We conclude that to fulfill the requirements of Beal's conjecture of positive integer solutions and perfect power terms, Beal's function must be single variable identity and that the intersection of Beal's function with the line $x - y = 0$ produces the only integer solution.

To graphically identify the common factor of Beal's equation (integer solution), it is sufficient to find where Beal's function intersects the line $x - y = 0$. In the example above, Beal's equation $2^3x^3 + y^3 = 3^5$ intersects the line $x - y = 0$ at $(3, 3)$; see figure 1.

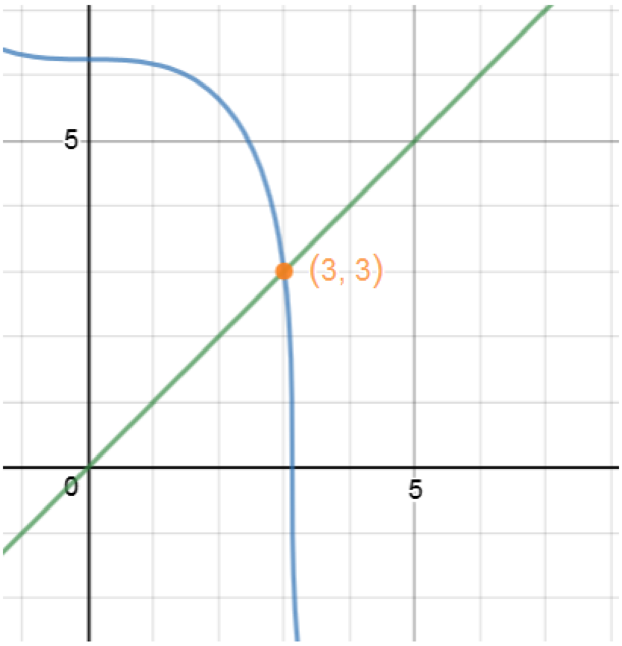

Fig. 1. An integer solution of the Beal's equation $2^3x^3 + y^3 = 243$ is where Beal's function intersects the line $x - y = 0$ at $(3, 3)$. The coordinates of the point of intersection make a common factor of the equation.

The general equation that describes Beal's equation is,

$$\alpha x^l + \beta x^l = \delta x^l \tag{2}$$

In equation (2) we are combining two same degree in single variable monomial functions by the rules of addition of polynomials and multiplication by a scalar. As mentioned earlier, the equation represents a simple algebraic identity which expands the equality $(\alpha + \beta)x^l = \delta x^l$ by the distributive law. It represents addition of two vectors in the vector space ax^l as a subspace of the general vector space of all polynomials over $\mathbb{Q}$ with basis $1, x, x^2, x^3$ ... Any of Beal's equations then is a solution in the proper subspace. In the example above, the existence of the common basis element x^3 in the equation is a must since we are adding vectors in the subspace ax^3, and the polynomial variable x is a bound number that must have a value since the polynomial equation is identified as an identity. Any other solution to the equation of $x \neq 3$ or different coefficients that satisfy the condition $(\alpha + \beta) = \delta$ but do not all successfully combine with the characteristic term x^l by the power rules to produce perfect powers, satisfies the general polynomial identity but does not comply with Beal's equation of perfect power terms. Integer solutions then to Beal's equation only occur on the line of $x - y = 0$. We can compare Beal's identity (2) with the two-variable identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ that produces Pythagorean triples by which we choose the proper value of the variables x and y to comply with the evenness and oddness of the numbers on the LHS of the identity to produce the proper triples. Simplifying the terms on both sides separately we get the same expression of $x^4 + 2\,x^2y^2 + y^4$. Likewise, we can simplify the LHS of equation (2) and get the same expression of δx^l by exploiting the power rules and choosing the proper value of the (bound) variable x in Beal's identity as well as the proper coefficients that produce perfect power terms.

