A time vector and simultaneity in TSR (v9, 2018-12-22)

Per Hokstad

phokstad@gmail.com

Abstract. We specify a time vector for an event in the theory of special relativity (TSR). This vector is well suited to specify various types of simultaneity. Moving (possibly imagined) clocks, which are synchronized at a common 'point of initiation', play a crucial role. We can present the time vector as a complex variable, and there is a close relation to the Minkowski distance. We exemplify the approach by including a short discussion of the 'travelling twin'.

Key words: Time dilation, simultaneity, Lorentz transformation, time vector, Minkowski distance, travelling twin.

1 Introduction

The concept of *simultaneity* is crucial in the theory of special relativity (TSR). Within a single inertial reference frame (RF) simultaneity is easily established by the synchronization of clocks, *e.g.* using light rays; for instance see textbooks like Giulini (2005) and Mermin (2005). Then we say that events with the same clock reading, 'time' (t) on a specific RF, are *simultaneous in the perspective* of this frame. So this type of simultaneity depends on the chosen RF.

The situation is more complex when we have several inertial RFs, which are moving relative to each other. However, if we have just a single event, these will of course have different (time, space) parameters, (t, x), within the different RFs. We refer to this trivial case as *basic simultaneity*.

However, when we have moving RFs, there is in TSR no unique definition of simultaneity of events occurring 'at a distance'. The various RFs will disagree with respect to simultaneity. One refers to *relativity of simultaneity*, *e.g.* see the discussion in Debs and Redhead (1996). They argue for the *conventionality of simultaneity*; the definition of simultaneity is essentially a matter on convention.

We will here argue that one can provide a single, sensible and consistent definition of simultaneity also 'at a distance'. Hokstad (2018) applied a symmetry argument to obtain such a simultaneity for two RFs; postulating an auxiliary RF with origin always located at the midpoint between the two main RFs. In the present paper, we pursue a slightly different approach.

First, we point out that an essential requirement for the use of the fundamental Lorentz transformation (LT) for two RFs, moving relative to each other, is that we start out with three sets of synchronizations:

1. All clocks on the first RF are synchronized; (so they are simultaneous in the perspective of this RF);
2. Similarly, all clocks on the second RF are synchronized;
3. The clocks at the origins of the two RFs: At time 0 these are at the same location, and they are then synchronized; this 'point of initiation' is a fundamental initial condition for the experiment. We refer to these two clocks as *basic clocks* (BCs), and this synchronization provides an example of *basic simultaneity* (see above).

Further, one implicitly assumes that the clocks on each of the RFs *remain* synchronized. We will here argue that also the two BCs at the origins of the RFs – which we synchronized at time 0 - will remain synchronized. They move away from each other at constant speed, v; but there is a symmetric situation; so there is no way to claim that one of the two clocks goes faster than the other.

So our claim is that when the two 'basic clocks' at the origins of the two RFs show the same time, this corresponds (in some sense) to simultaneous events 'at a distance'. Actually, we could consider this a consequence of the standard assumption of symmetry between the two RFs. We will find that this leads to a rather strong form of simultaneity, as all observers can agree on this. Further, in the above argument there is no need to restrict to consider just two RFs (or two BCs), so we can get simultaneity for any number of events.

In this paper, we start out by introducing a two dimensional time vectors related to any event (*t,* x). The clock reading of the BC at the location of the event is an essential element of the vector, which proves useful for defining simultaneity 'at a distance'. We restrict to consider just a single space parameter.

This paper gives a purely mathematical description of the phenomenon, investigating implications of the LT, and there is no attempt of a physical interpretation.

2 Time as a two-dimensional variable

We now introduce a two-dimensional state vector. We refer to this as a time vector, and will use it to define simultaneity.

2.1 Time vector of a single RF

We consider a RF, *K*. At virtually any position, there is a synchronized clock with a clock reading denoted, *t*. When at position, *x* there is a clock reading, *t,* we will simply refer to (*t*, *x*) as an event. Further, we introduce the parameter

$$w = x/t \tag{1}$$

which we of course can interpret as the velocity of an object (a point on a RF) that has moved from the origin at time 0 to the position, *x* at 'time' (clock reading), *t.*

We have previously (Hokstad (2017)) suggested the following time vector for this event

$$\vec{t}(t,x) = \begin{pmatrix} \sqrt{t^2-(x/c)^2} \\ x/c \end{pmatrix} \tag{2}$$

(We might use $|x|/c$ as the second component, but also x/c works.) Using, $w = x/t,$ we can write this time vector as

$$\vec{t}(t,w) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t \tag{3}$$

Now, both components of this vector has a specific interpretation. The first component equals

$$t^{BC} = t\sqrt{1-(w/c)^2} \tag{4}$$

Here we recognize the standard time dilation formula, (see App. A). We just imagine a RF, K_w moving relative to *K* at velocity *w*, and assume that at time 0 the origins of the two RFs where at the same location, and that the two clocks at this position where then synchronized. This synchronization implies that these clocks are of particular interest, and we call them Basic Clocks (BC). We also refer to this event of synchronization (at time, 0) as the 'point of initiation'.

