On velocity gauge field approach of interactions

Jean Pierre Rukundo

Email: rukundojp@hendrix.edu

November 26th, 2017

Abstract

A unified Mathematical framework which describes particles interactions both in classical field theory of gravity and quantum field theory is presented, by introducing a velocity gauge field. The velocity gauge field Lagrangian is developed and shown to be equivalent to the Dirac field and related to the Electromagnetic field. The velocity fields interactions are shown to produce the Einstein-Hilbert action Lagrangian. Similar extension of the Mathematical framework to the Electromagnetic field produces the Quantum electrodynamics and Yang-Mills fields Lagrangians. Finally, a unified Lagrangian describing both velocity field and Electromagnetic field types is presented and alternative formulation suggested.

I. Introduction

In this work, a velocity gauge field Lagrangian is developed by extending the classical free particle Lagrangian. It is shown to be equivalent to the Dirac field and obeying an equation similar to that of Dirac and the relation to the Electromagnetic field is shown. The velocity fields interactions are shown to produce the Einstein-Hilbert action Lagrangian, which describe the classical gravitational field interaction with matter, in a special case. Similar extension of the Mathematical framework to the Electromagnetic field produces the Quantum electrodynamics and Yang-Mills fields Lagrangians, which are known to describe fundamental particles, in a special case as well. Finally, a unified Lagrangian describing both velocity field and Electromagnetic field types is presented.

To describe the velocity gauge field, the Mathematics described in article [1] is used. Vectors like four-velocity U, four-potential A and field ϕ have the form

$$U=\begin{bmatrix}U_{ct}\\ iU_x\\ iU_y\\ iU_z\end{bmatrix}, A=\begin{bmatrix}A_{ct}\\ iA_x\\ iA_y\\ iA_z\end{bmatrix}\ and\ \phi=\begin{bmatrix}\phi_{ct}\\ i\phi_x\\ i\phi_y\\ i\phi_z\end{bmatrix} \tag{1}$$

Alpha matrices α^{μ} described in article [1] are used and they are

$$\alpha^1=\begin{bmatrix}0&-i&0&0\\ i&0&0&0\\ 0&0&0&-i\\ 0&0&i&0\end{bmatrix}, \alpha^2=\begin{bmatrix}0&0&-i&0\\ 0&0&0&i\\ i&0&0&0\\ 0&-i&0&0\end{bmatrix}, \alpha^3=\begin{bmatrix}0&0&0&-i\\ 0&0&-i&0\\ 0&i&0&0\\ i&0&0&0\end{bmatrix} \text{ and}$$

$$\alpha^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where α^1, α^2 and α^3satisfies the commutation and anticommutation relations similar to that of Pauli spin matrices

$$\left[\alpha^i, \alpha^j\right] = 2i\epsilon_{ijk}\alpha^k \ \ and \ \left\{\alpha^i, \alpha^j\right\} = 2\delta^{ij} I \tag{3}$$

Let us introduce matrices Γ^μ similar to the Dirac gamma matrices, but with 64 components (8 by 8 matrices)

$$\Gamma^\mu = \begin{bmatrix} 0 & \alpha^\mu \\ \alpha^{\mu T} & 0 \end{bmatrix} \tag{4}$$

where the new gamma matrices Γ^μ satisfy anticommutations relations just like Dirac gamma matrices

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} I \tag{5}$$

II. Free particle

1. Lagrangian

Let us introduce the velocity gauge field Lagrangian of a free particle with four-velocity U,

$$\mathcal{L} = \rho U^T U - \frac{c^2}{4\pi G}\left[\alpha^\mu \partial_\mu U\right]^T \left[\alpha^\nu \partial_\nu U\right] \tag{6}$$

If $\left[\alpha^\mu \partial_\mu U\right] = 0$, the Lagrangian in equation (6) is reduced to the classical free particle Lagrangian. Also, the Lagrangian in equation (6) is invariant under a boost $e^{\frac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\eta}}U$ as described in article [1], which is equivalent to the gauge invariance.

Solving the Euler–Lagrange equation for equation (6), yields the wave equation

$$\Box^2 U = -\frac{4\pi G}{c^2}\rho U \tag{7}$$

2. Maxwell's equations form

Let us introduce the field F with

$$\alpha^\mu \partial_\mu U = F \tag{8}$$

$$\alpha^{\mu T} \partial_\mu F = -\frac{4\pi G}{c^2}\rho U \tag{9}$$

Equation (9) follow from equation (8) and (7). Also, equation (8) is analogous to the definition of the Electromagnetic four potential. Similarly, equation (9) is analogous to the Maxwell's equation and equivalent to the Gravitoelectromagnetic field equations introduced by Oliver Heaviside in 1893.

