DERIVATION OF INTERACTIONS FROM DIRAC EQUATION

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Abstract: In this work, we show that a massless physical field that accompanies a massive particle can be derived from Dirac equation, such as an electron is accompanied by the Coulomb electrostatic field, and we show that Dirac equation can also be generalised to form a field equation to describe internal dynamics of massless physical fields by considering the components of the momentum operators as matrix operators rather than scalar operators as in the original Dirac equation. One of many remarkable results that can be obtained from the generalised Dirac field equation is a linear potential that may be used to describe the quark confinement at large distances in the quark model.

Dirac highly influential relativistic wave equation, which brought together Einstein special theory of relativity and quantum mechanics, has played a significant role in the development of quantum physics. Even though originally Dirac equation was formulated to describe the dynamics of an electron, the consequences of the equation are far-reaching such as it correctly predicted the existence of antimatter. This result raises the question of whether Dirac equation is in fact a more fundamental equation in physics than it was originally formulated for, for example, whether Dirac equation can play a similar role in quantum mechanics to Newton's second law of classical mechanics. Along this line of speculation, in this work first we will show that Dirac equation can also be used to describe a massless physical field that accompanies a massive particle whose dynamics is also described by Dirac equation, such as an electron is accompanied by the Coulomb electrostatic field or the Newton gravitational field. Then we will show that Dirac equation can be generalised to describe internal dynamics of massless physical fields by considering the components of the momentum operators as matrix operators rather than scalar operators as in the original Dirac equation. One of many different solutions that can be obtained from the generalised Dirac equation is Hooke's law. This is a remarkable result since Dirac equation not only provides an inverse square law but also a linear law that can be combined to represent the potential between two quarks in the form $V = k_1/r + k_2 r$, where the first term representing a gluon exchange at short distances and the second term associating with quark confinement at large distances. In the Minkowski spacetime of Einstein special relativity with the pseudo-Euclidean metric defined by $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$, the energy-momentum relationship is given as [1]

$$E^2 = (mc^2)^2 + (pc)^2 \tag{1}$$

From this relationship, the Dirac relativistic first order partial differential equation can be formulated by proposing that it is of the form [2]

$$E\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m\right)\psi \tag{2}$$

where the unknown operators α_i and β are assumed to be independent of the momentum p and the mass m. From Equation (2), we obtain

$$E^2\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m\right)^2\psi \tag{3}$$

By expanding Equation (3), and if we assume that all linear momentum operators commute mutually then in order to reduce to the form of the relationship between the energy and the momentum given in Equation (1), the operators α_i and β must satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{4}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{5}$$

$$\alpha_i^2 = 1 \tag{6}$$

$$\beta^2 = 1 \tag{7}$$

As shown in Appendix 1, to satisfy the conditions given by the relations (4-7), the operators α_i and β can be represented as matrices as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{8}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

where σ_i are Pauli matrices given by $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If we multiply Equation (2) by the operator β on the left and apply the method of quantisation in quantum mechanics in which the energy E, the momentum p and the mass m are replaced by differential operators

$$E \to i\frac{\partial}{\partial t}, \quad p_x \to -i\frac{\partial}{\partial x}, \quad p_y \to -i\frac{\partial}{\partial y}, \quad p_z \to -i\frac{\partial}{\partial z} \quad \text{and} \quad m \to m \tag{10}$$

then with the mathematical units in which $\hbar = c = 1$, Dirac equation given in Equation (2) can be rewritten in a covariant form as

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0 \tag{11}$$

where $\partial_\mu = \left(\partial_t, \partial_x, \partial_y, \partial_z\right)$, $\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$. Using the γ^i operators given in Equation (15) in Appendix 1, Dirac equation given in Equation (11) can be written out in full form for the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ as

$$i\frac{\partial\psi_1}{\partial t}+i\frac{\partial\psi_3}{\partial z}+i\frac{\partial\psi_4}{\partial x}+\frac{\partial\psi_4}{\partial y}=m\psi_1 \tag{12}$$

$$i\frac{\partial\psi_2}{\partial t}-i\frac{\partial\psi_4}{\partial z}+i\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_3}{\partial y}=m\psi_2 \tag{13}$$

$$-i\frac{\partial\psi_3}{\partial t}-i\frac{\partial\psi_1}{\partial z}-i\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_2}{\partial y}=m\psi_3 \tag{14}$$

$$-i\frac{\partial\psi_4}{\partial t}+i\frac{\partial\psi_2}{\partial z}-i\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_1}{\partial y}=m\psi_4 \tag{15}$$

