On spacetime transformations

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March 29th, 2016

Abstract

In this paper, a general spacetime transformation from biquaternion left and right multiplications is introduced. First, it is shown that the transformation leaves the spacetime length invariant. Furthermore, the Lorentz transformation is derived as a special case of the general biquaternion transformation. Surprisingly, some commutation and anticommutation properties generally found in quantum mechanics are manifested by the biquaternion transformation.

I. Introduction

In 1843, William Rowan Hamilton introduced the concept of quaternion which extended the concept of complex number to three imaginary components. One year late, he introduced the concept of biquaternion which are combination of ordinary complex numbers with quaternion. More than half a century later, Albert Einstein introduced the special theory of relativity which was based on a transformation introduced by Hendrik Lorentz earlier. In this paper, it is shown that the special theory of relativity is generally based on the concept of biquaternion multiplication and a general biquaternion transformation in the form of biquaternion left and right multiplication is introduced, where the Lorentz transformation is shown to be a special case. Finally, some useful properties of the biquaternion transformation such as commutation and anticommutation relations known for Pauli spin matrices as well as a derivation of an equivalent set of Dirac matrices are also shown.

II. Biquaternion Transformation Matrices

a. Boost along arbitrary direction

Consider the four velocities $U = [u_{ct}, u_x, u_y, u_z]$ and $V = [v_{ct}, v_x, v_y, v_z]$ which satisfies

$u_{ct}^2 - u_x^2 - u_y^2 - u_z^2 = c^2$ and $v_{ct}^2 - v_x^2 - v_y^2 - v_z^2 = c^2$.

Let us intruduce a boost along an arbitrary direction with transformations

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \frac{1}{c} \begin{bmatrix} u_{ct} & -iu_x & -iu_y & -iu_z \\ iu_x & u_{ct} & -iu_z & iu_y \\ iu_y & iu_z & u_{ct} & -iu_x \\ iu_z & -iu_y & iu_x & u_{ct} \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \tag{1}$$

and

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \frac{1}{c}\begin{bmatrix} v_{ct} & -iv_x & -iv_y & -iv_z \\ iv_x & v_{ct} & iv_z & -iv_y \\ iv_y & -iv_z & v_{ct} & iv_x \\ iv_z & iv_y & -iv_x & v_{ct} \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \quad (2)$$

Note that transformations in equations (1) and (2) are equivalent to the right and left multiplication of biquaternions with real time component and pure imaginary spatial components.

From both equations (1) and (2), computing

$$[ct', ix', iy', iz'] \cdot \begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} \quad (3)$$

yields

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = c^2 t^2 - x^2 - y^2 - z^2 \quad (4)$$

Which means that transformations in equations (1) and (2) leave the spacetime length invariant.

A combination of transformations of the left and right multiplication of the biquaternions in equations (1) and (2) can be written as

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \frac{1}{c^2}\begin{bmatrix} u_{ct} & -iu_x & -iu_y & -iu_z \\ iu_x & u_{ct} & -iu_z & iu_y \\ iu_y & iu_z & u_{ct} & -iu_x \\ iu_z & -iu_y & iu_x & u_{ct} \end{bmatrix} \cdot \begin{bmatrix} v_{ct} & -iv_x & -iv_y & -iv_z \\ iv_x & v_{ct} & iv_z & -iv_y \\ iv_y & -iv_z & v_{ct} & iv_x \\ iv_z & iv_y & -iv_x & v_{ct} \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \quad (5)$$

Equation (5) also satisfies the Minkowski spacetime length invariance in equation (4).

b. Boost along the x axis case

Consider the case $u_y = u_z = v_y = v_z = 0$. The left and the right transformations in equations (1) and (2) become respectively

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \frac{1}{c}\begin{bmatrix} u_{ct} & -iu_x & 0 & 0 \\ iu_x & u_{ct} & 0 & 0 \\ 0 & 0 & u_{ct} & -iu_x \\ 0 & 0 & iu_x & u_{ct} \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \quad (6)$$

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \frac{1}{c}\begin{bmatrix} v_{ct} & -iv_x & 0 & 0 \\ iv_x & v_{ct} & 0 & 0 \\ 0 & 0 & v_{ct} & iv_x \\ 0 & 0 & -iv_x & v_{ct} \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \quad (7)$$

Furthermore, consider the case $u_{ct} = v_{ct}$ and $u_x = v_x$. The combined left and right transformation in equation (5) become

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \frac{1}{c^2}\begin{bmatrix} u_{ct}^2 + u_x^2 & -2iu_{ct}u_x & 0 & 0 \\ 2iu_{ct}u_x & u_{ct}^2 + u_x^2 & 0 & 0 \\ 0 & 0 & u_{ct}^2 - u_x^2 & 0 \\ 0 & 0 & 0 & u_{ct}^2 - u_x^2 \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \tag{8}$$

We can use the hyperbolic functions as $\frac{u_{ct}}{c} = \cosh\left(\frac{\eta}{2}\right)$ and $\frac{u_x}{c} = sinh\left(\frac{\eta}{2}\right)$. Then, we have

