A Motion Paradox from Einstein's Relativity of Simultaneity

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Abstract

We are describing a new and potentially important paradox related to Einstein's theories of special relativity and relativity of simultaneity.[1] We fully agree on all of the mathematical derivations in Einstein's special relativity theory and his result of relativity of simultaneity when using Einstein-Poincaré synchronized clocks. The paradox introduced shows that Einstein's special relativity theory leads to a motion paradox, where a train moving relative to the ground (and the ground moving relative to the train) must stand still and be moving at the same time. We will see that one reference frame will claim that the train is moving and that the other reference frame must claim that the train is standing still in the time window "between" two distant events. This goes against common sense and logic. However, looking back at the history of relativity theory, even time dilation was going against common sense and a series of academics attempted to refute it.[2] Still, based on this new paradox we have to ask ourselves if the world really can be that bizarre, or if Einstein's special relativity could be incomplete in some way? We are not going to give an answer to the second question in this paper, but we will simply present the new paradox.

Key words: Special relativity theory, relativity of simultaneity, paradox, motion.

1 The Motion Paradox

A cornerstone prediction in Einstein's theory of special relativity is the relativity of simultaneity. This basically predicts that two distant events that happen simultaneously in one reference frame cannot happen simultaneously as observed from another reference frame. In other words, under special relativity theory there is no absolute simultaneity. We will see here that relativity of simultaneity leads to a strange paradox that we have not seen mentioned before.

Assume that two events L distance apart happen simultaneously in reference frame one. The distance between the two events is L, as measured from this frame. Stating that the events happen simultaneously basically means there can be no time difference between the two events as measured in that frame, as they indeed happened at precisely the same time. In other words, the time window is zero. However, under the concept of the relativity of simultaneity, Einsteins special relativity maintains that the two events happening simultaneously in reference frame one will, as observed from reference frame two, have happened with the following time difference apart:

$$\Delta t = \frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}} \tag{1}$$

where v is the speed of frame one relative to frame two as measured with Einstein synchronized clocks.[3] In special relativity this speed is reciprocal. Both frames will observe the other frame moving

[1] We have not seen this paradox discussed before. Still, after having studied special relativity for many years and having access to one of the worlds largest libraries on special relativity literature, we do not claim to have read every publication on the topic. By 1922 already, there were supposedly more than 3,400 papers written about relativity, according to Maurice LeCat in "Bibliographie de la Relativité," Bruxelles 1924.

[2] Today, the concept that time is affected by motion has been confirmed by a series of experiments, see for example Bailey and et al. (1977) and Haefele and Keating (1971b,a). Time dilation was introduced by Larmor (1900) who combined it with length contraction to get a mathematical theory consistent with the Michelson and Morley experiment.

[3] Some prefer to call them Einstein-Poincaré synchronized clocks, as Henry Poincaré suggested a very similar way of synchronizing the clocks, yet with a slightly different interpretation. Poincare'e assumed that the "true" one-way speed of light was a function of its velocity against the ether, but that this could not be detected and so for clock synchronization purposes, one could assume the one-way time of light was half of the round trip time of light. Einstein on the other hand simply abandoned the ether and assumed that the one-way speed of light was the same as the round-trip speed of light. Einstein's theory is the simplest as long as it truly is impossible to measure velocity against a preferred frame, such as the ether frame.

at speed v relative to their own frame as measured with Einstein synchronized clocks. Equation 1 can be derived directly from the Lorentz transformation and is well known from a series of sources in the special relativity theory literature, see for example Comstock (1910), Carmichael (1913), Bohem (1965), Shadowitz (1969), and Krane (2012), see also Appendix A.

Since the two distant events happened simultaneously in frame one, then no time can have gone by between the events. This also means that from the perspective of frame one, the two reference frames cannot have moved relative to each other in the instant that the two events happened.

Still, from frame two the time between the two events (happening simultaneously in frame one) must, in Einstein's special relativity theory, happen Δt apart. This means from the perspective of frame two the two reference frames must have moved the following distances relative to each other during the time difference between the two events

$$\Delta t v = \frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}} v = \frac{Lv^2}{c^2\sqrt{1-\frac{v^2}{c^2}}} \tag{2}$$

But again from frame one's perspective the two distant events happened simultaneously, that is to say at the same point in time and therefore the two frames cannot have moved relative to each other "during" the two events. One could try to argue that no event takes zero time. However, we could make the event itself as short as we would like. It is the time difference between the two events that is important here, this time difference is zero in one frame and Δt in the other frame.