Thus, we interpret t^{BC} of *eq.* (4) as the clock reading of the BC (located at the origin of K_w) at the instant when this clock has reached the position, $x = wt$ on *K,* (at an instant when the local clock on *K* reads *t*). We may say that t^{BC} defines the 'basic time' of the event (*t,* x) on *K*. Thus, we have the following alternative expression for our time vector:

$$\vec{t}(t,x) = \begin{pmatrix} t^{BC} \\ x/c \end{pmatrix} \tag{5}$$

To summarize, *eqs.* (2), (3) and (5) are all valid expression for the time vector of the event (*t*, *x*) on *K*:

$$\vec{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sqrt{t^2-(x/c)^2} \\ x/c \end{pmatrix} = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t = \begin{pmatrix} t^{BC} \\ x/c \end{pmatrix} \tag{6}$$

We note that the first component of the time vector is a valid expression only when $|x|/c < t,$ (that is $|w|<c$). So at a given position, *x* the time vector (6) is only defined for clock readings, $t > |x|/c$. Here $|x|/c$ is the time required for a light flash, which occurred at the point of initiation, to reach the position, *x*.

Actually, in the limit, when $|w| \to c$, the moving BC arriving x will read $t^{BC} = 0$ (*eq.* (4)), and thus, no time has elapsed on this BC, even if the local clock on K reads t. This is valid for an arbitrarily large $|x|$.

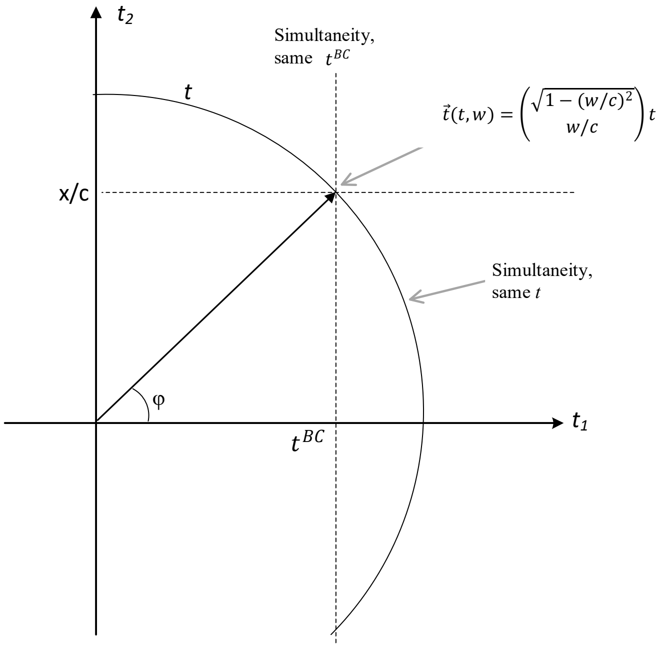

Figure 1 One specific time vector, $\vec{t}(t,w) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t = \begin{pmatrix} t^{BC} \\ x/c \end{pmatrix}$ for a specific clock reading, *t* at the position, $x = w \cdot t$, when $|t|>x/c$; (here sin $\varphi = w/c$).

However, for $t > |x|/c$ we can present the time vector, $\vec{t} = \vec{t}(t,w) = \vec{t}(t,x)$ as a point on the semicircle with radius, t in the (t^{BC}, x/c) space; see Fig. 1. In summary, the components of the time vector have simple interpretations:

1. The first component, $t^{BC} = t\sqrt{1-(w/c)^2}$ equals the clock reading of the BC of the (possibly imagined) RF, K_w which has now reached the position $x = w\,t$ on K. We call this the 'basic time' (or 'BC reading') at this position.
2. The second component, x/c, equals the time required for a light flash to go to the distance, x from the origin of K (with its own BC) to the given position. So this equals the distance in time between the BC located at the origin of K and the BC at position x.

Thus, both components refer to aspects of 'distance in time' from the 'point of initiation', ($x = t = 0$).

