A geometric presentation of the position and momentum representations in quantum mechanics

Spiros Konstantogiannis
spiroskonstantogiannis@gmail.com
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Abstract

Making an intuitive assumption, and using the completeness of the position and momentum eigenstates, along with the postulates of quantum mechanics, we provide a geometric presentation of the position and momentum representations in quantum mechanics, in the hope of offering a perspective complementary to those given in standard textbooks.

1. The position representation – The position space

The position is an observable quantity, and thus the respective operator, i.e. the position operator, is Hermitian and its eigenstates – the position eigenstates – form an orthogonal basis in the state space (Hilbert space) of the examined particle (or system).
We can use this basis to represent the states and operators of the state space.
This representation is called the position representation.
In other words, the position representation is the representation of the states and operators of the state space of the particle (system) in the basis of the position eigenstates.
More abstractly, we can think of the position representation as the representation of the whole state space in the basis of the position eigenstates. In this sense, we'll refer to the position representation as the position space.

1.1 The position and momentum operators in the position representation

We'll examine the one-dimensional case, as the generalization to three dimensions is straightforward.
Let $|x\rangle$ be an arbitrary position eigenstate with eigenvalue x, i.e.

$$\hat{x}|x\rangle = x|x\rangle$$

This means that if the particle is in the state $|x\rangle$, it is located at the position x, and, likewise, if the particle is located at x, it is described by the state $|x\rangle$.
Obviously, the eigenvalue x is non-degenerate, since if the eigenstate $|x'\rangle$ also has eigenvalue x, then it also describes a particle located at x, and then $|x'\rangle = |x\rangle$.
The position eigenstates are then non-degenerate, and thus, since the position operator is Hermitian, two different position eigenstates are orthogonal.
The spectrum of the position operator is continuous and the orthogonality and completeness of its eigenstates are expressed by the equations

$\langle x|x'\rangle = \delta(x-x')$ and $\int_{-\infty}^{\infty} dx|x\rangle\langle x| = 1$, respectively.

> The norm of a position eigenstate $|x\rangle$ is infinite, since
>
> $$\||x\rangle\| = \sqrt{\langle x|x\rangle} = \sqrt{\delta(0)} \to \infty$$
>
> The position eigenstates are then not normalizable, and thus the basis of the position eigenstates, i.e. the basis $\{|x\rangle\}_{x\in\mathbb{R}}$, is not orthonormal, it is orthogonal.

Also, since the position operator is Hermitian, its eigenvalues x are real, as they should be, since they are the possible positions of the examined particle.
Let now $|\psi\rangle$ be an arbitrary state of the particle. Using the completeness relation of the position eigenstates, we expand the state $|\psi\rangle$ in the basis of the position eigenstates.
We have

$$\left|\psi\right\rangle = \left(\underbrace{\int_{-\infty}^{\infty} dx' \left|x'\right\rangle\left\langle x'\right|}_{1} \right)\left|\psi\right\rangle = \int_{-\infty}^{\infty} dx' \left|x'\right\rangle\left\langle x'\middle|\psi\right\rangle$$

> As an integration variable, the variable in the completeness relation is a dummy variable, and thus we can change it from x to x'.

Thus

$$\left|\psi\right\rangle = \int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle\left|x'\right\rangle \quad (1)$$

This is the expansion of the arbitrary state $\left|\psi\right\rangle$ in the basis of the position eigenstates.
Using (1), the action of the position operator $\hat{x}$ on the state $\left|\psi\right\rangle$ yields

$$\hat{x}\left|\psi\right\rangle = \hat{x}\int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle\left|x'\right\rangle$$

We note that the inner product $\left\langle x'\middle|\psi\right\rangle$ is, generally, a complex number depending on x', i.e. it is a complex function of x'.
The position operator $\hat{x}$ acts on states, and thus

$$\hat{x}\int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle\left|x'\right\rangle = \int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle\hat{x}\left|x'\right\rangle = \int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle x'\left|x'\right\rangle$$

In the last equality, we used that the state $\left|x'\right\rangle$ is a position eigenstate with eigenvalue x'.
Thus

$$\hat{x}\left|\psi\right\rangle = \int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle x'\left|x'\right\rangle \quad (2)$$

This is the expansion of the state $\hat{x}\left|\psi\right\rangle$ in the basis of the position eigenstates.
We now want to project the state $\hat{x}\left|\psi\right\rangle$ on an arbitrary position eigenstate $\left|x\right\rangle$, i.e. we want to calculate the inner product $\left\langle x\middle|\hat{x}\middle|\psi\right\rangle$.
Using (2), we have

$$\left\langle x\middle|\hat{x}\middle|\psi\right\rangle = \left\langle x\right|\int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle x'\left|x'\right\rangle = \int_{-\infty}^{\infty} dx' \left\langle x'\middle|\psi\right\rangle x'\left\langle x\middle|x'\right\rangle$$

Since the eigenstates $\left|x\right\rangle$ and $\left|x'\right\rangle$ are orthogonal,

$$\left\langle x\middle|x'\right\rangle = \delta\left(x - x'\right),$$

and thus

$$\langle x|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} dx' \langle x'|\psi\rangle x'\delta(x-x') = \langle x|\psi\rangle x = x\langle x|\psi\rangle$$

That is

$\langle x|\hat{x}|\psi\rangle = x\langle x|\psi\rangle$ (3)

The inner product $\langle x|\psi\rangle$ is the projection of the state $|\psi\rangle$ on the arbitrary position eigenstate $|x\rangle$, or else, it is the projection of the state $|\psi\rangle$ on the position space.
Similarly, the inner product $\langle x|\hat{x}|\psi\rangle$ is the projection of the state $\hat{x}|\psi\rangle$ on the arbitrary position eigenstate $|x\rangle$, or else, it is the projection of the state $\hat{x}|\psi\rangle$ on the position space.
Since the state $\hat{x}|\psi\rangle$ results from the action of the position operator on the state $|\psi\rangle$, it is legitimate to assume that the projection of the state $\hat{x}|\psi\rangle$ on the position space is equal to the action of

- the expression of the position operator in the position space

on

- the projection of the state $|\psi\rangle$ on the position space, i.e.

$\langle x|\hat{x}|\psi\rangle = \hat{x}(x)\langle x|\psi\rangle$ (4)

where $\hat{x}(x)$ is the expression – we'll also call it "projection" – of the position operator in the position space, i.e. it is the position representation of the position operator.

Generalizing the previous reasoning, we'll assume that the projection of the state resulting from the action of an operator $\hat{O}$ on a state $|\psi\rangle$ is equal to the result of the action of the projection of the operator $\hat{O}$ – let us denote it by $\hat{O}(x)$ – on the projection of the state $|\psi\rangle$, i.e. $\langle x|\hat{O}|\psi\rangle = \hat{O}(x)\langle x|\psi\rangle$.

We'll use the previous assumption in both the position and momentum spaces.
Comparing (3) and (4), we obtain

$\hat{x}(x)\langle x|\psi\rangle = x\langle x|\psi\rangle$ (5)

Since the state $|\psi\rangle$ is arbitrary, and so is the position eigenstate $|x\rangle$, from (5) we derive that

$\hat{x}(x) = x$ (6)

This is the expression of the position operator in the position space, i.e. it is the position operator in the position representation.
We see that, in the position space, the position operator is the position coordinate, or else, it is equal to its eigenvalues, which is expected, since in the basis of its eigenstates, the position operator is diagonal.