The system of equations given in Equations (12-15) is normally used to describe massive quantum particles with spin of half-integral values. This system of equations can also be applied to massless particles that leads to Weyl equation written out as follows [3]

$$\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_4}{\partial x}-i\frac{\partial\psi_4}{\partial y}+\frac{\partial\psi_3}{\partial z}=0 \tag{16}$$

$$\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_3}{\partial x}+i\frac{\partial\psi_3}{\partial y}-\frac{\partial\psi_4}{\partial z}=0 \tag{17}$$

$$\frac{\partial\psi_3}{\partial t}+\frac{\partial\psi_2}{\partial x}-i\frac{\partial\psi_2}{\partial y}+\frac{\partial\psi_1}{\partial z}=0 \tag{18}$$

$$\frac{\partial\psi_4}{\partial t}+\frac{\partial\psi_1}{\partial x}+i\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial z}=0 \tag{19}$$

The Weyl relativistic wave equation given in Equations (16-19) is a time-dependent system of equations that is used to describe massless particles with spin of half-integral values. Now, if the wavefunction $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T$ is time-independent then the system reduces to the following system of equations

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_4+\frac{\partial\psi_3}{\partial z}=0 \tag{20}$$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_3-\frac{\partial\psi_4}{\partial z}=0 \tag{21}$$

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_2+\frac{\partial\psi_1}{\partial z}=0 \tag{22}$$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_1-\frac{\partial\psi_2}{\partial z}=0 \tag{23}$$

We would like to show that the system of time-independent equations given in Equations (20-23) can be solved to give solutions that can be interpreted in terms of classical fields such as the Coulomb electrostatic field or the Newton gravitational field. By applying the differential operator $(\partial/\partial x + i\,\partial/\partial y)$ to Equation (20) and using Equation (21), we obtain Laplace equation for the component ψ_4 of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ as given below

$$\frac{\partial^2 \psi_4}{\partial x^2} + \frac{\partial^2 \psi_4}{\partial y^2} + \frac{\partial^2 \psi_4}{\partial z^2} = 0 \tag{24}$$

Similarly, it can be shown that all components of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ satisfy Laplace equation $\nabla^2 \psi_\mu = 0,\ \mu = 1, 2, 3, 4$. In this case it is seen that all components of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ can be used to describe an irrotational and incompressible flow similar to a steady fluid flow in fluid dynamics where the potential flow satisfies Laplace equation. If $\mathbf{v}$ is the velocity of an irrotational flow then it can be defined in terms of a potential flow φ as $\mathbf{v} = \boldsymbol{\nabla}\varphi$. If the flow is incompressible then $\boldsymbol{\nabla}.\mathbf{v} = 0$, therefore $\nabla^2 \varphi = 0$ [4]. Furthermore, since the components are time-independent, they are basically three-dimensional physical objects and solutions to Laplace equation $\nabla^2 \psi_\mu = 0$ can be written in the form

$$\psi_\mu(x, y, z) = \frac{k}{\sqrt{x^2 + y^2 + z^2}}, \quad \mu = 1, 2, 3, 4 \tag{25}$$

Therefore it could be suggested that accompanying to each massive quantum particle whose dynamics is described by Dirac equation is a massless physical field that that can be described by a massless time-independent Weyl equation. For example, for the case of an electron, the electron itself is described by the massive Dirac equation but the physical field that accompanies the electron is described the field given by Equation (25), which is either the Coulomb electrostatic field or the Newton gravitational field. Now, a more subtle question that arises is whether other forms of physical interactions can be derived from Dirac equation. It seems in order to answer this question we would need to generalise Dirac equation in some way to obtain extra dynamical variables and the generalised formulation will reduce to the original Dirac equation when the new dynamical variables vanish. In the following we will address this particular question.