Equations of the form (2) that do not comply with Beal's condition of perfect powers intersect the line $x - y = 0$ at a rational point i.e. the equation $2^3x^3 + y^3 = 299$ (made by slightly changing the numerical value on the right side of $2^3x^3 + y^3 = 3^5$) has a rational solution of $(3.24, 3.24)$ at the intersection point with the line $x - y = 0$ and is not Beal's identity since the coefficient of the term on the right side does not combine with the characteristic term x^l by the power rules to form perfect power; see figure 2.

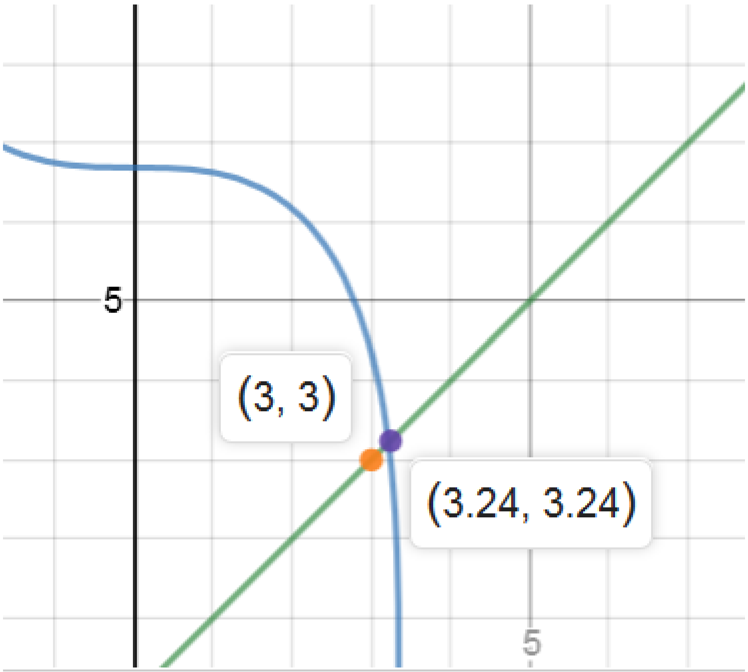

Fig. 2. A rational solution of the equation $2^3x^3 + y^3 = 299$ at the intersection with the line $x - y = 0$.

For any polynomial identity, a common factor must exist. In Pythagorean identity, the GCD polynomial is x^0 derived from a primitive polynomial of basis-variable of the vector space ax^l where l is the nonnegative integer zero and the coefficient terms of equation (2) form single terms of power 2 as special case. For higher powers of l, the GCD of the basis-element in the polynomial identity constitutes an essential contribution to Beal's equations, or any polynomial identity with higher number of addends, abiding with the condition of perfect power terms. The trickiest question that is easy to fall in is, why the coefficients α, β and δ of equation (2) do not themselves make a polynomial identity if powers of Beal's terms are > 2 by taking the basis-element of power $l = 0$? The answer is straight forward. If Beal's terms' powers are > 2, the powers must be contributed by the basis-element of the vector space involved, which effectively upgrades one or more of the terms to $l > 0$. This process is a direct consequence to the basic arithmetic operations of the laws of exponentiation. On the other hand, special identities such as that of Pythagorean identity and those of higher number of addends such as $3^3 + 4^3 + 5^3 = 6^3$ can be simply added by addition of terms in the subspace of ax^0 and therefore are trivial identities of Beal's. They can also be summed in a higher degree of the polynomial subspace of $p(x) = ax^l$ that abides with exponentiation rules, i.e. Pythagorean identity by multiplying the identity equation by CF of x^2 and the later one by x^3. In other words we upgrade them as addition of two vectors in the subspaces of ax^2 and ax^3 respectively.