As we mentioned, the inner product $\langle x|\psi\rangle$, which is the projection of the state $|\psi\rangle$ on the position space, is a complex function of the real variable x.
We identify this function as the wave function in the position space – or in the position representation – and we denote it by $\psi(x)$, i.e.

$$\psi(x)=\langle x|\psi\rangle \quad (7)$$

To find the momentum operator in the position representation, we use the commutation relation $[\hat{x},\hat{p}]=i\hbar$, which holds in the state space and in every representation of it.
In the position representation, the previous commutator is written as

$$[x,\hat{p}(x)]=i\hbar$$

where $\hat{p}(x)$ is the momentum operator in the position representation.
Thus, if $\psi(x)$ is an arbitrary wave function in the position representation, then

$$[x,\hat{p}(x)]\psi(x)=i\hbar\psi(x)\Rightarrow(x\hat{p}(x)-\hat{p}(x)x)\psi(x)=i\hbar\psi(x)\Rightarrow$$
$$\Rightarrow x\hat{p}(x)\psi(x)-\hat{p}(x)(x\psi(x))=i\hbar\psi(x)$$

That is

$$\hat{p}(x)(x\psi(x))-x\hat{p}(x)\psi(x)=-i\hbar\psi(x) \quad (8)$$

We observe that

$$\frac{d}{dx}(x\psi(x))=\psi(x)+x\frac{d}{dx}\psi(x)\Rightarrow\frac{d}{dx}(x\psi(x))-x\frac{d}{dx}\psi(x)=\psi(x)$$

Multiplying both sides of the last equation by $-i\hbar$, we obtain

$$-i\hbar\frac{d}{dx}(x\psi(x))-x\left(-i\hbar\frac{d}{dx}\right)\psi(x)=-i\hbar\psi(x)$$

Comparing the last equation with (8), we derive that

$$\hat{p}(x)=-i\hbar\frac{d}{dx} \quad (9)$$

This is the momentum operator in the position representation.

Let us check if the operator $-i\hbar\dfrac{d}{dx}$ is Hermitian, as it should be.
Consider two wave functions $f_1(x), f_2(x)$ that are square integrable on $\mathbb{R}$.
The inner product $(f_1,\hat{p}f_2)$ is then written as – using integration by parts –

$$\left(f_1, \hat{p}f_2\right) = \int_{-\infty}^{\infty} dx f_1^*\left(x\right)\left(-i\hbar\frac{d}{dx}f_2\left(x\right)\right) = -i\hbar\int_{-\infty}^{\infty} dx f_1^*\left(x\right)\frac{df_2\left(x\right)}{dx} =$$

$$= -i\hbar\int_{-\infty}^{\infty} dx\left(\frac{d}{dx}\left(f_1^*\left(x\right)f_2\left(x\right)\right) - \frac{df_1^*\left(x\right)}{dx}f_2\left(x\right)\right) =$$

$$= -i\hbar\left(f_1^*\left(x\right)f_2\left(x\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\frac{df_1^*\left(x\right)}{dx}f_2\left(x\right)\right)$$

Since the functions $f_1\left(x\right), f_2\left(x\right)$ are square integrable,

$$\left|f_1\left(\pm\infty\right)\right| = \left|f_2\left(\pm\infty\right)\right| = 0$$

Thus

$$f_1\left(\pm\infty\right) = f_2\left(\pm\infty\right) = 0$$

and thus

$$f_1^*\left(\pm\infty\right) = f_2\left(\pm\infty\right) = 0$$

Then, the inner product $\left(f_1, \hat{p}f_2\right)$ becomes

$$\left(f_1, \hat{p}f_2\right) = i\hbar\int_{-\infty}^{\infty} dx\frac{df_1^*\left(x\right)}{dx}f_2\left(x\right) = \left(-i\hbar\right)^*\int_{-\infty}^{\infty} dx\frac{df_1^*\left(x\right)}{dx}f_2\left(x\right) =$$

$$= \int_{-\infty}^{\infty} dx\left(-i\hbar\right)\frac{df_1^*\left(x\right)}{dx}f_2\left(x\right) = \int_{-\infty}^{\infty} dx\left(-i\hbar\frac{d}{dx}f_1^*\left(x\right)\right)f_2\left(x\right) = \left(\hat{p}f_1, f_2\right)$$

That is

$$\left(f_1, \hat{p}f_2\right) = \left(\hat{p}f_1, f_2\right)$$

But, by definition,

$$\left(f_1, \hat{p}f_2\right) \equiv \left(\hat{p}^\dagger f_1, f_2\right)$$

Comparing the last two equations, we conclude that

$$\hat{p} = \hat{p}^\dagger$$

The operator $-i\hbar\frac{d}{dx}$ is thus Hermitian, as it should be.

As a consequence, the operator $\frac{d}{dx}$ is anti-Hermitian.

Indeed, using that the operator $-i\hbar\frac{d}{dx}$ is Hermitian, we obtain

$$\left(-i\hbar\frac{d}{dx}\right)^\dagger = -i\hbar\frac{d}{dx} \Rightarrow i\hbar\left(\frac{d}{dx}\right)^\dagger = -i\hbar\frac{d}{dx} \Rightarrow \left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$

In other words, the derivative is anti-Hermitian operator.

We found that, in the position representation, $\hat{x}\left(x\right) = x$ and $\hat{p}\left(x\right) = -i\hbar\frac{d}{dx}$.

Then, an operator $f\left(\hat{x}, \hat{p}\right)$ is represented by the differential operator $f\left(x, -i\hbar\frac{d}{dx}\right)$.

1.2 Position representation of states

As we saw, in the position representation, a state $|\psi\rangle$ of the state space is represented by the wave function $\psi(x)=\langle x|\psi\rangle$, which is derived by projecting the state on the position eigenstates.
The wave function $\psi(x)$ is then the projection of the state $|\psi\rangle$ on the position space.
Consider now the state $f(\hat{x},\hat{p})|\psi\rangle$, i.e. the state resulting from the action of an operator $f(\hat{x},\hat{p})$ on a state $|\psi\rangle$.
In the position space, this state is represented by its projection on the position eigenstates, i.e. it is represented by the inner product $\langle x|f(\hat{x},\hat{p})|\psi\rangle$, for every (real) value of the position x.
Using the assumption we made in 1.1, the inner product $\langle x|f(\hat{x},\hat{p})|\psi\rangle$ is written as

$$\langle x|f(\hat{x},\hat{p})|\psi\rangle = f(\hat{x}(x),\hat{p}(x))\langle x|\psi\rangle$$

where $\hat{x}(x)$ and $\hat{p}(x)$ are, respectively, the position and momentum operators in the position representation, i.e. $\hat{x}(x)=x$ and $\hat{p}(x)=-i\hbar\dfrac{d}{dx}$.
Also, $\langle x|\psi\rangle=\psi(x)$ is the wave function of the state $|\psi\rangle$ in the position space.
Thus

$$\langle x|f(\hat{x},\hat{p})|\psi\rangle = f\left(x,-i\hbar\frac{d}{dx}\right)\psi(x)$$

Therefore, the position representation of the state $f(\hat{x},\hat{p})|\psi\rangle$ is the wave function $f\left(x,-i\hbar\dfrac{d}{dx}\right)\psi(x)$, or, in other words, the wave function $f\left(x,-i\hbar\dfrac{d}{dx}\right)\psi(x)$ is the projection of the state $f(\hat{x},\hat{p})|\psi\rangle$ on the position space.

1.3 Position and momentum eigenfunctions in the position representation

In the position representation, the position and momentum eigenstates are also represented by wave functions, which are called, respectively, position and momentum eigenfunctions.
We saw that the orthogonality of the position eigenstates is expressed by the relation $\langle x|x'\rangle=\delta(x-x')$.
But, $\langle x|x'\rangle$ is also the projection of the position eigenstate $|x'\rangle$ on the arbitrary position eigenstate $|x\rangle$, which is identified as the wave function of the position eigenstate $|x'\rangle$ in the position representation.
Therefore, the delta function $\delta(x-x')$ is the position eigenfunction with eigenvalue x'. It represents, in the position representation or in the position space, the state $|x'\rangle$.

Thus, in the position space, a particle located at x' is described by the delta function $\delta(x-x')$.