In the above discussions we assumed that all linear momentum operators are treated as normal differential operators therefore they commute mutually. However, it is observed that it is possible to generalise to formulate the momentum operators p_i so that they can be represented in the same formulation as the operators α_i and β if we write the momentum operators p_i in the original Dirac equation as follows

$$p_i = -i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial x^i} \tag{26}$$

With this new form of formulation now we can generalise the momentum operators to take a more general form as

$$p_i = -i\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix}\frac{\partial}{\partial x^i} \tag{27}$$

where a_i are undetermined constants. As shown in Appendix 2, if the momentum operators p_i satisfy the following conditions

$$p_i p_j = p_i p_j \tag{28}$$

$$\alpha_i p_j = p_i \alpha_j \tag{29}$$

$$\beta p_j = p_i \beta \tag{30}$$

then the momentum operators given in Equation (27) reduce to the original form given in Equation (26) for massive particles. However, for a massless physical field with $m = 0$, the condition given in Equation (30) is not required and in this case the momentum operators p_i can take the following form

$$p_i = -i\begin{pmatrix} 1 & 0 & a_i & 0 \\ 0 & 1 & 0 & a_i \\ a_i & 0 & 1 & 0 \\ 0 & a_i & 0 & 1 \end{pmatrix}\frac{\partial}{\partial x^i} \tag{31}$$

It is seen that massless physical fields that are described by the generalised Dirac equation with the momentum operators given by Equation (31) may have internal geometric dynamics. Using the momentum operators given in Equation (31), the generalised Dirac equation for massless physical fields can be written as

$$\frac{\partial \psi}{\partial t} = -\left(\begin{pmatrix} 0 & a_x & 0 & 1 \\ a_x & 0 & 1 & 0 \\ 0 & 1 & 0 & a_x \\ 1 & 0 & a_x & 0 \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & -ia_y & 0 & -i \\ ia_y & 0 & i & 0 \\ 0 & -i & 0 & -ia_y \\ i & 0 & ia_y & 0 \end{pmatrix}\frac{\partial}{\partial y} + \begin{pmatrix} a_z & 0 & 1 & 0 \\ 0 & -a_z & 0 & -1 \\ 1 & 0 & a_z & 0 \\ 0 & -1 & 0 & -a_z \end{pmatrix}\frac{\partial}{\partial z}\right)\psi \tag{32}$$

In terms of components, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, the generalised Dirac equation for massless physical fields given in Equation (32) written out out in full form as follows

$$\frac{\partial \psi_1}{\partial t} + a_x\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_4}{\partial x} - ia_y\frac{\partial \psi_2}{\partial y} - i\frac{\partial \psi_4}{\partial y} + a_z\frac{\partial \psi_1}{\partial z} + \frac{\partial \psi_3}{\partial z} = 0 \tag{33}$$

$$\frac{\partial \psi_2}{\partial t} + a_x\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_3}{\partial x} + ia_y\frac{\partial \psi_1}{\partial y} + i\frac{\partial \psi_3}{\partial y} - a_z\frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_4}{\partial z} = 0 \tag{34}$$

$$\frac{\partial \psi_3}{\partial t} + \frac{\partial \psi_2}{\partial x} + a_x\frac{\partial \psi_4}{\partial x} - i\frac{\partial \psi_2}{\partial y} - ia_y\frac{\partial \psi_4}{\partial y} + \frac{\partial \psi_1}{\partial z} + a_z\frac{\partial \psi_3}{\partial z} = 0 \tag{35}$$

$$\frac{\partial \psi_4}{\partial t} + \frac{\partial \psi_1}{\partial x} + a_x\frac{\partial \psi_3}{\partial x} + i\frac{\partial \psi_1}{\partial y} + ia_y\frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z} - a_z\frac{\partial \psi_4}{\partial z} = 0 \tag{36}$$

If the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is time-independent then Equations (33-36) reduce to the following system of equations

$$\left(a_x \frac{\partial}{\partial x} - i a_y \frac{\partial}{\partial y}\right)\psi_2 + \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)\psi_4 + a_z \frac{\partial \psi_1}{\partial z} + \frac{\partial \psi_3}{\partial z} = 0 \tag{37}$$

$$\left(a_x \frac{\partial}{\partial x} + i a_y \frac{\partial}{\partial y}\right)\psi_1 + \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)\psi_3 - a_z \frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_4}{\partial z} = 0 \tag{38}$$

$$\left(a_x \frac{\partial}{\partial x} - i a_y \frac{\partial}{\partial y}\right)\psi_4 + \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)\psi_2 + a_z \frac{\partial \psi_3}{\partial z} + \frac{\partial \psi_1}{\partial z} = 0 \tag{39}$$

$$\left(a_x \frac{\partial}{\partial x} + i a_y \frac{\partial}{\partial y}\right)\psi_3 + \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)\psi_1 - a_z \frac{\partial \psi_4}{\partial z} - \frac{\partial \psi_2}{\partial z} = 0 \tag{40}$$