Further, the absolute value of the time vector equals the clock reading of the event itself:

$$\left|\vec{t}(t,w)\right| = t \tag{7}$$

Thus, events on K, with time vectors having the same absolute value, also have identical clock readings, and thus are simultaneous 'in the perspective' of K. The semicircle of Fig. 1 illustrates this.

We see Fig. 1 as an illustration of the time vectors of one specific RF. Different semicircles (of radius, t) which we could draw, represent different clock readings. By specifying both a t and a position, x, we also obtain a corresponding t^{BC}. Both at position x and $-x$ there will be a BC with this reading, t^{BC}, (they have moved in opposite directions from the origin of K).

We observe that there is a strong link between the above approach and Minkowski's approach to space-time; *cf.* space-time distance given as $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$ in his four-dimensional space, Minkowski (1909). As stated in Petkov (2012), Minkowski refers to our BC reading (see $\sqrt{t^2 - (x/c)^2}$ of *eq.* (2)), as 'proper time, and our t as 'coordinate time'.

In analogy with the Minkowski space-time, we could generalize this approach to be valid for a three-dimensional space with coordinates (x, y, z). We would then define w (*eq.* (1)) by $w = \sqrt{x^2 + y^2 + z^2}/t$

2.2 Time formulated as a complex variable

We can of course formulate our time vector for the event (t, x) as a complex variable. In polar form, we can write the vector $\vec{t}(t, w)$ in (3) as:

$$\mathbf{t}(t, w) = te^{i\varphi}, \tag{8}$$

Here the argument, $\varphi \in (-\pi/2,\ \pi/2)$, is given by

$$\sin\varphi = w/c. \tag{9}$$

When $\varphi = 0$, we have $w = x = 0$. Then the corresponding event occurs at the origin of K, and the relevant BC is the one located on K itself. In this case only, the time variable becomes a real number. Further, the magnitude, t of $\mathbf{t}(t, w)$; its real part,

$$\mathrm{Re}(\mathbf{t}(t, w)) = t\sqrt{1 - (w/c)^2} = t\cos\varphi = t^{BC};$$

and its imaginary part,

$$\mathrm{Im}(\mathbf{t}(t, w)) = t \cdot (w/c) = t\sin\varphi = x/c$$

all have interpretations as described in Section 2.1.

2.3 Relating time vectors of different RFs. The Lorentz Transformation

Now consider a RF, K_v moving relative to a K_0 at a speed, v. An event on K_v is specified by the clock reading, t_v at the position, x_v, (and this applies also for $v = 0$). Further, $w_v = x_v/t_v$, and we define φ_v by

$$\sin\varphi_v = w_v/c, (w_v < c) \tag{10}$$

As seen in (6) above, there are various ways to write the time vector on K_v; one alternative being:

$$\vec{t_v} = \begin{pmatrix} t_v^{BC} \\ x_v/c \end{pmatrix}$$

In analogy with the formulation as a complex variable (see *eqs.* (8) and (9)) we can also write it as

$$\vec{t_v}(t_v, w_v) = \begin{pmatrix} t_{v,1} \\ t_{v,2} \end{pmatrix} = \begin{pmatrix} \cos\varphi_v \\ \sin\varphi_v \end{pmatrix} t_v = \begin{pmatrix} 1 \\ \tan\varphi_v \end{pmatrix} t_v^{BC} \tag{11}$$

Now consider the case that $\vec{t_v}(t_v, w_v)$ and $\vec{t_0}(t_0, w_0)$ describe the same event, just expressed by the co-ordinates of K_v and K_0, respectively; (thus, having 'basic simultaneity'). Then the LT (*cf.* Appendix A) will provide the relation between these vectors. It is easily verified, (and rather well known *cf. eq.* (2)), that the first component $t_{v,1}$ is then invariant under the LT, and so in this case

$$t_{v,1} = t^{BC}, \text{ (independent of } v\text{)}.$$

The point is simply that time vectors $\vec{t_0}(t_0, w_0)$ and $\vec{t_v}(t_v, w_v)$ refer to the same event, and thus experience the same BC reading, t^{BC}. Fig. 2 provides an illustration, where K^{BC} represents the RF of the BC being present at the event.

So applying the LT on the time vectors gives the simple result that the first component is unaffected. Thus, using the complex form of the vector, as in Section 2.2, and replacing w by x in the second argument of $\mathbf{t}$, we can write the LT as

$$\mathbf{t}_v(t_v, x_v) = \mathbf{t_0}(t_0, x_0) + i\,(x_v - x_0) \tag{12}$$

Thus, when we express the LT by the complex time vector it is only the imaginary part that is affected by the transformation. Fig. 3 also provides an illustration; here using the vector representation of time. We see that the time vector on K_0 (blue) and the on K_v (red) have identical first component, t^{BC}.