Furthermore, the other relation of the velocity field to the Electromagnetic field can be obtained using equation (7) and Electromagnetic field equation

$$\Box^2 A = \mu_0 \rho_c U \tag{10}$$

The other relation of the velocity field to the Electromagnetic field is

$$-\frac{4\pi G}{c^2}\rho \Box^2 A = \mu_0 \rho_c \Box^2 U \tag{11}$$

For constant ρ and ρ_c, one of the solutions of equation (11) is

$$-\frac{4\pi G}{c^2}\rho A = \mu_0 \rho_c U \tag{12}$$

Using equation (12) in the velocity field Lagrangian in equation (6), yields the Electromagnetic field lagrangian in the new formulation

$$\mathcal{L} = \rho_c U^T A + \frac{1}{\mu_0}\left[\alpha^\mu \partial_\mu A\right]^T \left[\alpha^\nu \partial_\nu A\right] \tag{13}$$

3. Dirac's equation form

Furthermore, let us introduce the wave function ψ by

$$\psi = \begin{bmatrix} i\hbar F \\ \frac{\hbar}{c}\sqrt{4\pi G\rho}U \end{bmatrix} \tag{14}$$

Using equations (7), (8) and (9), the wave function in equation (14) satisfies the analogous to the Dirac equation

$$\left[\mathrm{i}\hbar \Gamma^\mu \partial_\mu - \frac{\hbar}{c}\sqrt{4\pi G\rho}\right]\psi = 0 \tag{15}$$

From this analogue to the Dirac equation (15), the mass of the particle is obtained as

$$m = \frac{\hbar}{c^2}\sqrt{4\pi G\rho} \tag{16}$$

III. Two Particles interaction

1. Two velocity fields interaction

Since the Lagrangian in equation (6) is gauge invariant, a gauge covariant derivative is introduced to describe the interaction with another gauge field. In this case, the interacting gauge field is also a velocity field which also has an interaction term. The two velocity fields Lagrangian has the form

$$\begin{aligned}\mathcal{L} = \rho_1 U_1^T U_1 - \frac{c^2}{4\pi G}\left[\alpha^\mu \left[\partial_\mu - \frac{m_1}{i\hbar}U_{2\mu}\right]U_1\right]^T \left[\alpha^\nu \left[\partial_\nu + \frac{m_1}{i\hbar}U_{2\nu}\right]U_1\right] \\ + \rho_2 U_2^T U_2 - \frac{c^2}{4\pi G}\left[\alpha^\mu \left[\partial_\mu - \frac{m_2}{i\hbar}U_{1\mu}\right]U_2\right]^T \left[\alpha^\nu \left[\partial_\nu + \frac{m_2}{i\hbar}U_{1\nu}\right]U_2\right]\end{aligned} \tag{17}$$

A special case $U_1 = U_2 = U$ and $m_1 = m_2 = m$ has the Lagrangian

$$\mathcal{L} = [\rho_1 + \rho_2]U^T U - \frac{c^2}{2\pi G}\left[\alpha^\mu \left[\partial_\mu - \frac{m}{i\hbar}U_\mu\right]U\right]^T \left[\alpha^\nu \left[\partial_\nu + \frac{m}{i\hbar}U_\nu\right]U\right] \tag{18}$$

With $\alpha^{\mu^T}\alpha^\nu + \alpha^{\nu^T}\alpha^\mu = 2\eta^{\mu\nu}$,

$$\Delta = \eta^{\mu\nu}\left[\partial_\mu\partial_\nu + 2\frac{m}{i\hbar}U_\mu\partial_\nu\right] \tag{19}$$

$$R = U^T\left[\alpha^{\mu^T}\partial_\mu\left[\alpha^\nu\frac{m}{i\hbar}U_\nu\right] + \left[\alpha^{\mu^T}\frac{m}{i\hbar}U_\mu\right]\left[\alpha^\nu\frac{m}{i\hbar}U_\nu\right]\right]U \tag{20}$$

The Lagrangian in equation (18) can be evaluated to

$$\mathcal{L} = U^T\left[\rho_1 + \rho_2 + \frac{c^2}{2\pi G}\Delta\right]U + \frac{c^2}{2\pi G}R \tag{21}$$

Equation (21) is the Lagrangian of the Einstein-Hilbert action with matter field which yields the Einstein field equation of gravity.