If we impose the condition that the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ does not depend on the variable z, then Equations (37-40) reduce to the following system of equations

$$\left(a_x \frac{\partial}{\partial x} - i a_y \frac{\partial}{\partial y}\right)\psi_2 + \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)\psi_4 = 0 \tag{41}$$

$$\left(a_x \frac{\partial}{\partial x} + i a_y \frac{\partial}{\partial y}\right)\psi_1 + \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)\psi_3 = 0 \tag{42}$$

$$\left(a_x \frac{\partial}{\partial x} - i a_y \frac{\partial}{\partial y}\right)\psi_4 + \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)\psi_2 = 0 \tag{43}$$

$$\left(a_x \frac{\partial}{\partial x} + i a_y \frac{\partial}{\partial y}\right)\psi_3 + \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)\psi_1 = 0 \tag{44}$$

It is observed from Equations (41-44) that there are two pairs of components that couple to each other, the component $\psi_1$is coupled to the component ψ_3, and the component $\psi_2$is coupled to the component ψ_4. It is worth mentioning here that a two-dimensional space may play a significant role in Schrödinger wave mechanics because Schrödinger wave equation in two-dimensional space can be used to show that the angular momentum of a particle can take half-integral values [5]. Now, if we apply the differential operator $\left(a_x \partial/\partial x \pm i a_y \partial/\partial y\right)$ to one of the equations and use the other equation of the coupled component then it can be checked that all components ψ_μ, $i = 1, 2, 3, 4$ of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ satisfy the following equation

$$\left(\left(a_x \frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2 - \left(a_y \frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \pm 2 i a_x a_y \frac{\partial}{\partial x}\frac{\partial}{\partial y} \mp 2i \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)\psi_\mu = 0 \tag{45}$$

Depending on the values of a_x and a_y we have various solutions to Equation (45). However, since many solutions are similar therefore for simplicity we will consider only the following combinations of the values of a_x and a_y.

- If $a_x = 1$ and $a_y = 1$ then we can assume that $\psi_2 = -\psi_4 = f(x, y)$

and $\psi_1 = -\psi_3 = g(x, y)$, where f and g are arbitrary functions.

- If $a_x = 1$ and $a_y = -1$ then we have

$$\frac{\partial^2 \psi_\mu}{\partial x \partial y} = 0 \tag{46}$$

and in this case ψ_μ is a linear function of x and y, $\psi_\mu = ax + by$, where a and b are arbitrary constants.

- If $a_x = i$ and $a_y = 1$ then we have

$$\frac{\partial^2 \psi_\mu}{\partial x^2} \pm 2(1+i) \frac{\partial^2 \psi_\mu}{\partial x \partial y} = 0 \tag{47}$$

A general solution to Equation (47) is given in the form

$$\psi_\mu = f\big((2+2i)x \mp y\big) \tag{48}$$

where f is an arbitrary function.

- If $a_x = 1$ and $a_y = i$ then we have

$$\frac{\partial^2 \psi_\mu}{\partial y^2} \pm 2(1+i) \frac{\partial^2 \psi_\mu}{\partial x \partial y} = 0 \tag{49}$$

A general solution to Equation (49) is given in the form

$$\psi_\mu = f(x \mp (2+2i)y) \tag{50}$$

- If $a_x = i$ and $a_y = -i$ then we have

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \psi_\mu = 0 \tag{51}$$

This is a hyperbolic equation of second order which has a general solution given in terms of two arbitrary functions f and g as

$$\psi_\mu = f(x+y) + g(x-y) \tag{52}$$

A particular form of solutions to Equation (52) is given by the following

$$\psi_\mu(x, y) = \frac{1}{2} k r^2, \quad \text{where} \quad r^2 = x^2 + y^2 \tag{53}$$

Similar to the case when the components ψ_μ could be interpreted as the electrostatic or gravitational potential, it is seen that in the present situation the components ψ_μ could be interpreted a harmonic potential which results in the Hooke's law $F = -kr$. This is remarkable result since Dirac equation not only provides an inverse square law but also a linear law and if these two laws combine they can be used to represent the potential between two quarks in the form $V = k_1/r + k_2 r$, where the first term representing a gluon exchange at short distances and the second term associating with quark confinement at large distances [3].