The single variable identity of Beal's equation which evaluates to integer solutions is similar to addition of two fractions. In this case we are adding two vectors in the subspace ax_i of the general infinite-dimensional vector space $\mathbb{Q}$ over itself with basis $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ .... While addition of fractions in the physical sense is obvious, similar addition of numbers in exponential form as that of Beal's equation is not so obvious. The following is an example of adding two fractions in the vector space ax_i with the bound variable $x_i = \frac{1}{6}$,

$$\frac{5}{3} + \frac{7}{2} = \left(\frac{5}{3} + \frac{7}{2}\right)\frac{6}{6} \;\Rightarrow\; 10\left(\frac{1}{6}\right) + 21\left(\frac{1}{6}\right) = 31\left(\frac{1}{6}\right)$$

In the example above, the right side is the identity fractional-equation which represents addition of two vectors in the vector space ax_i of numerical values of the sum $10\left(\frac{1}{6}\right) + 21\left(\frac{1}{6}\right)$ of the general identity equation $\alpha x_i + \beta x_i = \delta x_i$. By the use of the rules of addition of fractions and multiplication by a scalar in the vector space ax_i, we take the basis-element as CF and combine the resulting two terms (coefficients) to obtain a single fractional term of $31\left(\frac{1}{6}\right)$. If the basis-element x_i is cancelled out from both sides of the equation, the equation reduces to the numerical trivial solution of $10 + 21 = 31$ that represents simple addition of natural numbers in the field of the rational numbers. Like the exponential example above (or any solution to Beal's equation), this example of adding fractions, as adding two vectors in a vector space, shows that the existence of the basis-element in the equation is an integrated part of the addition process of elements in the vector space and presents a valid justification of the intrinsic existence of a GCD monomial and the corresponding numerical GCD on the LHS of Beal's equation once the equation is identified as a single polynomial identity.

Basically then we can convert Beal's terms to improper fractions and proceed to add two fractions as above which necessarily includes a common factor of the base in exponential form as that of the denominator of the same number expressed as a fraction, i.e. the LHS of Beal's equation $3^3 + 6^3 = 3^5$ can be added as fractions as $\frac{81}{3} + \frac{1296}{6} = \frac{81}{3} + \frac{648}{3} = \frac{729}{3} = \frac{3^6}{3} = 3^5$. Addition of different fractions than those of Beal's equations produces fractions that simply cannot be expressed as perfect power terms. Therefore, Beal's binomial identity is in fact a special fractional identity equation whose terms can be converted to perfect powers.

Beal's equation then is a single polynomial identity such that both of the LHS and the RHS are equal polynomial functions for every x in their domain. In other words, the numerical solution of Beal's equation is a particular solution to the general polynomial identity equation $\alpha x^l + \beta x^l \equiv \delta x^l$ that represents addition of two vectors in the vector space ax^l as subspace of the general vector space of all polynomials over the field $\mathbb{Q}$ with basis $1, x, x^2, x^3$ ..., where l is positive integer specific to a particular numerical solution of Beal's equation.

2. Beal's equation as single polynomial identity

Lemma 2.1. *Let z^c be any number in exponential form such that $c \geq 3$; z, c are positive integers . Then the term is intrinsically a product of two numbers in exponential form.*
Proof. The proof is obvious by the rules of exponentiation. ☐

Corollary 2.1. Let Beal's equation be $x^a + y^b = z^c$. Then each term of its numerical solution can be represented as a product of two numbers in exponential form.
Proof. The proof follows from Lemma 1 and the restriction of Beal's conjecture that a, b, c are integers ≥ 3. ☐

Proposition 2.1. *Let **P** be the vector space of all polynomials over $\mathbb{Q}$ and power basis $1, x, x^2, \ldots,$ with the addition operation and scalar multiplication are defined as the usual polynomial operations. Further, let the set of all polynomials of the form $p(x) = ax^l$ for $a \in \mathbb{Q}$ and $l > 2$ is positive integer. Then for any particular l the set of polynomials $p(x)$ is a subspace of **P**.*
Proof. We check the criteria of $p(x) = ax^l$ for $l = 3$,

a. Contains the zero vector
   For all $a \in \mathbb{Q}$
   Let $a = 0$; $p(x) = 0$ is a vector in the set.
   For any $p(x)$: $p(x) + 0 = p(x)$

b. Closed under addition
   Choose a_1x^3 and a_2x^3
   $a_1x^3 + a_2x^3 = (a_1 + a_2)x^3 \in ax^3$

c. Closed under scalar multiplication
   Choose a_1x^3 and the scalar b
   $ba_1x^3 \in ax^3$