Now, we'll find the wave function representing, in the position space, the momentum eigenstate $|p\rangle$.
That is, we'll find the momentum eigenfunction with eigenvalue p, in the position space.
The meaning of the state $|p\rangle$ is that it describes a particle with momentum p.
Since the state $|p\rangle$ is the momentum eigenstate with eigenvalue p, it satisfies the eigenvalue equation

$$\hat{p}|p\rangle = p|p\rangle$$

To write the previous eigenvalue equation in the position representation, we project both sides of the equation on an arbitrary position eigenstate $|x\rangle$.
Thus, we have

$$\langle x|\hat{p}|p\rangle = \langle x|p|p\rangle \quad (10)$$

Since p is a number,

$$\langle x|p|p\rangle = p\langle x|p\rangle,$$

where $\langle x|p\rangle$ is the wave function of the state $|p\rangle$ in the position representation, i.e. it is the momentum eigenfunction with eigenvalue p, in the position representation.
Besides, using the assumption we made in 1.1, the inner product $\langle x|\hat{p}|p\rangle$ is written as

$$\langle x|\hat{p}|p\rangle = \hat{p}(x)\langle x|p\rangle,$$

where $\hat{p}(x)$ is the momentum operator in the position representation.
Thus, (10) is written as

$$\hat{p}(x)\langle x|p\rangle = p\langle x|p\rangle$$

Denoting by $p(x)$ the momentum eigenfunction $\langle x|p\rangle$ and using (9), the last equation is written as

$$-i\hbar\frac{d}{dx}p(x) = pp(x)$$

This is an easily solved differential equation.
Indeed, we have

$$-i\hbar p'(x) = pp(x) \Rightarrow \frac{p'(x)}{p(x)} = \frac{ip}{\hbar} \Rightarrow \frac{d}{dx}\ln\left(p(x)\right) = \frac{ip}{\hbar} \Rightarrow \ln\left(p(x)\right) = \frac{ipx}{\hbar} + C \Rightarrow$$

$$\Rightarrow p(x) = \exp\left(\frac{ipx}{\hbar} + C\right) = \underbrace{\exp(C)}_{A}\exp\left(\frac{ipx}{\hbar}\right) = A\exp\left(\frac{ipx}{\hbar}\right)$$

That is

$$p(x) = A\exp\left(\frac{ipx}{\hbar}\right) \quad (11)$$

where A is a complex constant.

Note that the constant A does not depend either on the position x or on the momentum p.

We observe that the momentum eigenfunctions are not square integrable, since from (11) we have $|p(x)| = |A|$, and thus $|p(\pm\infty)| = |A| > 0$.
Therefore, the momentum eigenfunctions are not normalizable.
Thus, the constant A cannot be calculated by the normalization condition.
However, it can be calculated using the orthogonality of the momentum eigenstates.
From (11), we see that each eigenvalue p of the momentum corresponds to only one momentum eigenfunction $p(x)$. Thus, the momentum eigenfunctions, and the momentum eigenstates too, are non-degenerate.
Then, since the momentum operator is Hermitian, as it describes the momentum, which is an observable quantity, its non-degenerate eigenstates are orthogonal.
Thus, since the spectrum of the momentum operator is continuous, the orthogonality of the momentum eigenstates $|p\rangle$ and $|p'\rangle$ is expressed by the relation

$$\delta(p'-p) = \langle p'|p\rangle \quad (12)$$

Using the completeness relation of the position eigenstates, i.e.

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = 1,$$

(12) is written as

$$\delta(p'-p) = \langle p'|\left(\underbrace{\int_{-\infty}^{\infty} dx|x\rangle\langle x|}_{1}\right)|p\rangle = \int_{-\infty}^{\infty} dx\langle p'|x\rangle\langle x|p\rangle = \int_{-\infty}^{\infty} dx\langle x|p'\rangle^* \langle x|p\rangle$$

That is

$$\delta(p'-p) = \int_{-\infty}^{\infty} dx\langle x|p'\rangle^* \langle x|p\rangle \quad (13)$$

But $\langle x|p\rangle = p(x)$ is the momentum eigenfunction with eigenvalue (momentum) p, and, obviously, $\langle x|p'\rangle = p'(x)$ is the momentum eigenfunction with eigenvalue (momentum) p'.
Then, (13) is written as

$$\delta(p'-p) = \int_{-\infty}^{\infty} dxp'^*(x)p(x) \quad (14)$$

This is the orthogonality relation of the momentum eigenfunctions in the position space.
By means of (11), (14) becomes

$$\delta(p'-p)=\int_{-\infty}^{\infty}dxA^*\exp\left(-\frac{ip'x}{\hbar}\right)A\exp\left(\frac{ipx}{\hbar}\right)=|A|^2\int_{-\infty}^{\infty}dx\exp\left(\frac{i(p-p')x}{\hbar}\right)$$

> We remind that the constant A in (11) does not depend either on the position or on the momentum.

That is

$$\delta(p'-p)=|A|^2\int_{-\infty}^{\infty}dx\exp\left(\frac{i(p-p')x}{\hbar}\right) \quad (15)$$

Now, we'll use one of the integral representations of the delta function, and particularly the relation

$$\delta(v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}du\exp(iuv),$$

where v is a real parameter.
For $v\equiv p'-p$, we obtain

$$\delta(p'-p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}du\exp\left(iu(p'-p)\right) \quad (16)$$

Comparing (15) and (16), we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}du\exp\left(iu(p'-p)\right)=|A|^2\int_{-\infty}^{\infty}dx\exp\left(\frac{i(p-p')x}{\hbar}\right) \quad (17)$$

To bring both integrals in the same form, we change the integration variable x, i.e. the position, to $y=\frac{x}{\hbar}$.
Then, we have

$$x=\hbar y\Rightarrow dx=\hbar dy$$

and

since $x:-\infty\to\infty$, then $y:-\infty\to\infty$

Thus, the integral on the right hand side of (17) becomes

$$\int_{-\infty}^{\infty}dx\exp\left(\frac{i(p-p')x}{\hbar}\right)=\int_{-\infty}^{\infty}\hbar dy\exp\left(iy(p-p')\right)$$

Then, (17) is written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}du\exp\left(iu(p'-p)\right)=|A|^2\hbar\int_{-\infty}^{\infty}dy\exp\left(iy(p-p')\right)$$

The integration variables are dummy variables and we change y to u or u to y to show that the two integrals are equal. Then, the last equation gives

$$\frac{1}{2\pi} = |A|^2 \hbar \Rightarrow |A| = \frac{1}{\sqrt{2\pi\hbar}}$$

Omitting the physically unimportant phase of A, we end up to

$$A = \frac{1}{\sqrt{2\pi\hbar}} \quad (18)$$

By means of (18), (11) is written as

$$p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) \quad (19)$$

This is the momentum eigenfunction with eigenvalue (momentum) p, in the position space or in the position representation.
Also, since $p(x) = \langle x|p\rangle$, (19) is also written as

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) \quad (20)$$

1.4 An example: the time-independent Schrödinger equation (TISE)

Consider a particle of mass m moving in a time-independent potential $V(x)$. Its Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad (21)$$

If $|E\rangle$ is an arbitrary energy eigenstate of the particle, with eigenvalue E, then the particle's energy eigenvalue equation is written as

$$\hat{H}|E\rangle = E|E\rangle$$

Projecting both sides of the previous equation on an arbitrary position eigenstate $|x\rangle$, we obtain

$$\langle x|\hat{H}|E\rangle = \langle x|E|E\rangle = E\langle x|E\rangle$$

That is

$$\langle x|\hat{H}|E\rangle = E\langle x|E\rangle \quad (22)$$

Using the assumption we made in 1.1, we have

$$\langle x|\hat{H}|E\rangle = \hat{H}(x)\langle x|E\rangle \quad (23)$$

where $\hat{H}(x)$ is the Hamiltonian (21) in the position representation.
Comparing (22) and (23) yields

$$\hat{H}(x)\langle x|E\rangle = E\langle x|E\rangle \quad (24)$$

To write the Hamiltonian in the position representation, we replace the position and momentum operators with their expressions in the position representation, i.e. with x and $-i\hbar\dfrac{d}{dx}$, respectively.

Thus

$$\hat{H}(x)=\frac{\left(-i\hbar\dfrac{d}{dx}\right)^2}{2m}+V(x)=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$$

That is

$$\hat{H}(x)=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$$

Also, $\langle x|E\rangle=\psi_E(x)$ is the energy eigenfunction with eigenvalue, i.e. with energy, E.