Appendix 1

Assume the operators α_i are represented in terms of the operators σ_i in the forms

$$\alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{1}$$

then we obtain

$$\alpha_i^2 = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} \tag{2}$$

If $\sigma_i^2 = 1$ then $\alpha_i^2 = 1$. On the other hand, if $\sigma_i^2 = -1$ then $\alpha_i^2 = -1$. Now, if the operators α_i are given in the forms

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3}$$

then in this case we have

$$\alpha_i^2 = \begin{pmatrix} -\sigma_i^2 & 0 \\ 0 & -\sigma_i^2 \end{pmatrix} \tag{4}$$

If $\sigma_i^2 = 1$ then $\alpha_i^2 = -1$. On the other hand, if $\sigma_i^2 = -1$ and then $\alpha_i^2 = 1$.

Now, if we write the operator σ_i as a two by two matrix in the form

$$\sigma_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

then from the requirement $\sigma_i^2 = 1$, we arrive at the following system of equations for the unknown quantities a, b, c and d

$$a^2 + bc = 1 \tag{6}$$

$$b(a + d) = 0 \tag{7}$$

$$c(a + d) = 0 \tag{8}$$

$$d^2 + bc = 1 \tag{9}$$

From Equations (6) and (9) we require $d = \pm a$. If $d = a \neq 0$ then $b = c = 0$ and the operator σ_i can take the values $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. If $d = -a$ and if $b = c = 0$, then the operator σ_i can be as $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. If $d = -a$ but $b \neq 0$ and $c \neq 0$, then the operator σ_i can be written in the form $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. These are only a few standard representations of the operators σ_i. It is also seen from the representations of the operators α_i given in Equations (1) that there are many different combinations that can be chosen for the operators α_i and β to satisfy the following relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \qquad \text{for } i \neq j \tag{10}$$

$$\beta \alpha_i + \alpha_i \beta = 0 \tag{11}$$

$$\alpha_i^2 = 1 \tag{12}$$

$$\beta^2 = 1 \tag{13}$$

The most common use of the forms of the operators α_i is that they are defined in terms of Pauli matrices $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$. In this case the operators α_i are found as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{14}$$

In addition, if the operator β is defined in terms of the operators σ_i as $\beta = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$ then with $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the operator β takes the form

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

Using $\gamma^i = \beta \alpha_i$ and $\gamma^0 = \beta$, the γ^i operators are calculated and arranged in the following order

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{15}$$

Appendix 2

In Appendix 2 we will consider the case when the momentum operators are assumed to be represented in the same formulation as the operators α_i, β. If we assume Dirac equation of the form

$$E\psi = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)\psi \tag{1}$$

then we have

$$E^2\psi = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)^2\psi \tag{2}$$

By expanding Equation (2) we obtain

$$\begin{aligned} E^2\psi = (&\alpha_1 p_1 \alpha_1 p_1 + \alpha_1 p_1 \alpha_2 p_2 + \alpha_1 p_1 \alpha_3 p_3 + \alpha_1 p_1 \beta m + \alpha_2 p_2 \alpha_1 p_1 + \alpha_2 p_2 \alpha_2 p_2 \\ &+ \alpha_2 p_2 \alpha_3 p_3 + \alpha_2 p_2 \beta m + \alpha_3 p_3 \alpha_1 p_1 + \alpha_3 p_3 \alpha_2 p_2 + \alpha_3 p_3 \alpha_3 p_3 + \alpha_3 p_3 \beta m \\ &+ \beta m \alpha_1 p_1 + \beta m \alpha_2 p_2 + \beta m \alpha_3 p_3 + \beta m \beta m)\psi \end{aligned} \tag{3}$$