We conclude that polynomials $p(x) = ax^l$, where $l = 3$ are subspace of **P**. Similarly we can prove that $p(x) = ax^l$ is a subspace for any value of $l > 3$. ☐

Proposition 2.2. *Let the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ represent addition of two polynomials in the infinite vector space $p(x)$ as in proposition 1 with coefficients in $\mathbb{Q}$. Then the solution to the polynomial equation is every integer value of the indeterminate x.*
Proof. The proof is obvious by the rules of addition of polynomials since the polynomial equation is identified as identity and represents the sum of two elements in the subspace $p(x)$ of the general vector space **P**. ☐

Corollary 2.2. Let Beal's equation represent the solution to the general polynomial identity equation $\alpha x^l + \beta x^l = \delta x^l$ with the numerical valuation leads to perfect power terms. Then, there exists a particular solution to Beal's equation where the coefficients of each of the polynomial terms α, β, δ must combine with the numerical value of the basis x^l to produce perfect power number.

Proof. The proof is straight forward by employing Lemma 1 and corollary 1 and the use of exponential rules since the polynomial identity defines terms in one-variable and that the numerical solution requires combining the coefficients of the terms with x^l. ☐

Corollary 2.3. Let Beal's equation represent the solution of the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ as in Corollary 2.2. Then the solution to the equation has a common polynomial GCD of x^l and a common base of x with numerical value.
Proof. Since the general polynomial identity that represents Beal's equation must have a common factor of x^l as the basis element in the vector space $p(x)$ as in proposition 1, it follows that the specific solution of Beal's equation must have a common base of the numerical value corresponding to that of the base x. ☐

Corollary 2.4. Let Beal's equation represent the solution of the general polynomial identity equation of $\alpha x^l + \beta x^l \equiv \delta x^l$ as in Corollary 2.2. Then, for $l > 2$ there exists a solution to the equation where x^l is a GCD polynomial.
Proof. The proof is clear by corollary 3 and the argument above. ☐

3. Proof of Beal's equation as univariate binomial identity

Proposition 3.1. *Beal's equation is a univariate binomial identity with its expanded terms reduced to two terms.*
Proof. Firstly, Suppose Beal's equations are not comprime. Then the general polynomial identity equatio*n that represents Beal's equation must be in the form of,*

$$\alpha x^l + \beta x^l = \delta x^l \qquad 3.1$$

where α, β, δ, l are positive integers.

Secondly, let the binomial identity be univariate as a special case. Its general form is,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Expanding the RHS we get,

$$(\lambda x + \delta y)^n = \lambda^n x^n + \cdots + \delta^n y^n \qquad 3.2$$

For a special case, substitute x for y to get,

$$(\lambda + y)^n \, x^n = \lambda^n x^n + \cdots + \delta^n x^n \qquad 3.3$$

All the terms on the RHS have the same exponent since they are all have the form $\alpha x^b x^c$; where $a + b = n$. Taking the first or last term to be the leading term because it is perfect power and summing over the rest of the terms we can reduce the RHS to two terms representing Beal's equation as in proposition 1. The equation becomes,

$$(\lambda + y)^n \, x^n = \lambda^n x^n + kx^n \qquad 3.4$$

By comparing equations (3.1) and (3.4) and equating coefficients we obtain $\alpha = (\lambda + y)^n$, $\beta = \lambda^n$ and $\delta = k$. The two equations are then equivalent. ☐

Corollary 3.1. For > 2 , all solutions to Beal's general equation (1) are solutions to the binomial identity equation (3.4).

Proof. *It is easy to see that the powers of the terms of equation (3.4) are $\geq n$. This is because they are evaluated according to the value of the coefficient k in the last term to produce Beal's equation by setting the variable $x = k$. In the case of the binomial $(\lambda x^l + \delta y^l)^n$ of power 1, k can take the value of either λ or δ and the final powers are dependent on the power of the variables l. This produces powers of the terms of Beal's equation ≥ 3 if $l \geq 3$ according to Beal's conjecture.* ☐

4. Appendix A: Beal's examples from direct expansion of binomial identities (1)

The following are examples of numerical solutions of Beal's equation derived from direct expansion of binomial identity (1).