Substituting into (24) yields

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi_E(x)=E\psi_E(x)\Rightarrow-\frac{\hbar^2}{2m}\psi_E{}''(x)+V(x)\psi_E(x)=E\psi_E(x)\Rightarrow$$

$$\Rightarrow-\frac{\hbar^2}{2m}\psi_E{}''(x)-\left(E-V(x)\right)\psi_E(x)=0\Rightarrow\psi_E{}''(x)+\frac{2m}{\hbar^2}\left(E-V(x)\right)\psi_E(x)=0$$

That is, the energy eigenvalue equation in the position representation, for a particle with mass m moving in a time-independent potential $V(x)$, is

$$\psi_E{}''(x)+\frac{2m}{\hbar^2}\left(E-V(x)\right)\psi_E(x)=0$$

This is the well-known time-independent Schrödinger equation (TISE).

Therefore, the time-independent Schrödinger equation is the energy eigenvalue equation in the position space.

2. The momentum representation – The momentum space

As in the case of the position, the momentum is also an observable quantity, and thus the respective operator, i.e. the momentum operator, is Hermitian and its eigenstates – the momentum eigenstates – form an orthogonal basis in the state space (Hilbert space) of the examined particle (or system).
We can use this basis to represent the states and operators of the state space.
This representation is called the momentum representation.
In other words, the momentum representation is the representation of the states and operators of the state space of the particle (system) in the basis of the momentum eigenstates.
More abstractly, we can think of the momentum representation as the representation of the whole state space in the basis of the momentum eigenstates. In this sense, we'll refer to the momentum representation as the momentum space.

2.1 The position and momentum operators in the momentum representation

As in the case of the position representation, we'll examine the one-dimensional case, as the three-dimensional case follows easily.
Similarly to what we did to find the position operator in the position representation, we'll calculate the inner product $\langle p|\hat{p}|\psi\rangle$ and from this we'll derive the momentum operator (in the momentum representation).
Using the completeness relation of the momentum eigenstates, i.e. the relation

$$\int_{-\infty}^{\infty} dp'|p'\rangle\langle p'|=1,$$

the previous inner product is written as

$$\langle p|\hat{p}|\psi\rangle=\langle p|\hat{p}\left(\underbrace{\int_{-\infty}^{\infty} dp'|p'\rangle\langle p'|}_{1}\right)|\psi\rangle=\int_{-\infty}^{\infty} dp\langle p|\hat{p}|p'\rangle\langle p'|\psi\rangle$$

> We remind that the spectrum of the momentum operator is continuous, as it happens with the spectrum of the position operator too. Thus, the completeness relation of the momentum eigenstates is expressed by an integral.

Thus

$$\langle p|\hat{p}|\psi\rangle=\int_{-\infty}^{\infty} dp\langle p|\hat{p}|p'\rangle\langle p'|\psi\rangle \quad (1)$$

Since the state $|p'\rangle$ is a momentum eigenstate with eigenvalue, i.e. with momentum, p', then

$$\hat{p}|p'\rangle=p'|p'\rangle$$

Thus

$\langle p|\hat{p}|p'\rangle = \langle p|p'|p'\rangle = p'\langle p|p'\rangle$

The eigenstates $|p\rangle$ and $|p'\rangle$ are orthogonal, and thus

$\langle p|p'\rangle = \delta(p-p')$

Therefore

$\langle p|\hat{p}|p'\rangle = p'\delta(p-p')$ (2)

By means of (2), (1) becomes

$$\langle p|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} dpp'\delta(p-p')\langle p'|\psi\rangle = p\langle p|\psi\rangle$$

That is

$\langle p|\hat{p}|\psi\rangle = p\langle p|\psi\rangle$ (3)

Reading the inner product $\langle p|\hat{p}|\psi\rangle$ as the projection of the state $\hat{p}|\psi\rangle$ on the momentum space, and using the assumption we made in 1.1, we write it as

$\langle p|\hat{p}|\psi\rangle = \hat{p}(p)\langle p|\psi\rangle$ (4)

where $\hat{p}(p)$ is the expression – the "projection" – of the momentum operator in the momentum space, i.e. it is the momentum representation of the momentum operator. Comparing (3) and (4) yields

$\hat{p}(p)\langle p|\psi\rangle = p\langle p|\psi\rangle$

Since the state $|\psi\rangle$ is arbitrary, and so is the momentum eigenstate $|p\rangle$, the last equation gives

$\hat{p}(p) = p$ (5)

This is the expression of the momentum operator in the momentum space, i.e. it is the momentum operator in the momentum representation.
We see that, in the momentum space, the momentum operator is the momentum coordinate, or else, it is equal to its eigenvalues, which is expected, since in the basis of its eigenstates, the momentum operator is diagonal.
Besides, the inner product $\langle p|\psi\rangle$, which is the projection of the state $|\psi\rangle$ on the momentum space, is a complex function of the real variable p.
We identify this function as the wave function in the momentum space – or in the momentum representation – and we denote it by $\tilde{\psi}(p)$, i.e.

$\tilde{\psi}(p) = \langle p|\psi\rangle$ (6)

> We put a tilde on the wave function in the momentum representation to distinguish it from the wave function of the same state in the position representation and, mainly, to indicate that the two wave functions, i.e. the functions $\psi(x)$ and

$\tilde{\psi}(p)$, are, generally, different. That is, the function $\tilde{\psi}(p)$ is NOT the function $\psi(x)$ with the momentum p in place of the position x.

Having calculated the momentum operator in the momentum representation, we'll use the commutator $[\hat{x},\hat{p}]=i\hbar$, which holds in every representation, to derive the position operator in the momentum representation.
Similarly to what we did to find the momentum operator in the position representation, choosing an arbitrary wave function $\phi(p)$ in the momentum representation, we have

$$[\hat{x}(p),p]\phi(p)=i\hbar\phi(p)$$

where $\hat{x}(p)$ is the position operator in the momentum representation.
The previous equation is written as

$$\hat{x}(p)(p\phi(p))-p\hat{x}(p)\phi(p)=i\hbar\phi(p) \quad (7)$$

We observe that

$$\frac{d}{dp}(p\phi(p))=\phi(p)+p\frac{d}{dp}\phi(p)\Rightarrow\frac{d}{dp}(p\phi(p))-p\frac{d}{dp}\phi(p)=\phi(p)$$

Multiplying both sides of the previous equation by $i\hbar$, we obtain

$$i\hbar\frac{d}{dp}(p\phi(p))-p\left(i\hbar\frac{d}{dp}\right)\phi(p)=i\hbar\phi(p) \quad (8)$$

Comparing (7) and (8) yields

$$\hat{x}(p)=i\hbar\frac{d}{dp} \quad (9)$$

This is the expression of the position operator in the momentum space, i.e. it is the position operator in the momentum representation.

2.2 Momentum representation of states

As in the case of the position representation, the states of the state space are represented by wave functions derived by projecting the states on the basis states, i.e. on the momentum eigenstates.
Thus, a state $|\psi\rangle$ is represented by the wave function $\tilde{\psi}(p)=\langle p|\psi\rangle$, which is the projection of the state $|\psi\rangle$ on the momentum space.
The wave function $\tilde{\psi}(p)$ is the wave function in the momentum representation or in the momentum space.
Consider now the state $f(\hat{x},\hat{p})|\psi\rangle$, i.e. the state resulting from the action of an operator $f(\hat{x},\hat{p})$ on a state $|\psi\rangle$.
In the momentum space, this state is represented by its projection on the momentum eigenstates, i.e. it is represented by the inner product $\langle p|f(\hat{x},\hat{p})|\psi\rangle$, for every (real) value of the momentum p.

Using the assumption we made in 1.1, the inner product $\langle p|f(\hat{x},\hat{p})|\psi\rangle$ is written as

$$\langle p|f(\hat{x},\hat{p})|\psi\rangle = f(\hat{x}(p),\hat{p}(p))\langle p|\psi\rangle$$

By means of (5), (6), and (9), the last equation is written as

$$\langle p|f(\hat{x},\hat{p})|\psi\rangle = f\left(i\hbar\frac{d}{dp},p\right)\tilde{\psi}(p)$$

Therefore, the momentum representation of the state $f(\hat{x},\hat{p})|\psi\rangle$ is the wave function $f\left(i\hbar\frac{d}{dp},p\right)\tilde{\psi}(p)$, or, in other words, the wave function $f\left(i\hbar\frac{d}{dp},p\right)\tilde{\psi}(p)$ is the projection of the state $f(\hat{x},\hat{p})|\psi\rangle$ on the momentum space.