If the operators α_i, β and p_i satisfy the conditions

$$\alpha_i p_j = p_i \alpha_j \tag{4}$$

$$\beta p_j = p_i \beta \tag{5}$$

then Equation (3) becomes

$$\begin{aligned} E^2\psi = (&\alpha_1 \alpha_1 p_1 p_1 + \alpha_1 \alpha_2 p_1 p_2 + \alpha_1 \alpha_3 p_1 p_3 + \alpha_1 \beta p_1 m + \alpha_2 \alpha_1 p_2 p_1 + \alpha_2 \alpha_2 p_2 p_2 \\ &+ \alpha_2 \alpha_3 p_2 p_3 + \alpha_2 \beta p_2 m + \alpha_3 \alpha_1 p_3 p_1 + \alpha_3 \alpha_2 p_3 p_2 + \alpha_3 \alpha_3 p_3 p_3 + \alpha_3 \beta p_3 m \\ &+ \beta \alpha_1 m p_1 + \beta \alpha_2 m p_2 + \beta \alpha_3 m p_3 + \beta \beta m m)\psi \end{aligned} \tag{6}$$

If furthermore p_i satisfy the conditions $p_i p_j = p_i p_j$ then Equation (6) becomes

$$\begin{aligned} E^2\psi = (&\alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 + \alpha_3^2 p_3^2 + (\alpha_1\alpha_2 + \alpha_2\alpha_1) p_1 p_2 + (\alpha_1\alpha_3 + \alpha_3\alpha_1) p_1 p_3 \\ &+ (\alpha_2\alpha_3 + \alpha_3\alpha_2) p_2 p_3 + (\alpha_1\beta + \beta\alpha_1) p_1 m + (\alpha_2\beta + \beta\alpha_2) p_2 m \\ &+ (\alpha_3\beta + \beta\alpha_3) p_3 m + \beta^2 m^2)\psi \end{aligned} \tag{7}$$

Using the conditions given in Equations (10-13) in Appendix 1, Equation (7) reduces to the relativistic energy-momentum relationship. Now, if we write the momentum operators p_i in the original Dirac equation as follows

$$p_i = -i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial x^i} \tag{8}$$

then with this new form of formulation we can generalise the momentum operators to take a more general form as

Now we assume the operator p_i to take a more general form given below

$$p_i = -i\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix}\frac{\partial}{\partial x^i} \tag{9}$$

where a_i are undetermined constants. If $\alpha_1 p_i = p_i \alpha_1$ then the operator p_i given in Equation (9) reduces to

$$p_i = -i\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_8 & a_7 & a_6 & a_5 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix}\frac{\partial}{\partial x^i} \tag{10}$$

If $\alpha_2 p_i = p_i \alpha_2$ then the operator p_i given in Equation (10) is simplified further to

$$p_i = -i\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & 0 & a_8 \\ a_8 & 0 & a_6 & 0 \\ 0 & a_3 & 0 & a_1 \end{pmatrix}\frac{\partial}{\partial x^i} \tag{11}$$

If $\alpha_3 p_i = p_i \alpha_3$ then the operator p_i satisfying the conditions $\alpha_i p_j = p_i \alpha_j,\ i, j = 1, 2, 3$ takes the general form

$$p_i = -i\begin{pmatrix} b & 0 & a & 0 \\ 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \end{pmatrix}\frac{\partial}{\partial x^i} \tag{12}$$

where a and b are undetermined constants. With the form given in Equation (12) it can be checked that the operators p_i satisfy the conditions $p_i p_j = p_i p_j,\ i, j = 1, 2, 3$. Finally, we need to check the conditions $\beta p_j = p_i \beta$. Using the operator p_i given in Equation (12) and the operator β, the conditions $\beta p_j = p_i \beta$ require $a = -a$. For massive particles we need to use the operator β therefore $a = 0$ and in this case the p_i must be of the form

$$p_i = -i\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}\frac{\partial}{\partial x^i} \tag{13}$$

and we recover Dirac equation by setting $b = 1$. However, for massless particles the p_i can take the more general form as

$$p_i = -i\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ a & 0 & 1 & 0 \\ 0 & a & 0 & 1 \end{pmatrix}\frac{\partial}{\partial x^i} \tag{14}$$

If we write $a = a_i$ then we have

$$p_i = -i\begin{pmatrix} 1 & 0 & a_i & 0 \\ 0 & 1 & 0 & a_i \\ a_i & 0 & 1 & 0 \\ 0 & a_i & 0 & 1 \end{pmatrix}\frac{\partial}{\partial x^i} \tag{22}$$

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