The paradox gets even more interesting when one could argue that since Δt must have gone by between the two events as observed from frame two, then the following time must have gone by between the two events as measured from frame one (simply taking into account time dilation):

$$\Delta \hat{t} = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

where Δt is the time gone by between the two events as observed from frame two and converted into frame one by using time dilation. This leads to the observation that the two reference frames must have moved the following distance relative to each other as measured from frame one:

$$\Delta \hat{t} v = \frac{\frac{Lv^2}{c^2\sqrt{1-\frac{v^2}{c^2}}}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta \hat{t} v = \frac{Lv^2}{c^2\left(1-\frac{v^2}{c^2}\right)} \tag{3}$$

This is actually the same formula as given by Comstock (1910), and is consistent with formula 1 when taking into account length contraction when observed from the other frame. Based on this, both frames will agree: the two frames moved relative to each other in the time between the two distant events, but then again from frame one's perspective both events happened simultaneously and no time can have gone by between the two events.

We could have set up the entire problem the other way around and the paradox and the calculations would be reciprocal. Two events happening simultaneously in frame two as measured from frame one cannot happen simultaneously as measured from frame one. However, even if the paradox is reciprocal between the two frames this does not solve the paradox described above, this just means the paradox is also reciprocal.

The paradox is naturally that both frames cannot be right, at least not if we follow common sense. The two frames either have moved relative to each other or they are standing still relative to each other. The motion between the two frames could naturally be checked. For example, if frame one was a train and frame two was the embankment, then the train could have accelerated to a given speed v relative to the embankment. This speed could be checked both from the embankment frame and the train frame using clocks that were Einstein synchronized in each frame respectively. Both frames would then agree that the frames were moving relative to each other with speed v. The paradox first comes when we look at two events that are happening simultaneously as observed in one frame and not simultaneously as observed from another frame.

The Minkowski perspective

Could the paradox be solved by simply looking at it from the Minkowski space-time perspective? Minkowski (1908) showed that the space-time interval was invariant:

$$ds^2 = c^2t^2 - dx^2 - dy^2 - dz^2 \tag{4}$$

Assume the two distant simultaneous events happen L meters apart on board the train as observed from the train. From this standpoint, the Minkowski space-time interval is given by

$$ds^2 = c^2 \times 0^2 - L^2 = -L^2 \tag{5}$$

while from the standpoint of the train platform the Minkowski space-time interval is given by

$$\begin{aligned} ds^2 &= c^2\left(\frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{L-0\times v}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 \\ ds^2 &= \frac{L^2v^2}{c^2\left(1-\frac{v^2}{c^2}\right)} - \frac{L^2}{\left(1-\frac{v^2}{c^2}\right)} \\ ds^2 &= \frac{-L^2(c^2-v^2)}{c^2\left(1-\frac{v^2}{c^2}\right)} \\ ds^2 &= -L^2 \end{aligned} \tag{6}$$

Both the train observer and the train platform observer agree on the space-time interval that has gone by between the two events. We would have obtained the same result if we used the Lorentz transformation for the dt term alternatively in our derivation as shown in Appendix B, this naturally gives the same result. We are not questioning that Minkowski is consistent with special relativity theory and Einstein synchronized clocks. The fact that the train observer and the embankment observer both agree on the space-time interval does not remove or solve the paradox: that from the viewpoint of the train observer, the train was standing still during the time between the two distant events and from the viewpoint of the embankment observer, the train was moving relative to the embankment. They do not agree on whether or not the train has moved relative to the train platform in between the two events.

It is well known that two reference frames not will agree on the time interval and the space interval, but with regard to the space-time interval, this is expected, as we naturally accept that there is time dilation and length contraction and that Minkowski is consistent with SR. It seems, however, when pushed to the limit of two events that happen simultaneously in one of the reference frames that we must also accept that the train did not at all move relative to the embankment, while from the embankment it must be moving. This is not new from a pure mathematical perspective; it is the deeper philosophical and logical part we are questioning here.

One solution to the paradox is simply to accept that relativity of simultaneity leads to the following conclusion: we must agree that a train is both moving relative to the embankment and not moving in between two distant events, that is if the two events happen simultaneously as observed from one of the reference frames with Einstein synchronized clocks. Alternatively, we could claim that there must something deeper here that is not fully covered by Einstein's special relativity theory. We look forward to a discussion around this paradox.[4].

2 Conclusion

Einstein's relativity of simultaneity says that two events that are observed to happen simultaneously in one frame not are happening simultaneously as observed from another frame. We fully agree that this is what is predicted and what one will observe when using Einstein synchronized clocks. Still, this leads to the paradox that when we have two frames moving relative to each other, for example a train and the embankment, then if two distant events are happening simultaneously on the train, then no time can have gone by in between the two distant events as observed from the train and the train cannot have moved relative to the embankment during a zero time interval. On the other hand, although there is no time difference between the two events that are happening simultaneously on the train, they are observed to happen with a time difference from the embankment, and the train must have moved relative to the embankment during this time period. Either one must accept that Einstein's special relativity leads to results that are far from common sense (and we would say rather bizarre) or one should consider that there is possibly a deeper reality not fully uncovered by Einstein's special relativity theory.

[4]This is not the first time someone has raised a discussion on the relativity of simultaneity, see for example the recent discussion by Bolós, Liern, and Olivert (2001), Brogaard and Marlow (2013) and Manson (2014).