2.3 Position and momentum eigenfunctions in the momentum representation

In the momentum representation, the position and momentum eigenstates are also represented by wave functions, as it happens in the position representation too. These wave functions are called, respectively, position and momentum eigenfunctions in the momentum representation, and they are, respectively, the projection of the position and momentum eigenstates on the momentum space.

The wave function representing the arbitrary momentum eigenstate $|p'\rangle$ is derived by projecting the eigenstate $|p'\rangle$ on the momentum eigenstates $|p\rangle$, for every (real) value of the momentum p, i.e. it is the wave function $\langle p|p'\rangle$.

Since the momentum eigenstates are orthogonal and the spectrum of the momentum operator is continuous, then

$$\langle p|p'\rangle = \delta(p-p')$$

Therefore, the delta function $\delta(p-p')$ is the momentum eigenfunction with eigenvalue p'. It represents, in the momentum representation or in the momentum space, the state $|p'\rangle$.

Likewise, the wave function representing the arbitrary position eigenstate $|x\rangle$ in the momentum space is derived by projecting the state $|x\rangle$ on the momentum eigenstates $|p\rangle$, for every (real) value of the momentum p, i.e. it is the wave function $\langle p|x\rangle$.

In 1.3, we showed that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right)$$

Thus, since $\langle p|x\rangle = \langle x|p\rangle^*$, we obtain

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{ipx}{\hbar}\right) \quad (10)$$

This is the wave function representing the position eigenstate $|x\rangle$ in the momentum space, i.e. it is the position eigenfunction with eigenvalue x in the momentum space, and it describes, in the momentum space, a particle being at x. It is a function of the momentum p and we denote it by $x(p)$.
Then, (10) is also written as

$$x(p)=\frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{ipx}{\hbar}\right) \quad (11)$$

In 1.3, we showed that, in the position space, the momentum eigenfunction with eigenvalue p is

$$p(x)=\frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right)$$

Note
From the relation $\langle p|x\rangle=\langle x|p\rangle^*$, we see that the position eigenfunctions in the momentum space are the complex conjugates of the momentum eigenfunctions in the position space.
Although, in the position space, the position is a variable and the momentum is a parameter, while, in the momentum space, the momentum is a variable and the position is a parameter.
Thus, the position eigenfunction $\langle p|x\rangle$ in the momentum space, i.e. the function $x(p)$, is a function of the momentum p, while the momentum eigenfunction $\langle x|p\rangle$ in the position space, i.e. the function $p(x)$, is a function of the position x.

3. The relation between the wave functions in the one-dimensional position and momentum spaces – The Fourier transform of the wave function

Let $|\psi\rangle$ be an arbitrary state.

Then, $\psi(x) = \langle x|\psi\rangle$ and $\tilde{\psi}(p) = \langle p|\psi\rangle$ are, respectively, the wave functions of the state $|\psi\rangle$ in the position and momentum spaces.

Using the completeness relation of the position eigenstates, i.e. the relation

$$\int_{-\infty}^{\infty} dx |x\rangle\langle x| = 1,$$

the wave function $\tilde{\psi}(p)$ is written as

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \langle p|\left(\underbrace{\int_{-\infty}^{\infty} dx |x\rangle\langle x|}_{1}\right)|\psi\rangle = \int_{-\infty}^{\infty} dx \langle p|x\rangle\langle x|\psi\rangle$$

That is

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx \langle p|x\rangle\langle x|\psi\rangle \quad (1)$$

But

$$\langle x|\psi\rangle = \psi(x)$$

and

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ipx}{\hbar}\right)$$

Thus, (1) is written as

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{ipx}{\hbar}\right)\psi(x) \quad (2)$$

We see that the wave function in the momentum space is the Fourier transform of the wave function in the position space. That is, $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$.

With the same reasoning, using the completeness relation of the momentum eigenstates, i.e. the relation

$$\int_{-\infty}^{\infty} dp |p\rangle\langle p| = 1,$$

the wave function $\psi(x)$ is written as

$$\psi(x) = \langle x|\psi\rangle = \langle x|\left(\underbrace{\int_{-\infty}^{\infty} dp |p\rangle\langle p|}_{1}\right)|\psi\rangle = \int_{-\infty}^{\infty} dp \langle x|p\rangle\langle p|\psi\rangle$$

That is

$$\psi(x)=\int_{-\infty}^{\infty}dp\langle x|p\rangle\langle p|\psi\rangle \quad (3)$$

But

$$\langle p|\psi\rangle=\tilde{\psi}(p)$$

and

$$\langle x|p\rangle=\frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right)$$

Thus, (3) is written as

$$\psi(x)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dp\exp\left(\frac{ipx}{\hbar}\right)\tilde{\psi}(p) \quad (4)$$

As should have been expected, the wave function in the position space is the inverse Fourier transform of the wave function in the momentum space, i.e. $\psi(x)$ is the inverse Fourier transform of $\tilde{\psi}(p)$.

Using (2), we derive the wave function in the momentum space from the wave function in the position space and, using (4), we derive the wave function in the position space from the wave function in the momentum space.

3.1 Two useful properties

i) We'll show that $\langle\psi|\psi\rangle=\int_{-\infty}^{\infty}dx|\psi(x)|^2=\int_{-\infty}^{\infty}dp|\tilde{\psi}(p)|^2$.

Proof

Using the completeness relation of the position eigenstates, the inner product $\langle\psi|\psi\rangle$ is written as

$$\langle\psi|\psi\rangle=\langle\psi|\left(\underbrace{\int_{-\infty}^{\infty}dx|x\rangle\langle x|}_{1}\right)|\psi\rangle=\int_{-\infty}^{\infty}dx\langle\psi|x\rangle\langle x|\psi\rangle=$$

$$=\int_{-\infty}^{\infty}dx\langle x|\psi\rangle^{*}\langle x|\psi\rangle=\int_{-\infty}^{\infty}dx|\langle x|\psi\rangle|^2$$

Using that $\langle x|\psi\rangle=\psi(x)$, we end up to

$$\langle\psi|\psi\rangle=\int_{-\infty}^{\infty}dx|\psi(x)|^2 \quad (5)$$

In the same way, using the completeness relation of the momentum eigenstates, we write the inner product $\langle\psi|\psi\rangle$ as

$$\langle\psi|\psi\rangle = \langle\psi|\left(\underbrace{\int_{-\infty}^{\infty} dp|p\rangle\langle p|}_{1}\right)|\psi\rangle = \int_{-\infty}^{\infty} dp\langle\psi|p\rangle\langle p|\psi\rangle =$$

$$= \int_{-\infty}^{\infty} dp\langle p|\psi\rangle^{*}\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp\left|\underbrace{\langle p|\psi\rangle}_{\tilde{\psi}(p)}\right|^{2} = \int_{-\infty}^{\infty} dp\left|\tilde{\psi}(p)\right|^{2}$$

That is

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dp\left|\tilde{\psi}(p)\right|^{2} \quad (6)$$

Combining (5) and (6) yields

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\left|\psi(x)\right|^{2} = \int_{-\infty}^{\infty} dp\left|\tilde{\psi}(p)\right|^{2}$$

Notes

1. Since $\langle\psi|\psi\rangle = \left\||\psi\rangle\right\|^{2}$, the previous property is also written as

$$\left\||\psi\rangle\right\| = \sqrt{\int_{-\infty}^{\infty} dx\left|\psi(x)\right|^{2}} = \sqrt{\int_{-\infty}^{\infty} dp\left|\tilde{\psi}(p)\right|^{2}}$$

Observe that only if the state $|\psi\rangle$ is bound, i.e. if its norm is finite, the wave functions $\psi(x)$ and $\tilde{\psi}(p)$ are square integrable.