2. Velocity field and Electromagnetic fields interaction

Following the reasoning in section III.1, the Lagrangian for the velocity field interaction with the Electromagnetic field whose Lagrangian in the new formulation as described in equation (13) has the form

$$\begin{aligned}\mathcal{L} = \rho U^T U - \frac{c^2}{4\pi G}\left[\alpha^\mu\left[\partial_\mu - \frac{q}{i\hbar}A_\mu\right]U\right]^T\left[\alpha^\nu\left[\partial_\nu + \frac{q}{i\hbar}A_\nu\right]U\right] \\ + \rho_c U_c^T A + \frac{1}{\mu_0}\left[\alpha^\mu\left[\partial_\mu - \frac{m_c}{i\hbar}U_\mu\right]A\right]^T\left[\alpha^\nu\left[\partial_\nu + \frac{m_c}{i\hbar}U_\nu\right]A\right]\end{aligned} \tag{22}$$

If $\rho_c = 0$ and $m_c = 0$, the Quantum electrodynamics Lagrangian is obtained as

$$\mathcal{L} = \rho U^T U - \frac{c^2}{4\pi G}\left[\alpha^\mu\left[\partial_\mu - \frac{q}{i\hbar}A_\mu\right]U\right]^T\left[\alpha^\nu\left[\partial_\nu + \frac{q}{i\hbar}A_\nu\right]U\right] + \frac{1}{\mu_0}\left[\alpha^\mu\partial_\mu A\right]^T\left[\alpha^\nu\partial_\nu A\right] \tag{23}$$

3. Two Electromagnetic fields interaction

Similarly to sections III.1 and III.2, the Lagrangian for two Electromagnetic fields interaction is given by

$$\begin{aligned}\mathcal{L} = \rho_{c1}U_1^T A_1 + \frac{1}{\mu_0}\left[\alpha^\mu\left[\partial_\mu - \frac{q_1}{i\hbar}A_{2\mu}\right]A_1\right]^T\left[\alpha^\nu\left[\partial_\nu + \frac{q_1}{i\hbar}A_{2\nu}\right]A_1\right] \\ + \rho_{c2}U_2^T A_2 + \frac{1}{\mu_0}\left[\alpha^\mu\left[\partial_\mu - \frac{q_2}{i\hbar}A_{1\mu}\right]A_2\right]^T\left[\alpha^\nu\left[\partial_\nu + \frac{q_2}{i\hbar}A_{1\nu}\right]A_2\right]\end{aligned} \tag{24}$$

Similarly to section III.1, a special case $U_1 = U_2 = U$, $A_1 = A_2 = A$ and $q_1 = q_2 = q$ has the Lagrangian

$$\mathcal{L} = [\rho_{c1} + \rho_{c2}]U^T A + \frac{2}{\mu_0}\left[\alpha^\mu\left[\partial_\mu - \frac{q}{i\hbar}A_\mu\right]A\right]^T\left[\alpha^\nu\left[\partial_\nu + \frac{q}{i\hbar}A_\nu\right]A\right] \tag{25}$$

Equation (25) is the Yang-Mills field form.

IV. Many Particles interaction

For N interacting particles with fields ϕ_i's of the form described in section II and III, the Lagrangian is given by

$$\mathcal{L} = \sum_i^N \left[\rho_i U_i^T \Phi_i + \frac{1}{\kappa_i} \left[\alpha^\mu \left[\partial_\mu - \sum_j^N \frac{\epsilon_{ij}}{i\hbar} \phi_{j\mu} [1 - \delta_{ij}] \right] \Phi_i \right]^T \left[\alpha^\nu \left[\partial_\nu + \sum_j^N \frac{\epsilon_{ij}}{i\hbar} \phi_{j\nu} [1 - \delta_{ij}] \right] \Phi_i \right] \right] \quad (26)$$

V. Conclusion

The Mathematical framework presented in this work provides a unified framework to describe particles interactions both in classical field theory of gravity and quantum field theory. However, the framework may not be unique, other representation using biquaternions such as the Pauli spin matrices as they are used in Dirac equation are also plausible. In Pauli matrices formulation, determinants may be used to compute vector amplitudes. It is in the author's hope that this work sparks more research in this unified field point of view to truly understand the universe.

VI. Reference

[1] Rukundo, JPR, 2016. On spacetime transformations. Quaternion Physics, [Online]. 1, 5. Available at: https://quaternionphysics.files.wordpress.com/2016/04/on-spacetime-transformations3.pdf [Accessed 29 March 2016].