From binomial of power 3 with signature $a, b, c = (3, 4, 3)$:

Example 4.1 $(3x + y)^3$ gives,

$$(3x + x)^3 = 3^3x^3 + 3 \cdot 3^2x^3x + 3x^2 \cdot 3x + x^3$$

$$(4x)^3 = 3^3x^3 + 37x^3$$

Which upon substituting x with numerical value 37 gives Beal's equation,

$$111^3 + 37^4 = 148^3$$

Example 4.2 $(4x + y)^3$ gives,

$$(4x + x)^3 = 4^3x^3 + 3 \cdot 16 \cdot x^2x + 3x^2 \cdot 4x + x^3$$

Taking 4^3x^3 as the leading term since it makes a perfect power and simplifying we get,

$$(5x)^3 = 4^3x^3 + 61x^3$$

Which upon substituting x with numerical value 61 gives Beal's equation,

$$244^3 + 61^4 = 305^3$$

If we take the perfect power term x^3 as the leading term we get Beal's equation,

$$124^3 + 124^4 = 620^3$$

Example 4.3 $(5x + 3y)^3$ gives,

$$(5x + 3x)^3 = 5^3x^3 + 3 \cdot 9x^2 \cdot 5x + 3 \cdot 25x^2 \cdot 3x + 3^3x^3$$

$$(8x)^3 = 5^3x^3 + 387x^3$$

Which upon substituting x with numerical value 387 gives Beal's equation,

$$1935^3 + 387^4 = 3096^3$$

From binomial of power 4 with signature $\boldsymbol{a, b, c = (4, 5, 4)}$:

Example 4.4 $(x + y)^4$ gives,

$$(x + x)^4 = x^4 + 4x^3 \cdot x + 6x^2 \cdot x^2 + 4x \cdot x^3 + x^4$$

$$(2x)^4 = x^4 + 15x^4$$

Which upon substituting x with numerical value 15 gives Beal's equation,

$$15^4 + 15^5 = 30^4$$

Example 4.5 $(2x + y)^4$ gives,

$$(2x + x)^4 = 2^4x^4 + 4 \cdot 2^3 \cdot x^3 \cdot x + 6 \cdot 4x^2 \cdot x^2 + 4 \cdot 2x \cdot x^3 + x^4$$

$$(3x)^4 = 2^4x^4 + 65x^4$$

Which upon substituting x with numerical value 65 gives Beal's equation,

$$130^4 + 65^5 = 195^4$$

From binomial of power 5 with signature $\boldsymbol{a, b, c} = (\mathbf{5, 6, 5})$:

Example 4.6 $(x+y)^5$ gives,

$$(x+x)^5 = x^5 + 5x^4 \cdot x + 10x^3 \cdot x^2 + 10x^2 \cdot x^3 + 5x \cdot x^4 + x^5$$

$$(2x)^5 = x^5 + 31x^5$$

Which upon substituting x with numerical value 31 gives Beal's equation,

$$31^5 + 31^6 = 62^5$$

Example 4.7 $(3x+y)^5$ gives,

$$(3x+x)^5 = 3^5x^5 + 5 \cdot 3^4x^4 \cdot x + 10 \cdot 3^3x^3 \cdot x^2 + 10 \cdot 3^2x^2 \cdot x^3 \\ +5 \cdot 3x \cdot x^4 + x^5$$

$$(4x)^5 = 3^5x^5 + 781x^5$$

Which upon substituting x with numerical value 781 gives Beal's equation,

$$2343^5 + 781^6 = 3124^5$$

From binomial of power 7 with signature $\boldsymbol{a, b, c} = (\mathbf{7, 8, 7})$:

Example 4.8 $(2x+y)^7$ gives,

$$(2x+x)^7 = 2^7x^7 + 7 \cdot 2^6x^6 \cdot x + 21 \cdot 2^5x^5 \cdot x^2 + 35 \cdot 2^4x^4 \cdot x^3 + 35 \cdot 2^3x^3 \cdot x^4 + 21 \cdot \\ 2^2x^2 \cdot x^5 + 7 \cdot 2x \cdot x^6 + x^7$$

$$(3x)^7 = 2^7x^7 + 2059x^7$$

Which upon substituting x with numerical value 2059 gives Beal's equation,

$$4118^7 + 2059^8 = 6177^7$$

Beal's equations of orders different than $(n, n+1, n)$ may be produced by binomial of power 1, characterized by equation (2) as shown in appendix B.

5. Appendix B: Beal's examples following the general equation (2)

The following are examples of numerical solutions of Beal's general equation $\alpha x^l + \beta x^l = \delta x^l$. The examples show how we can take the GCD of x^l on the LHS of the equation and combine it with the sum of the coefficients by the power rules to produce the RHS.

Example 5.1 The equation $70^3 + 105^3 = 35^4$ complies with Beal's conjecture. Factoring the GCD of 35^3 from the LHS we obtain $(3^3 + 2^3)\,35^3$. Simplifying we obtain $35 \cdot 35^3 = 35^4$; the RHS of the equation. The characteristic x^l here is 35^3 and the point of intersection with the line $x - y = 0$ is $(35, 35)$, and the corresponding binomial of power 1 is $(3^3\,x^3 + 2^3 y^3)^1$.

Example 5.2 For the equation $7^6 + 7^7 = 98^3$, taking CD of 7^3 from the LHS we get $(7^3 + 7^4)\,7^3$. The sum of the coefficient terms yields 2744 which can be shaped to 14^3 by taking the third root, which produces the RHS of the equation upon combining the terms by the power rule of the product of two numbers having the same exponent. If we factor out the GCD of 7^6 from the LHS of the equation, the expression becomes $(1 + 7)7^6$ and can further be expressed as $2^3 \cdot 7^6$. Simplifying we get $2^3 \cdot 49^3 = 98^3$; the RHS. This example works with two possible CF because the GCF of the characteristic x^l can be shaped to $x^l = x^{2n}$ representing $x^l = 7^6$ and $x^n = 7^3$.

Example 5.3 For the equation $34^5 + 51^4 = 85^4$, factoring the GCD 17^4 gives $2^5 \cdot 17 \cdot 17^4 + 3^4 \cdot 17^4 = 5^4 \cdot 17^4 = 85^4$; the RHS. The characteristic x^l here is 17^4 and the point of intersection with the line $x - y = 0$ is $(17, 17)$.

Example 5.4 The LHS of the equation $760^3 + 456^3 = 152^4$ can be factored to the product of base primes and becomes $5^3 \cdot 2^9 \cdot 19^3 + 3^3 \cdot 2^9 \cdot 19^3$. The two terms now can be combined to yield $(3^3 + 5^3)\,2^9 \cdot 19^3$, and by shaping 2^9 to 8^3 the expression becomes $(3^3 + 5^3)\,8^3 \cdot 19^3$ with a GCD of 152^3 to yield $152 \cdot 152^3 = 152^4$; the RHS. The characteristic x^l here is 152^3 and the point of intersection with the line $x - y = 0$ is $(152, 152)$.

Example 5.5 Let's consider the equation $27^4 + 162^3 = 9^7$. By factoring the GCD 27^4 we get $(1 + 2^3)27^4$, which becomes $3^2 \cdot 3^{12}$ and produces 3^{14}, which can be shaped to produce 9^7; the RHS of the equation. The characteristic x^l here is 27^4 and the point of intersection with the line $x - y = 0$ is $(27, 27)$. It is important to make sure that the sum-term on the RHS of the equation has not been shaped differently before we judge whether the resulting equation is identical to the given one.