2. You should also keep in mind the useful relation (Parseval-Plancherel formula)

$$\int_{-\infty}^{\infty} dx\left|\psi(x)\right|^{2} = \int_{-\infty}^{\infty} dp\left|\tilde{\psi}(p)\right|^{2}$$

ii) We'll show that if $\psi(x)$ is even/odd, then and only then $\tilde{\psi}(p)$ is even/odd too.

Proof

In (2), we change the integration variable to $-x$ and the integral is written as

$$\int_{-\infty}^{\infty} dx\exp\left(-\frac{ipx}{\hbar}\right)\psi(x) = \int_{\infty}^{-\infty} d(-x)\exp\left(-\frac{ip(-x)}{\hbar}\right)\psi(-x) =$$

$$= -\int_{\infty}^{-\infty} dx\exp\left(-\frac{i(-p)x}{\hbar}\right)\psi(-x) = \int_{-\infty}^{\infty} dx\exp\left(-\frac{i(-p)x}{\hbar}\right)\psi(-x)$$

That is

$$\int_{-\infty}^{\infty} dx\exp\left(-\frac{ipx}{\hbar}\right)\psi(x) = \int_{-\infty}^{\infty} dx\exp\left(-\frac{i(-p)x}{\hbar}\right)\psi(-x) \quad (7)$$

If $\psi(-x) = \pm\psi(x)$, (7) is written as

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{ipx}{\hbar}\right)\psi(x) = \pm\int_{-\infty}^{\infty} dx \exp\left(-\frac{i(-p)x}{\hbar}\right)\psi(x)$$

or

$$\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{ipx}{\hbar}\right)\psi(x)}_{\tilde{\psi}(p)} = \pm\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{i(-p)x}{\hbar}\right)\psi(x)}_{\tilde{\psi}(-p)}$$

That is

$$\tilde{\psi}(p) = \pm\tilde{\psi}(-p)$$

or, multiplying both sides by ± 1, we obtain

$$\tilde{\psi}(-p) = \pm\tilde{\psi}(p)$$

Thus, we showed that if $\psi(-x) = \pm\psi(x)$, then $\tilde{\psi}(-p) = \pm\tilde{\psi}(p)$. In other words, if $\psi(x)$ is even/odd, then $\tilde{\psi}(p)$ is even/odd too.

In the same way, changing the integration variable in (4) to $-p$, the integral is written as

$$\int_{-\infty}^{\infty} dp \exp\left(\frac{ipx}{\hbar}\right)\tilde{\psi}(p) = \int_{\infty}^{-\infty} d(-p) \exp\left(\frac{i(-p)x}{\hbar}\right)\tilde{\psi}(-p) =$$

$$= -\int_{\infty}^{-\infty} dp \exp\left(\frac{ip(-x)}{\hbar}\right)\tilde{\psi}(-p) = \int_{-\infty}^{\infty} dp \exp\left(\frac{ip(-x)}{\hbar}\right)\tilde{\psi}(-p)$$

That is

$$\int_{-\infty}^{\infty} dp \exp\left(\frac{ipx}{\hbar}\right)\tilde{\psi}(p) = \int_{-\infty}^{\infty} dp \exp\left(\frac{ip(-x)}{\hbar}\right)\tilde{\psi}(-p)$$

Thus, if $\tilde{\psi}(-p) = \pm\tilde{\psi}(p)$, then

$$\int_{-\infty}^{\infty} dp \exp\left(\frac{ipx}{\hbar}\right)\tilde{\psi}(p) = \pm\int_{-\infty}^{\infty} dp \exp\left(\frac{ip(-x)}{\hbar}\right)\tilde{\psi}(p)$$

or

$$\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp \exp\left(\frac{ipx}{\hbar}\right)\tilde{\psi}(p)}_{\psi(x)} = \pm\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp \exp\left(\frac{ip(-x)}{\hbar}\right)\tilde{\psi}(p)}_{\psi(-x)}$$

That is

$$\psi(x) = \pm\psi(-x)$$

or, multiplying both sides by ± 1, we obtain

$$\psi(-x) = \pm\psi(x)$$

Thus, if $\tilde{\psi}(p)$ is even/odd, then $\psi(x)$ is even/odd too.

4. The three-dimensional position and momentum spaces

The generalization to three dimensions is straightforward. In three dimensions, the arbitrary position eigenstate is the state $|\vec{r}\rangle$, where the position vector $\vec{r}$ defines a possible position of the particle, and, likewise, the arbitrary momentum eigenstate is the state $|\vec{p}\rangle$, where the momentum vector $\vec{p}$ defines a possible momentum of the particle. That is, the position eigenstate $|\vec{r}\rangle$ describes a particle being at $\vec{r}$, while the momentum eigenstate $|\vec{p}\rangle$ describes a particle with momentum $\vec{p}$.
The completeness relation of the position eigenstates is written as

$$\int_{-\infty}^{\infty} d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}| = 1,$$

where, in Cartesian position coordinates, the integration limits are from $-\infty$ to ∞ on each axis x, y, z.
Similarly, the completeness relation of the momentum eigenstates is written as

$$\int_{-\infty}^{\infty} d^3\vec{p}\,|\vec{p}\rangle\langle\vec{p}| = 1,$$

where, in Cartesian momentum coordinates, the integration limits are from $-\infty$ to ∞ on each axis p_x, p_y, p_z.
As in one-dimension, the position and momentum eigenstates are non-degenerate and thus, since the respective operators, i.e. the position and momentum operators, are Hermitian, the position and momentum eigenstates are orthogonal.
The orthogonality of the position eigenstates is written as

$$\langle\vec{r}'|\vec{r}\rangle = \delta(\vec{r}' - \vec{r})$$

and, similarly, the orthogonality of the momentum eigenstates is written as

$$\langle\vec{p}'|\vec{p}\rangle = \delta(\vec{p}' - \vec{p})$$

Since the spectra of both operators are continuous, the orthogonality relations are expressed by delta functions.
In Cartesian coordinates, the three-dimensional delta function $\delta(\vec{r}' - \vec{r})$ is

$$\delta(\vec{r}' - \vec{r}) = \delta(x' - x)\delta(y' - y)\delta(z' - z)$$

and, similarly,

$$\delta(\vec{p}' - \vec{p}) = \delta(p'_x - p_x)\delta(p'_y - p_y)\delta(p'_z - p_z)$$

In the position space, the position operator is the position vector, i.e.

$$\hat{\vec{r}}(\vec{r}) = \vec{r},$$

while the momentum operator is

$$\hat{\vec{p}}(\vec{r}) = -i\hbar\vec{\nabla},$$

where, in Cartesian position coordinates, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z$.

In the momentum space, the momentum operator is the momentum vector, i.e.

$$\hat{\vec{p}}(\vec{p}) = \vec{p},$$

while the position operator is

$$\hat{\vec{r}}(\vec{p}) = i\hbar\vec{\nabla}_p,$$

where, in Cartesian momentum coordinates, $\vec{\nabla}_p = \frac{\partial}{\partial p_x}\hat{e}_{p_x} + \frac{\partial}{\partial p_y}\hat{e}_{p_y} + \frac{\partial}{\partial p_z}\hat{e}_{p_z}$.

In three dimensions, the commutation relation between position and momentum is written as

$$\left[\hat{r}_i, \hat{p}_j\right] = i\hbar\delta_{ij},\ i, j = 1,2,3,$$

where 1 stands for x, 2 for y, and 3 for z.

The previous commutators hold in every representation, i.e. they hold in both the position and momentum spaces.

As in one dimension, the projection of a state on the position eigenstates gives the wave function of the state in the position space, while the projection of the state on the momentum eigenstates gives the wave function of the state in the momentum space.

Thus, for a state $|\psi\rangle$, the wave function in the position space is

$$\psi(\vec{r}) = \langle\vec{r}|\psi\rangle$$

and the wave function in the momentum space is

$$\tilde{\psi}(\vec{p}) = \langle\vec{p}|\psi\rangle$$

4.1 Position and momentum eigenfunctions in the three-dimensional position space

The position eigenfunction with eigenvalue $\vec{r}'$ is the function $\langle\vec{r}|\vec{r}'\rangle$, which is the delta function $\delta(\vec{r}-\vec{r}')$, as the position eigenstates $|\vec{r}\rangle$ and $|\vec{r}'\rangle$ are orthogonal and the spectrum of the position operator is continuous.