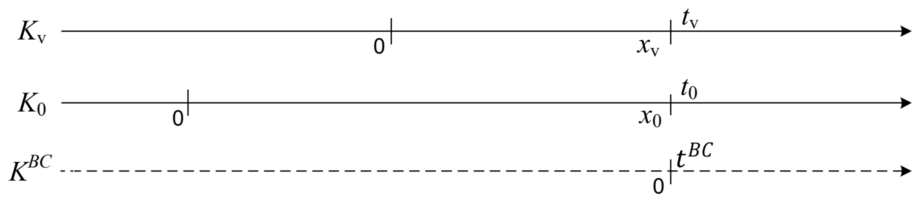

Figure 2 Two events (x_0, t_0) and (x_v, t_v) representing basic simultaneity, and the clock reading, t^{BC} of a BC at the same position. The origins of all RFs are marked with a zero, 0.

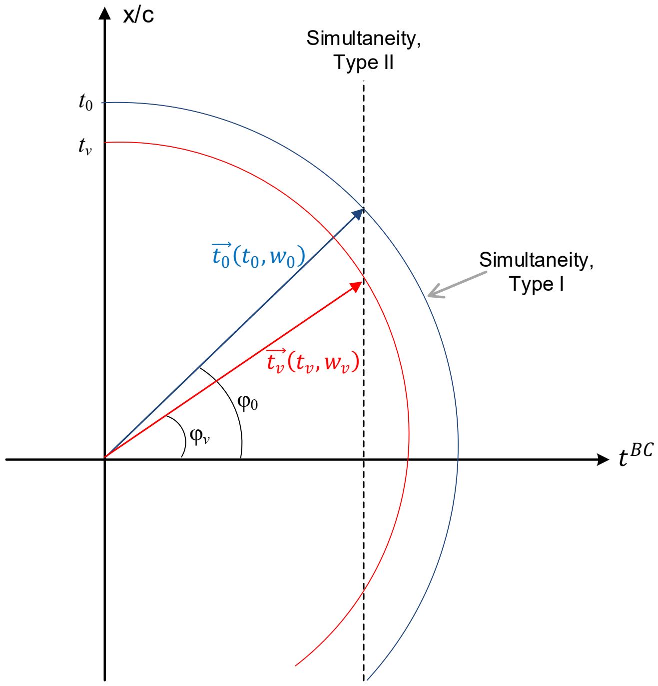

Figure 3 Time vector, $\vec{t_0}(t_0, w_0)$ on K_0 when its clocks read t_0, (blue); and time vector, $\vec{t_v}(t_v, w_v)$ on K_v at the same position, (red). Here we actually have one event, described by two different RFs.

3 Simultaneity and the time vector

It should be quite clear by now that there is a rather close connection between the time vector and simultaneity. As indicated in Figs. 1 and 3 we consider two types of simultaneity. We will now refer to these as Type I and II, respectively, and sum up the main features of these.

3.1 Simultaneity, Type I

The absolute value of the time vector is equal to the clock reading, t of the corresponding event. When we consider the events of a single RF, K, (but only then) this provides a measure of simultaneity. We say that events with identical, t are simultaneous '*in the perspective of K*', and refer to this as Simultaneity, Type I. So this occurs when the time vectors for events on a specific RF have the same *absolute value*; *cf.* the vectors on the semicircles of Figs. 1 and 3.

However, this is a very weak form of simultaneity. The various RFs will even disagree on the 'time' of a single, specific event, (what we have called basic simultaneity). The two time vectors given in Fig. 3 provide an example. Thus, different RFs will disagree regarding simultaneity, Type I.

3.2 Simultaneity, Type II

We use our concept of Basic Clocks (BCs) to define Simultaneity, Type II. At the point of initiation (t=0, all RF) all BCs are located at the common origin, and they are moving relative to each other at various speeds. Thus, for every event (t, x) on any RF (with $t > |x|/c$) there is a BC present. Events that have the same BC reading, t^{BC} are simultaneous in this sense (Type II). It is not required that the BCs actually exist; we are just stating what these clocks would read, *if* they were present at the location of an events. We distinguish between the following cases:

i. We consider a single RF, *cf.* Fig. 1. Given a fixed t^{BC}, we have that all events (t, x) on this RF, having a clock reading, t that equals

$$t = \sqrt{(t^{BC})^2 + (x/c)^2} \qquad (12)$$

are simultaneous, and have BC time, t^{BC}. We refer to this as Simultaneity, *Type II, Single RF*.