Example 5.6 Another example to beware of the end result as deemed different is the equation $33^5 + 66^5 = 1089^3$. The GCD on the LHS of the equation is 33^5. Simplifying we get $(1 + 32)33^5 = 33^6$, which can easily be shaped to 1089^3; the RHS of the equation. The characteristic x^l here is 33^5 and the point of intersection with the line $x - y = 0$ is $(33, 33)$.The same goes with the equation $8^3 + 8^3 = 4^5$; we get the sum as 2^{10} or 32^2 which can be shaped to 4^5. The last example simply can be simplified by shaping the terms to $2^9 + 2^9 = 2^{10}$, which simplifies as $2 \cdot 2^9 = 2^{10}$. It is obvious here that both terms on the LHS have the same prime base-unit of 2 no matter what shape the terms take. Compare here with the special case of the equation of a circle in the introduction of $x^2 + y^2 = c^z$ that produced the equation $2^2 + 2^2 = 2^3$ at the intersection point of (2,2) with the line $x - y = 0$, and by Beal's condition of single-power integer terms. Also compare with the case of the vector space $a3^l$, of variable equals 3, we can add three terms and get $3^l + 3^l + 3^l = 3^{l+1}$ and we can add four elements in the vector space $a4^l$, of variable equals 4, to get $4^l + 4^l + 4^l + 4^l = 4^{l+1}$ and so on.

Example 5.7 By factoring the GCD of 19^3 from the LHS of the equation $19^4 + 38^3 = 57^3$ we obtain $(19 + 8)\ 19^3$. Simplifying we get $27 \cdot 19^3$ which by shaping 27 becomes $3^3 \cdot 19^3$ and yields the RHS of the equation. The characteristic x^l here is 19^3 and the point of intersection with the line $x - y = 0$ is $(19, 19)$. This is the example in the introduction with corresponding binomial of power 3 is $(2x + y)^3$.

Example 5.8 By factoring out the GCD of 80^{12} from the LHS of the equation $80^{12} + 80^{13} = 1536000^4$ we obtain $(1 + 80)\ 80^{12}$. Simplifying we get $81 \cdot 80^{12}$ which becomes $3^4 \cdot 80^{12}$, and by shaping 80^{12} as 512000^4 we get the RHS of the equation. The characteristic x^l here is 80^{12} and the point of intersection with the line $x - y = 0$ is $(80, 80)$.

Example 5.9 By factoring out the GCD of 28^3 from the LHS of the equation $84^3 + 28^3 = 28^4$ we obtain $(27 + 1)\ 28^3$. Simplifying, we get $28 \cdot 28^3$ which becomes the RHS. The characteristic x^l here is 28^3 and the point of intersection with the line $x - y = 0$ is $(28, 28)$.

Example 5.10 By factoring out the GCD of 1838^3 from the LHS of the equation $1838^3 + 97414^3 = 5514^4$ we obtain $(1 + 148877)\ 1838^3$. By borrowing 1838 factor from the coefficient term and simplifying we get $81 \cdot 1838^4$. The 81 can be shaped to 3^4 and the product yields the RHS. The characteristic x^l here is 1838^4 and the point of intersection with the line $x - y = 0$ is $(1838, 1838)$.

Remark The author has checked many of Beal's numerical equations and found that they all comply with the assumption that Beal's equation is an identity with characteristic equation as expressed in equations (1) and (2).

6. Conclusion

The general equation of Beal's conjecture is identified as a univariate algebraic identity derived from algebraic expansion of powers of binomials and defined at the points of intersection of the identity equation with the line $x - y = 0$. Therefore the equation is characterized by a common bound variable that defines a common factor. The identity was represented by the addition of two vectors in the vector space of the set of all polynomials in the form $p(x) = a\,x^l$ for $a \in \mathbb{Q}$ as a subspace of the infinite vector space over $\mathbb{Q}$ of all polynomials with basis $1, x, x^2$ ... The identity was contrasted with the addition of two fractions that produces similar fractional-identity equation. As an integer solution, it was found that Beal's general equation is a binomial identity. It is concluded that identifying Beal's equation as a univariate binomial identity presents a proof to Beal's conjecture.

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