Therefore, as in the one-dimensional position space, the position eigenfunctions in the three-dimensional position space are delta functions.

The momentum eigenfunction with eigenvalue $\vec{p}$ is derived by solving the momentum eigenvalue equation

$$\hat{\vec{p}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$$

in the position space.

Projecting both sides of the previous equation on an arbitrary position eigenstate $|\vec{r}\rangle$, we obtain

$$\langle\vec{r}|\hat{\vec{p}}|\vec{p}\rangle = \langle\vec{r}|\vec{p}|\vec{p}\rangle$$

Using the assumption we made in 1.1, the previous relation is written as

$$\hat{\vec{p}}(\vec{r})\langle\vec{r}|\vec{p}\rangle = \vec{p}\langle\vec{r}|\vec{p}\rangle \quad (1)$$

where $\langle\vec{r}|\vec{p}\rangle$ is the momentum eigenfunction with eigenvalue $\vec{p}$, in the position space.

For each $\vec{r}$ and $\vec{p}$, the inner product $\langle\vec{r}|\vec{p}\rangle$ is a complex number.

In the position space, the position vector $\vec{r}$ is variable, while the momentum vector $\vec{p}$ is fixed – it is a parameter.

Thus, the momentum eigenfunction $\langle\vec{r}|\vec{p}\rangle$ is a scalar complex function of $\vec{r}$, or, in Cartesian coordinates, it is a scalar complex function of the three real variables x, y, z.

We emphasize that, in both the one- and three-dimensional position spaces, the momentum eigenfunctions are scalar functions. But, in one-dimension, they are scalar functions of x only, while in three dimensions, they are scalar functions of x, y, z.

Denoting the momentum eigenfunction $\langle\vec{r}|\vec{p}\rangle$ by $p(\vec{r})$, i.e.

$$p(\vec{r}) = \langle\vec{r}|\vec{p}\rangle \quad (2)$$

and using that $\hat{\vec{p}}(\vec{r}) = -i\hbar\vec{\nabla}$, (1) is written as

$$-i\hbar\vec{\nabla}p(\vec{r}) = \vec{p}p(\vec{r}) \quad (3)$$

This is the momentum eigenvalue equation in the three-dimensional position space.

In Cartesian coordinates, (3) is written as

$$-i\hbar\left(\frac{\partial p(\vec{r})}{\partial x}\hat{e}_x + \frac{\partial p(\vec{r})}{\partial y}\hat{e}_y + \frac{\partial p(\vec{r})}{\partial z}\hat{e}_z\right) = \left(p_x\hat{e}_x + p_y\hat{e}_y + p_z\hat{e}_z\right)p(\vec{r}) \quad (4)$$

or, since $p(\vec{r}) \neq 0$, otherwise $p(\vec{r})$ wouldn't be an eigenfunction,

$$\frac{\dfrac{\partial p(\vec{r})}{\partial x}}{p(\vec{r})}\hat{e}_x + \frac{\dfrac{\partial p(\vec{r})}{\partial y}}{p(\vec{r})}\hat{e}_y + \frac{\dfrac{\partial p(\vec{r})}{\partial z}}{p(\vec{r})}\hat{e}_z = \frac{ip_x}{\hbar}\hat{e}_x + \frac{ip_y}{\hbar}\hat{e}_y + \frac{ip_z}{\hbar}\hat{e}_z \Rightarrow$$

$$\Rightarrow \frac{\partial \ln p(\vec{r})}{\partial x}\hat{e}_x + \frac{\partial \ln p(\vec{r})}{\partial y}\hat{e}_y + \frac{\partial \ln p(\vec{r})}{\partial z}\hat{e}_z = \frac{ip_x}{\hbar}\hat{e}_x + \frac{ip_y}{\hbar}\hat{e}_y + \frac{ip_z}{\hbar}\hat{e}_z \Rightarrow$$

$$\Rightarrow \left(\frac{\partial \ln p(\vec{r})}{\partial x} - \frac{ip_x}{\hbar}\right)\hat{e}_x + \left(\frac{\partial \ln p(\vec{r})}{\partial y} - \frac{ip_y}{\hbar}\right)\hat{e}_y + \left(\frac{\partial \ln p(\vec{r})}{\partial z} - \frac{ip_z}{\hbar}\right)\hat{e}_z = 0$$

Thus, since the unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are linearly independent, from the last equation we obtain

$$\frac{\partial \ln p(\vec{r})}{\partial x} - \frac{ip_x}{\hbar} = 0 \quad (5)$$

$$\frac{\partial \ln p(\vec{r})}{\partial y} - \frac{ip_y}{\hbar} = 0 \quad (6)$$

$$\frac{\partial \ln p(\vec{r})}{\partial z} - \frac{ip_z}{\hbar} = 0 \quad (7)$$

From (5) we obtain

$$\ln p(\vec{r}) = \frac{ip_x x}{\hbar} + f_1(y,z)$$

or

$$p(\vec{r}) = \exp\left(\frac{ip_x x}{\hbar} + f_1(y,z)\right) = \exp(f_1(y,z))\exp\left(\frac{ip_x x}{\hbar}\right)$$

That is

$$p(\vec{r}) = \exp(f_1(y,z))\exp\left(\frac{ip_x x}{\hbar}\right) \quad (8)$$

Similarly, from (6) and (7) we obtain, respectively,

$$p(\vec{r}) = \exp(f_2(z,x))\exp\left(\frac{ip_y y}{\hbar}\right) \quad (9)$$

$$p(\vec{r}) = \exp(f_3(x,y))\exp\left(\frac{ip_z z}{\hbar}\right) \quad (10)$$

Comparing (8), (9), and (10), we derive that

$$p(\vec{r}) = A\exp\left(\frac{ip_x x}{\hbar}\right)\exp\left(\frac{ip_y y}{\hbar}\right)\exp\left(\frac{ip_z z}{\hbar}\right)$$

or

$$p(\vec{r}) = A\exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \quad (11)$$

where A is a complex constant.

We could have well arrived at the equation (11) by separating the variables in (4), i.e. by setting $p(\vec{r}) = X(x)Y(y)Z(z)$, and converting the solution to the product of three one-dimensional solutions. Then, using the result of the one-dimensional case, the momentum eigenfunction $p(\vec{r})$ is written as

$$p(\vec{r}) = A_x \exp\left(\frac{ip_x x}{\hbar}\right) A_y \exp\left(\frac{ip_y y}{\hbar}\right) A_z \exp\left(\frac{ip_z z}{\hbar}\right) \quad (12)$$

Setting $A = A_x A_y A_z$, we end up to (11).

Using (11), we see that $|p(\vec{r})| = |A|$, i.e. the magnitude of the momentum eigenfunction is a positive constant, and thus, as in the one-dimensional position space, the momentum eigenfunctions in the three-dimensional position space are not normalizable.