Appendix A

The Einstein relativity of simultaneity equation can be derived directly from the Lorentz transformation. Assume two clocks A and B with a distance of L apart. A light signal is sent out simultaneously from points A and B as observed from the frame these clocks are at rest in, let's call it frame one. This means that the time it takes for the light signal going from A or from B to the midpoint in frame one as measured from frame one must be

$$t_{1,A} = t_{1,B} = \frac{\frac{1}{2}L}{c}$$

From frame two, which is moving at a speed of v relative to frame one, the time between the two events is given by the Lorentz time transformation. According to frame two, the two distant light signals are reaching the midpoint in frame one with a time difference of

$$t_{2,A} - t_{2,B} = \frac{\frac{\frac{1}{2}L}{c} + \frac{\frac{1}{2}Lv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{\frac{1}{2}L}{c} - \frac{\frac{1}{2}Lv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{Lv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}} \tag{7}$$

In other words, the two distant light signals that have happened simultaneously in frame one must have happened with a time difference of $\frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}}$ according to frame two. Under Einstein's special relativity theory this argument is also reciprocal. Two distant light signals that have happened simultaneously in frame two must have happened with a time difference of $\frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}}$ according to frame one.

Under special relativity what is happening simultaneously in one frame is not happening simultaneously in the other frame. There is no question about this result as long as we use Einstein-Poincarè synchronized clocks, one of the fundamental assumptions in Einstein's special relativity theory. However, as shown in this paper this leads to a rather bizarre paradox.

Appendix B

Assume the two distant simultaneous events happen L meters apart on board the train as observed from the train. Since the events happen simultaneously we have $t = 0$. The two events happen $x = L$ distance apart in the train frame. The Minkowski space-time interval as observed from the train frame is then given by

$$ds^2 = c^2 \times 0^2 - L^2 = -L^2 \tag{8}$$

while from the train platform the Minkowski space-time interval is given by the following equation, this time solved with the Lorentz transformation rather than the relativity of simultaneity equation:

$$\begin{aligned}
ds^2 &= c^2\left(\frac{t-\frac{Lv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{L-tv}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 \\
ds^2 &= c^2\left(\frac{0-\frac{Lv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{L-0\times v}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 \\
ds^2 &= \frac{L^2v^2}{c^2\left(1-\frac{v^2}{c^2}\right)} - \frac{L^2}{\left(1-\frac{v^2}{c^2}\right)} \\
ds^2 &= \frac{-L^2(c^2-v^2)}{c^2\left(1-\frac{v^2}{c^2}\right)} \\
ds^2 &= -L^2 \qquad (9)
\end{aligned}$$

which is the same as the result we derived directly from the relativity of simultaneity equation. This is as expected since the relativity of simultaneity time interval is found from the Lorentz transformation. Again this only shows that Minkowski is clearly consistent with Einstein‘s special relativity theory, it does not solve the paradox stated in this paper.

References

BAILEY, J., AND ET AL. (1977): "Measuring Time Dilation Via the Lifetime of High-Speed Muons in a Storage Ring," *Nature*, (268), 301–304.

BOHEM, D. (1965): *The Special Theory of Relativity*. Routledge.

BOLÓS, V. J., V. LIERN, AND J. OLIVERT (2001): "Relativistic Simultaneity and Causality," *International Journal of Theoretical Physics*, 6(41).

BROGAARD, B., AND K. MARLOW (2013): "Is The Relativity of Simultaneity a Temporal Illusion?," *Analysis*, 73(4).

CARMICHAEL, R. D. (1913): *The Theory of Relativity*. John Wiley & Sons.

COMSTOCK, D. (1910): "The Principle of Relativity," *Science*, 31, 767–772.

EINSTEIN, A. (1905): "Zur Elektrodynamik Bewegter Körper," *Annalen der Physik*, (17).

——— (1916): "Näherungsweise Integration der Feldgleichungen der Gravitation," *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin*.

HAEFELE, J. C., AND R. KEATING (1971a): "Around-the-World Atomic Clocks: Observed Relativistic Time Gains," *Science*, (177), 168–170.

——— (1971b): "Around-the-World Atomic Clocks: Predicted Relativistic Time Gains," *Science*, (177), 166–167.

KRANE, K. (2012): *Modern Physics, Third Edition*. John Wiley & Sons.

LARMOR, J. J. (1900): *Aether and Matter: A Development of the Dynamical Relations of the Aether to Material Systems*. Cambridge University Press.

MANSON, D. N. (2014): "The Relativity of Simultaneity is Not a Temporal Illusion: a Critique of Brogaard and Marlow," *Analysis*, 74(2).

MINKOWSKI, H. (1908): "Space and Time," *A Translation of an Address delivered at the 80th Assembly of German Natural Scientists and Physicians, at Cologne, 21 September, in the book "The Principle of Relativity", Dover 1923*.

SHADOWITZ, A. (1969): *Special Relativity*. Dover Publications.