ii. We have a single event; just described by two (or several) RFs, *cf.* Fig. 3. So these RFs have the same BC reading, t^{BC}. This is the situation described by the LT, and the red and the blue time vector of the figure are two representations (in two different RFs) of the same event. Thus, they have the same t^{BC}. In this rather trivial case, we apply just a single BC, and refer to Simultaneity, *Type II, Local*.

iii. We generalize the above two cases; thus, providing a means to give an objective definition of 'simultaneity at a distance', irrespective of RF. The main point is that the above argument also applies for various BCs 'at a distance'. For instance, looking at Fig. 3 we immediately see that the event described by the two given time vectors are also simultaneous with the corresponding event on the negative x-axis; where there is another BC with the same t^{BC}. However, we are not limited to such a special case. There will be BCs at any speed (<c) available at any position, $x<ct$, with the ability to specify simultaneity of events 'at a distance'. Thus, the simultaneous events, described by (12) do not have to be located on one specific RF! Equivalently, the requirement that various events (t_v, x_v) on a K_v have identical BC reading is obviously equivalent to

$$\sqrt{t_v^2 - (x_v^2/c)^2} = t_v^{BC} = \text{Const.} \qquad (13)$$

(*cf.* the Minkowski distance.) As known, this expression is invariant under the LT; and more important, the criterion is applicable also when we consider *several* BCs, K_v. So this is a useful concept, and we refer to it as Simultaneity, *Type II, At a distance*.

Now in general, (13) is the criterion for Simultaneity, *Type II*. This concept is obviously in conflict with Simultaneity, *Type I*. Events that are simultaneous according to their BC readings, t^{BC}, are usually *not* simultaneous 'in the perspective' of the relevant RF. Thus, the simultaneity concept II is more useful, than the concept, I, which is based entirely on the clock readings, t. In summary, Type II gives a consistent definition that applies across RFs, and thus is much more sensible.

Finally, one should note that we have defined simultaneity relative to a specific 'point of initiation', only. In this respect, it does not represent an 'absolute simultaneity'.

4 The travelling twin

The travelling twin paradox is frequently discussed, *e.g.* see Schuler and Robert (2014). As stated for instance in Mermin (2005) the paradox illustrates that two identical clocks, initially in the same place and reading the same time, can end up with different readings if they move apart from each other and then back together. We gave a lengthy discussion in Hokstad (2018), and now restrict to a comment on the simultaneity of events related to the actual arrival.

In this thought experiment we start out with two synchronized clocks at the origins of two reference frames: the RF of the earth, and the RF of the rocket of the travelling twin. We note that both clocks are located at the origin of their RFs, and so both are basic clocks (BCs) in our notation. This makes the case very well suited to illustrate the current approach. Actually, it is sufficient to point out that both clocks are BCs. So if the travelling twin's clock shows 4 years by his arrival, this is simultaneous with the event that the clock on the earth also shows 4 years. This follows from our concept, Simultaneity, *Type II, At a distance.*

However, to illustrate this further, we also consider the relevant time vectors. We use the numerical example of Mermin (2005). The distance from the earth to the star equals x_0 = 3 light years, *i.e.* x_0/c = 3 years. Further, the velocity of the rocket is $v = 0.6c$, giving $\sqrt{1-(v/c)^2} = 0.8$. It follows that by the arrival of the travelling twin, the clock at the star belonging to the earthbound twin will read $x_0/v = 3/0.6 = 5$ years, (assuming that the he has a clock, located on the star. being synchronized with his own). At the same instant the clock of the travelling twin reads 5·0.8= 4 years (time dilation). In the literature one now often just points out that 'time' equals 5 years 'in the perspective' of the earthbound twin, and 4 years 'in the perspective' of the travelling twin.

Fig. 4 illustrates the time vectors related to the arrival at the star: Red time vector for the travelling twin; his clock showing 4 years. Blue vector for the earthbound twin; his clock showing 5 years. We note that since $x_0/c = 3$ years at this position, we also directly get $t^{BC} = \sqrt{5^2 - 3^2} = 4$ years, as already stated. The two semicircles represents times 'in the perspectives' of the two twins; obviously representing very conflicting views. We note that Fig. 4 is a special case of Fig. 3, as the RF of the BC now is equal to that of the travelling twin, (red semicircle has radius t^{BC}=4).

However, we also have another BC, namely that of the earthbound twin. Thus, we conclude that when these two time vectors have the same $t^{BC} = 4$ years, this represents Simultaneity *Type II, at a distance*. Actually, the two twins then have identical time vectors, (but in different RFs).