As in the one-dimensional position space, the constant A in (11) is calculated using the orthogonality of the momentum eigenstates, i.e. using the relation

$$\delta(\vec{p}'-\vec{p})=\langle\vec{p}'|\vec{p}\rangle$$

Inserting into the inner product $\langle\vec{p}'|\vec{p}\rangle$ the unity, in the form of the integral $\int_{-\infty}^{\infty} d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}|$, i.e. using the completeness relation of the position eigenstates, the last equation is written as

$$\delta(\vec{p}'-\vec{p})=\langle\vec{p}'|\left(\int_{-\infty}^{\infty} d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}|\right)|\vec{p}\rangle=\int_{-\infty}^{\infty} d^3\vec{r}\,\langle\vec{p}'|\vec{r}\rangle\langle\vec{r}|\vec{p}\rangle=$$

$$=\int_{-\infty}^{\infty} d^3\vec{r}\,\langle\vec{r}|\vec{p}'\rangle^*\langle\vec{r}|\vec{p}\rangle=\int_{-\infty}^{\infty} d^3\vec{r}p'^*(\vec{r})\,p(\vec{r})$$

That is

$$\int_{-\infty}^{\infty} d^3\vec{r}p'^*(\vec{r})\,p(\vec{r})=\delta(\vec{p}'-\vec{p}) \quad (13)$$

This is the orthogonality relation of the momentum eigenfunctions in the three-dimensional position space.
By means of (12), the integral in (13) is written, in Cartesian coordinates, as

$$\int_{-\infty}^{\infty} d^3\vec{r}p'^*(\vec{r})\,p(\vec{r})=\int_{-\infty}^{\infty} dxdydz|A_x|^2\exp\left(\frac{i(p_x-p_x')x}{\hbar}\right)|A_y|^2\exp\left(\frac{i(p_y-p_y')y}{\hbar}\right)|A_z|^2\exp\left(\frac{i(p_z-p_z')z}{\hbar}\right)=$$

$$=\int_{-\infty}^{\infty} dx|A_x|^2\exp\left(\frac{i(p_x-p_x')x}{\hbar}\right)\int_{-\infty}^{\infty} dy|A_y|^2\exp\left(\frac{i(p_y-p_y')y}{\hbar}\right)\int_{-\infty}^{\infty} dz|A_z|^2\exp\left(\frac{i(p_z-p_z')z}{\hbar}\right)$$

Then, using that, in Cartesian coordinates, the three-dimensional delta function $\delta(\vec{p}'-\vec{p})$ is

$$\delta(\vec{p}'-\vec{p})=\delta(p_x'-p_x)\,\delta(p_y'-p_y)\,\delta(p_z'-p_z),$$

(13) is written as

$$\int_{-\infty}^{\infty} dx|A_x|^2\exp\left(\frac{i(p_x-p_x')x}{\hbar}\right)\int_{-\infty}^{\infty} dy|A_y|^2\exp\left(\frac{i(p_y-p_y')y}{\hbar}\right)\int_{-\infty}^{\infty} dz|A_z|^2\exp\left(\frac{i(p_z-p_z')z}{\hbar}\right)=$$

$$=\delta(p_x'-p_x)\,\delta(p_y'-p_y)\,\delta(p_z'-p_z)$$

From the last equation, we derive that

$$\int_{-\infty}^{\infty} dx|A_x|^2\exp\left(\frac{i(p_x-p_x')x}{\hbar}\right)=\delta(p_x'-p_x)$$

$$\int_{-\infty}^{\infty} dy|A_y|^2\exp\left(\frac{i(p_y-p_y')y}{\hbar}\right)=\delta(p_y'-p_y)$$

$$\int_{-\infty}^{\infty} dz |A_z|^2 \exp\left(\frac{i(p_z - p_z')z}{\hbar}\right) = \delta(p_z' - p_z)$$

We see that the magnitudes of A_x, A_y, A_z satisfy the same condition as the condition satisfied by the constant of the momentum eigenfunctions in the one-dimensional position space.

Thus, omitting the physically unimportant phases of the three constants A_x, A_y, A_z, we end up to

$$A_x = A_y = A_z = \frac{1}{\sqrt{2\pi\hbar}} \quad (14)$$

Using (14), (12) becomes

$$p(\vec{r}) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip_x x}{\hbar}\right) \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip_y y}{\hbar}\right) \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip_z z}{\hbar}\right)$$

or

$$p(\vec{r}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \quad (15)$$

This is the momentum eigenfunction with eigenvalue, i.e. with momentum, $\vec{p}$, in the three-dimensional position space.

We see that the constant of the momentum eigenfunctions in the position space is multiplied by $\frac{1}{\sqrt{2\pi\hbar}}$ for each spatial dimension.

Therefore, in an nth-dimensional position space, the momentum eigenfunctions are

$$p(\vec{r}) = \frac{1}{(2\pi\hbar)^{\frac{n}{2}}} \exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \quad (16)$$

where $\vec{r} = (r_1, ..., r_n)$ and $\vec{p} = (p_1, ..., p_n)$.

By means of (2), (15) becomes

$$\langle \vec{r} | \vec{p} \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \quad (17)$$

4.2 Position and momentum eigenfunctions in the three-dimensional momentum space

In the three-dimensional momentum space, the position eigenfunction with eigenvalue, i.e. with position, $\vec{r}$ is the function $\langle \vec{p} | \vec{r} \rangle$, which is a scalar function of the momentum $\vec{p}$, with the position $\vec{r}$ now being a parameter.

Using (17), the relation $\langle \vec{p} | \vec{r} \rangle = \langle \vec{r} | \vec{p} \rangle^*$ gives

$$\langle \vec{p} | \vec{r} \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \quad (18)$$

This is the position eigenfunction with eigenvalue $\vec{r}$ in the three-dimensional momentum space, which we'll denote by $r(\vec{p})$, i.e.

$$r(\vec{p}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \quad (19)$$

> Note
> As in the one-dimensional case, the position eigenfunctions in the three-dimensional momentum space are the complex conjugates of the momentum eigenfunctions in the three-dimensional position space.

Besides, the momentum eigenfunction with eigenvalue, i.e. with momentum, $\vec{p}'$ is the function $\langle\vec{p}|\vec{p}'\rangle$, which, since the momentum eigenstates $|\vec{p}\rangle$ and $|\vec{p}'\rangle$ are orthogonal and the spectrum of the momentum operator is continuous, is equal to the delta function $\delta(\vec{p}-\vec{p}')$.

Therefore, as in the one-dimensional momentum space, the momentum eigenfunctions in the three-dimensional momentum space are delta functions.

5. The relation between the wave functions in the three-dimensional position and momentum spaces

The relations between the wave functions $\psi(\vec{r})$ and $\tilde{\psi}(\vec{p})$ in the three-dimensional position and momentum spaces are easily derived using the completeness relations of the position and momentum eigenstates, i.e. the relations

$$\int_{-\infty}^{\infty} d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}| = 1 \text{ and } \int_{-\infty}^{\infty} d^3\vec{p}\,|\vec{p}\rangle\langle\vec{p}| = 1,$$

and the relations (17) and (18) of section 4, i.e. the relations

$$\langle\vec{r}\,|\,\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \text{ and } \langle\vec{p}\,|\,\vec{r}\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right).$$

Thus, the wave function in the three-dimensional momentum space is written as

$$\tilde{\psi}(\vec{p}) = \langle\vec{p}|\psi\rangle = \langle\vec{p}|\left(\int_{-\infty}^{\infty} d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}|\right)|\psi\rangle = \int_{-\infty}^{\infty} d^3\vec{r}\,\langle\vec{p}|\vec{r}\rangle\underbrace{\langle\vec{r}|\psi\rangle}_{\psi(\vec{r})} =$$

$$= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty} d^3\vec{r}\,\exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)\psi(\vec{r})$$

That is

$$\tilde{\psi}(\vec{p}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty} d^3\vec{r}\,\exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)\psi(\vec{r}) \quad (1)$$

We see that $\tilde{\psi}(\vec{p})$ is the three-dimensional Fourier transform of $\psi(\vec{r})$.
Similarly, the wave function in the three-dimensional position space is written as

$$\psi(\vec{r}) = \langle\vec{r}|\psi\rangle = \langle\vec{r}|\left(\int_{-\infty}^{\infty} d^3\vec{p}\,|\vec{p}\rangle\langle\vec{p}|\right)|\psi\rangle = \int_{-\infty}^{\infty} d^3\vec{p}\,\langle\vec{r}\,|\,\vec{p}\rangle\underbrace{\langle\vec{p}|\psi\rangle}_{\tilde{\psi}(\vec{p})} =$$

$$= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty} d^3\vec{p}\,\exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)\tilde{\psi}(\vec{p})$$

That is

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty} d^3\vec{p}\,\exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)\tilde{\psi}(\vec{p}) \quad (2)$$

We see that $\psi(\vec{r})$ is the three-dimensional inverse Fourier transform of $\tilde{\psi}(\vec{p})$.

6. References

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