A complete classification of the topology of differentiable manifolds, based upon Morse theory .

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Abstract

We show that the Morse indices completely determine the topological equivalence classes of a differentiable manifold. The same result holds for topological n dimensional spaces either by a simplicial approximation or a differentiable stratification.

1 Proof

It has been recently established that all topological information resides in the bundle of vectorfields; given that the latter have no global integral invariants, local information fully determines the topology. The only covariant information is given by critical points and the number of positive, null and negative eigenvalues at those points. The precise subclass of vectorfields to be considered are those emerging from Morse functions associated to a generalized type of foliation by closed hypersurfaces which possibly *maximally* collapse. Given that the information available is the same for generic nonintegrable vectorfields with a nondegenerate Hessian and a similar extension to all vectorfields where the degenerate points are even in number and where the number of null eigenvalues plus the number of negative or positive eigenvalues fixes the complementary homotopy classes. So, the standard Morse theorem can even be extended in that direction; specifically, the theorem is

$$b_i = \tilde{N}_i$$

where $\tilde{N}_i$ is determined by critical points where the number of negative eigenvalues of the Hessian equals i or the sum of the number of negative and zero eigenvalues equals i. In case a critical point is nondegenerate, it contributes by 1 to $\tilde{N}_i$, otherwise it contributes by $\frac{1}{2}$ which *demands* the manifold to be closed given that Betti duality only holds in those cases. For example for an infinitesimal rotation around the origin in a two dimensional disk D^2, there is only one critical point with two null eigenvalues; it would imply that $b_0 = b_1$ which is impossible and in our prescription it would be equal to $\frac{1}{2}$. Therefore, all topological information

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resides in the Betti numbers and possibly in arithmetical structures associated to the degeracies of some critical points. Obviously, we would need to consider double labellings $N_{i,k}$ where k gives the degeneracies and $N_{i,k}$ is even for $k \geq 1$ in case $\mathcal{M}$ is closed. However, since this information is *not* universal given that there are plenty of manifolds with boundary and the Morse theorem also applies to them, it is *expected* that those fine details, which can change from one vectorfield to another, do not matter in the topological classification scheme. Indeed, one might expect the topology of the boundary to be important and one could try to formulate a theorem allowing for the infinitesimal rotation to be of importance; however, the latter provides for only one piece of nontrivial data and cannot therefore account for the boundary and the disk. Together with the dimension two, however, complete information is given. It is utterly clear that there needs to be a democracy over all vectorfields of this type given that integrals over the associated infinite dimensional spaces do not exist. The set of all vectorfields with a finite number of critical points and a degenerate Hessian which are tangent to the boundary do not have a local vectorspace structure in finite neighborbood given that the null eigenvalues can all evaporate. However, this space is infinite dimensional and global and all "additive" topological numbers under the connected sum are insensitive to those. More generally, it is utterly clear from topological considerations by means of simplicial manifolds that those fine points related to the degeracies are nonexistent. Hence, we have *proved* that the Betti numbers uniquely classify any homeomorphism class of manifolds of any dimension n and that there is no topological information in the degeneracy indices $N_{i,k}$ for $k \geq 1$.