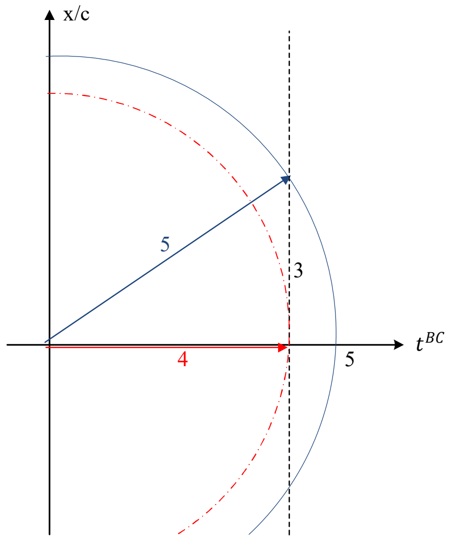

Figure 4. The time vectors at the star by the arrival: Red = RF of travelling twin. Blue = RF of earthbound twin. The relevant vectors are $\vec{t} = \binom{3}{4} = 5\binom{0,6}{0,8}$ and $\vec{t} = \binom{4}{0} = 4\binom{1}{0}$.

So the main finding here is that the arrival at the star is simultaneous with the event that the clock on the earth shows 4 years. Actually, 4 years is the only feasible result, considering the symmetry of the situation, *cf.* Hokstad (2018). However, we should point out that in spite of this, we will arrive at the standard result of 10 and 8 years, for the ages (or rather clock readings) by the reunion of the twins on the earth. In order to maintain the conditions of the TSR, the return of the travelling twin requires the introduction of a third RF with a third BC, (which is brought to the earth). So literally, it is not the twin himself - or even his own clock - that comes back to the earth, but rather a clock that had identical reading at the moment of the turning at the star. This dramatic discontinuity at the twins turning will have a significant effect, and is the reason that we after all arrive at the standard result of 10 and 8 years; (*cf.* Appendix A.3 of Hokstad (2018)). Thus, this result is actually rather 'inaccurate', without stressing this discontinuity, *i.e.* the violation of the assumptions of the TSR.

5 Conclusions

The above presentation is based on one fundamental claim. We postulate an infinite set of (possibly imagined) reference frames (RFs) moving relative to each other at constant speeds. At the origins of these RFs there is a clock, and initially these are all synchronized, (being at the same location!). From symmetry, we conclude that they remain synchronized, and refer to them as basic clocks (BCs).

Then for any event (*t, x*) on any RF, we define a time vector in two dimensions:

1. The clock reading of the (imagined) BC currently at this position, (t^{BC})
2. The time required for a light flash to go from the origin of the RF (where there also is a BC), to the current position,(x/c).

So, both components (dimensions) represent a 'distance in time' from the 'point of initiation', when the BCs were synchronized. We find that the absolute value of this time vector equals the clock reading, *t,* of the event, and see this as a measure for the overall distance in time from the 'point of initiation'.

This time vector provides a means to define various forms of simultaneity. Obviously, when the time vectors on a specific RF have the same absolute value, *t,* they will specify events that are simultaneous 'in the perspective' of this RF. This Simultaneity, *Type I*, follows as *t* also equals the clock reading of the event.

However, the main result is that time vectors, which have identical first component (t^{BC}) correspond to simultaneous events. This represents simultaneity in a much stronger sense, as we can use it as a holistic definition, valid for all RFs. We denote it Simultaneity *Type II*. The definition can be used locally: When we consider a specific event, described by various RFs, the vectors of all RFs of course have the same t^{BC}, as there is just one BC present.

However, the most useful application is to consider this simultaneity 'at a distance'. Events with the same t^{BC} will exhibit this form of simultaneity, also for distant events, irrespective of RF.

The travelling twin paradox represents a trivial application of this definition of simultaneity *Type II*.

The given results apply for simultaneity relative to a common 'point of initiation'. Finally, we observe that one can also formulate the time vector as a complex variable, and that there is a close link to the space-time of Minkowski.

References

Debs, Talal A. and Redhead, Michael L.G., The twin "paradox" and the conventionality of simultaneity. Am. J. Phys. **64** (4), April 1996, 384-392.

Giulini, Domenico, Special Relativity, A First Encounter, Oxford University Press, 2005.

Mermin. N. David, It's About Time. Understanding Einstein's Relativity. Princeton Univ. Press. 2005.

McCausland, Ian, A Question of Relativity. Apeiron, Vol. 15, No. 2, April 2008. 156-168.