$\frac{1}{c^2}(u_{ct}^2 + u_x^2) = cosh\left(\frac{\eta}{2}\right)^2 + sinh\left(\frac{\eta}{2}\right)^2 = \cosh\eta$, $\frac{2}{c^2}u_{ct}u_x = 2\cosh\left(\frac{\eta}{2}\right) sinh\left(\frac{\eta}{2}\right) = sinh\eta$ and of course, $\frac{1}{c^2}(u_{ct}^2 - u_x^2) = cosh\left(\frac{\eta}{2}\right)^2 - sinh\left(\frac{\eta}{2}\right)^2 = 1$. Equation (8) become

$$\begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix} = \begin{bmatrix} \cosh\eta & -isinh\eta & 0 & 0 \\ isinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix} \tag{9}$$

Which is the standard Lorentz transformation for a boost along the x axis.

III. Some Properties of the Biquaternion Transformation Matrices

a. Notation

Let us introduce the following notation:

$$X = \begin{bmatrix} ct \\ ix \\ iy \\ iz \end{bmatrix}, X' = \begin{bmatrix} ct' \\ ix' \\ iy' \\ iz' \end{bmatrix}, U_L = \begin{bmatrix} u_{ct} & -iu_x & -iu_y & -iu_z \\ iu_x & u_{ct} & -iu_z & iu_y \\ iu_y & iu_z & u_{ct} & -iu_x \\ iu_z & -iu_y & iu_x & u_{ct} \end{bmatrix} \text{ and } V_R = \begin{bmatrix} v_{ct} & -iv_x & -iv_y & -iv_z \\ iv_x & v_{ct} & iv_z & -iv_y \\ iv_y & -iv_z & v_{ct} & iv_x \\ iv_z & iv_y & -iv_x & v_{ct} \end{bmatrix} \tag{10}$$

Equations (1), (2) and (5) can now be written as

$$X' = \frac{1}{c}U_L \cdot X \,,\ X' = \frac{1}{c}V_R \cdot X \text{ and } X' = \frac{1}{c^2}U_L \cdot V_R \cdot X \tag{11}$$

U_L and V_R can also be written as

$$U_L = \alpha^{\mu} u_{\mu} \text{ and } V_R = \beta^{\mu} v_{\mu} \tag{12}$$

With

$$\alpha^0 = \beta^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$\alpha^1 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \alpha^2 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \text{ and } \alpha^3 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$\beta^1 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \beta^2 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \text{ and } \beta^3 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

b. Properties of Alphas and Betas Matrices

Alphas and Betas matrices in equations (14) and (15) have commutation relations

$$[\alpha^i, \alpha^j] = 2i\epsilon_{ijk}\alpha^k \tag{16}$$
$$[\beta^i, \beta^j] = -2i\epsilon_{ijk}\beta^k \tag{17}$$

Also, they have the following anticommutation relations

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij} I \tag{18}$$
$$\{\beta^i, \beta^j\} = 2\delta^{ij} I \tag{19}$$

These properties are also known for Pauli matrices and we can construct equivalent Dirac gamma matrices set

$$\gamma^0 = \beta^3 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

$$\gamma^1 = i\beta^1 \cdot \alpha^1 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \tag{21}$$

$$\gamma^2 = i\beta^1 \cdot \alpha^2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$\gamma^3 = i\beta^1 \cdot \alpha^3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \tag{23}$$

With anticommutation properties

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \tag{24}$$

Also, using the anticommutation relations in equations (18) and (19) and rapidity $\frac{1}{2}\eta$ and $\frac{1}{2}\theta$, transformation matrices in equation (11) can be written as

$$\frac{1}{c} U_L = e^{\frac{1}{2}\boldsymbol{\alpha} \cdot \boldsymbol{\eta}} \tag{25}$$
$$\frac{1}{c} V_R = e^{\frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\theta}} \tag{26}$$
$$\frac{1}{c^2} U_L \cdot V_R = e^{\frac{1}{2}\boldsymbol{\alpha} \cdot \boldsymbol{\eta}} \cdot e^{\frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\theta}} \tag{27}$$

IV. Discussion and Conclusion

The general biquaternion transformation matrix introduced in equation (5) leaves the spacetime length invariant and the Lorentz transformation in equation (9) is shown to be the special case of the general biquaternion transformation. Intuitively, the biquaternion transformation in equation (5) is equivalent to the rotation in four-dimensional Euclidian space where imaginary components for Minkowski spacetime are introduced and it is isomorphic to Van Elfrinkhof (1897) rotation formula in Euclidian space. Finally, section III on properties of the biquaternion transformation matrices shows the relation of the biquaternion transformations to quantum mechanics Pauli and Dirac matrices, so this work suggests a natural extension of biquaternion transformation to quantum mechanics as demonstrated in article [1].

V. Reference

[1] Rukundo, JPR, 2017. On velocity gauge field approach of interactions. Quaternion Physics, [Online]. 1, 5. Available at: https://quaternionphysics.files.wordpress.com/2017/11/on-velocity-gauge-field-approach-of-interactions8.pdf [Accessed 26 November 2017].