A simple proof of a "Betti - Morse" conjecture.

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Abstract

We show that for compact cobordisms, the Betti numbers are equal to the number of critical points of Morse functions whose Hessian has the appropriate number of positive eigenvalues. This theorem fails for manifolds which are not cobordisms such as the n dimensional disk.

1 Proof

This result is kind of obvious given that the number N_i of critical points with level surfaces of codimension i, where i is the number of positive eigenvalues is the only quantity which could interfere with the $n - i$'th Betti number. Given that both quantities are additive under a connected sum, the relationship is linear and obviously of the form

$$b_{n-i} = N_i.$$

Reflective properties of the $N_i = N_{n-i}$ induces the Betti relationship. This result fails for example in case of a two disk which can be regarded as a cone, where the top point represents the orgin. Considering the function r^2 for example gives a single critical point at the top with two positive eigenvalues and indeed $b_0 = 1, b_1 = b_2 = 0$ which still confirms the Euler formula which is valid under more general circumstances. Indeed, critical points where the level surfaces determined by the number of *negative* eigenvalues of the Hessian do not exist, basically because of an inappropriate dimension of the boundary related to the boundary critical points, should be discounted for in determining that Betti number and put in the conjugate one. In the latter cae, everything is allright; taking $-r^2$ on the other hand naively gives $b_0 = b_1 = 0$ and $b_2 = 1$ which is impossible and indeed the "level surface" associated to 0 is a point which has no two dimensional tangent space.

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