McCausland, Ian, A scientific Adventure: Reflections on the Riddle of Relativity. C. Roy Keys Inc, Montreal, Quebec, Canada 2011.

Minkowski, H., Raum und Zeit. Physikalische Zeitschrift 10, 75-88, 1909. English Translations in Wikisource: Space and Time

Hokstad, Per, An Approach for analysing Time Dilation in the TSR, **viXra:1706.0374**, v9. Category Relativity and Cosmology, June 2018.

Hokstad, Per, The Formulation and Interpretation of the Lorentz Transformation, viXra 1703.0281, v3, http://vixra.org/pdf/1703.0281v3.pdf**.** Category Relativity and Cosmology, 2017.

Petkov, V., Introduction to Space and Time: Minkowski's papers on relativity; translated by Fritz Lewertoff and Vesselin Petkov. Minkowski Inst. Press, Montreal 2012, pp. 39-55, Free version online.

Shuler Jr., Robert L., The Twins Clock Paradox History and Perspectives. Journal of modern Physics, 2014, 5, 1062-1078. https://file.scirp.org/Html/3-7501845_47747.htm.

Appendix A The Lorentz transformation (LT) and time dilation

This Appendix reproduces some material from Hokstad (2018).

A.1 Alternative formulation of the LT

The LT represents the fundament for our discussions. In our notation, the LT takes the form

$$t_v = \frac{t_0 - (v/c^2)x_0}{\sqrt{1-(v/c)^2}} \tag{A1}$$

$$x_v = \frac{x_0 - vt_0}{\sqrt{1-(v/c)^2}} \tag{A2}$$

We prefer a modified version of the LT. At any time, t_v and position, x_v we introduce w_v equal to $w_v = x_v/t_v$, (and therefore also $w_0 = x_0/t_0$). Then we insert $x_0 = w_0 t_0$, and (A1) directly gives that the clock reading on the RF, K_v at this position equals:

$$t_v = t_v(w_0) = \frac{1 - vw_0/c^2}{\sqrt{1-(v/c)^2}} t_0 \tag{A3}$$

Note that we here also write $t_v = t_v(w_0)$ to stress the dependence on w_0.

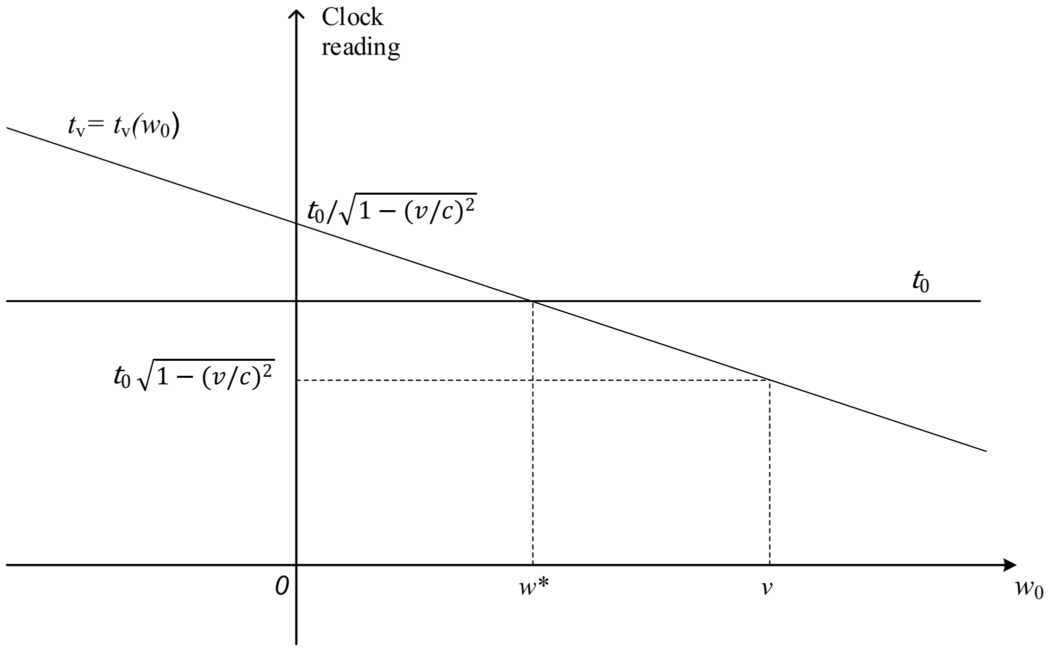

Figure A1. Clock readings in the perspective of K_0. Thus, 'time' all over K_0 equals t_0, while clock readings, $t_v(w_0)$ on the other RF is given as a function of w_0, where $w_0 = x_0/t_0$ provides the 'position' on K_0; *cf.* (3).

Further, by also inserting $x_0 = w_0 t_0$ and $x_v = w_v \cdot t_v$, we obtain

$$w_v = \frac{x_v}{t_v(w_0)} = \frac{w_0 - v}{1 - \frac{w_0}{c}\frac{v}{c}} \tag{A4}$$

So equations (A3), (A4) express the LT by parameters (t, w) rather than (t, x). We observe that clock readings, t_0 and t_v enters (A3) only!

Fig.A1 provides an illustration of the time dilation formula, (A3). This gives the clock reading both on K_0 and K_v *in the perspective of* K_0; (*i.e.* all clocks on K_0 having the same clock reading). Therefore, the figure illustrates an instant when clocks read t_0 all over this RF. The horizontal axis gives the 'position' $w_0 = x_0/t_0$ on K_0 at which the clock measurements are carried out. The vertical axis gives the actual clock readings. So as clocks on K_0 read t_0 at any 'position', w_0, the clock readings on K_v at this instant, $t_v = t_v(w_0)$, is a linear function of w_0, see (A3).

A.2 Two standard special cases (observational principles)

Two special cases are of particular interest. Recall that the first clock comparison is carried out at the origins $x_v = x_0 = 0$ when $t_v = t_0 = 0$. We specify two choices for the second comparison of clock readings.

First we compare the clock located at $x_v = 0$ on K_v (with the passing clocks on K_0 showing t_0). Thus, also $w_v = 0$, and (A4) implies $w_0 = v$, and (3) gives the relation between the two clock readings at this position, *cf.* Fig. A1:

$$t_v = t_v(v) = t_0\sqrt{1-(v/c)^2} \tag{A5}$$

This equals the standard 'time dilation formula'. Secondly, we can compare the clock located at $x_0 = 0$ on K_0 with a passing clock on K_v. For $x_0 = w_0 = 0$, *i.e.* following the basic clock at the origin of K_0, *eq.* (A3) gives the following relation, (again see Fig. A1):

$$t_v = t_v(0) = t_0/\sqrt{1-(v/c)^2} \tag{A6}$$

Apparently, the relations, (A5), (A6) are contradictory; *eq.* (A5) tells that the clock on K_v goes slower, and (A6) tells that the clock on K_0 goes slower; *cf.* the Dingle's question, (McCausland 2008, 2012). Thus, the time dilation is not a feature of the RF, but follows from which single clock we choose to follow when we perform the second clock comparisons. Therefore, we prefer to formulate the time dilation formulas (A5) and (A6) in compact form as

$$t^{BC} = t^{MC}\sqrt{1-(v/c)^2} \tag{A7}$$

Here we have introduced the notation regarding the second clock comparison.

t^{BC} = The clock reading of a basic clock (BC). Thus, on this RF we use the same clock in the second clock comparison. (We have previously used the concept Single Clock (SC) to denote this BC.)

t^{MC} = The clock reading at the same location, but on the other RF. Therefore, this is the clock reading on the RF, which use multiple clocks (MC) for the clock comparison; (*i.e.* it uses another clock in the second comparison).

Thus, both RFs can apply a BC for a certain clock comparison, and then conclude that 'time goes slower' on the RF which use BC. However, the same RF could also apply two clocks (MC) for a clock comparison with a BC on the other RF; and we would then conclude that 'time goes slower' on this other RF. Therefore, it is the *observational principle*, *i.e.* choice of clocks for the clock comparisons that matters; *cf.* discussion in Hokstad (2018). This is a well-known result. However, this duality has perhaps not received the attention it deserves in standard literature.

A.3 The symmetric case

There is another interesting special case of the LT, (A3), (A4). We can ask which value of w_0 (and thus w_v) will result in $t_v = t_0$. We easily find that this equality is obtained by choosing $w_0 = w^*$, where

$$w^* = \frac{c^2}{v}\left(1-\sqrt{1-(v/c)^2}\right) = \frac{v}{1+\sqrt{1-(v/c)^2}} \tag{A8}$$

Further, by this choice of w_0 we also get $w_v = -w^*$. This means that if we consistently consider the positions where simultaneously $x_0 = w^* t_0$ and $x_v = -w^* t_v = -w^* t_0$, then no time dilation will be observed at these positions. In other words (*cf.* Fig. A1):

$$t_v(w^*) = t_0$$

We also find $x_v = - x_0$, thus, providing a nice symmetry. Note that when we choose the observational principle, (A8), then absolutely everything is symmetric, and it should be no surprise that we get $t_v = t_0$.