On the Attempt to use a Stochastic Interpretation to compute the Trace of a Regular Representation U on $X = \mathrm{A}_K/K^*$

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Let $X = \mathrm{A}_K/K^*$, which is proposed by A. Connes, and think of the trace of a regular representation U on X of the idele class C_K. It is interesting for number theory whether the trace is computable or not. However, because of the non-compactness of X, it is hard to compute the trace of U. In this article, we try to show that the trace is computable. In order to show this, we will use a stochastic interpretation. We will ignore various subtle problems which the adele ring has and consider the adele ring as the Riemannian variety discussing on this problem.

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Let K be a global field, K_ν be a local field that is the completion of K at the place ν of K and A_K be the adele ring of K. Set

$$X = \mathrm{A}_K/K^*.$$

The $L^2(X, dx)$ is an interesting space for the number theory.

(A) For $\xi(x) \in L^2(X, dx)$, let $(\mathrm{T}\xi)(a) = |a|^{1/2}\xi(a);\ \forall a \in C_K$ be the restriction of $\xi(x)$ to the idele class group $C_K = \mathrm{A}_K^*/K^*$. Since $dx = |x|d^*x$, we will understand that

$$(\mathrm{T}\xi)(a) \in L^2(C_K, d^*x).$$

(B) The idele class group C_K naturally acts on $\xi(x)$ as follows;

$$(U(g)\xi)(x) = \xi(g^{-1}x) \quad \forall g \in C_K,\ x \in X,$$

since $X = \mathrm{A}_K/K^*$. Thus the restriction of $L^2(X)$:

$$\begin{aligned} \mathrm{T}(U(g)\xi)(a) &= \text{the restriction of } \xi(g^{-1}x) \\ &= |g|^{1/2}(V(g)\mathrm{T}\xi)(a) \qquad \forall a, g \in C_K, \end{aligned}$$

gives a regular representation $(V, \mathrm{T}(L^2(X)))$ of C_K which is unitary. Here, $\mathrm{T}(L^2(X)) = \{(\mathrm{T}\xi)(a) \mid \xi(x) \in L^2(X)\}$.

The space $L^2(X)$ gives a representation of C_K, which isn't always unitary. However if $L^2(X)$ is decomposed in irreducible representations then $\mathrm{T}(L^2(X))$ which is a subspace of $L^2(C_K, d^*x)$ is also decomposed in irreducible representations of C_K. Especially, it is important whether U is a trace-class operator or not. In this article, we try to consider this problem. We will think of the case $K = \mathbb{Q}$.

Set

$$\Delta = |x|^2 \frac{d^2}{dx^2}$$

which is a differential operator on X. Let

$$\langle \xi, \eta \rangle = \int_X \xi \overline{\eta}\, dx .$$

We can show that

$$\int_X \frac{d^2}{dx^2} \xi \cdot \overline{\eta}\, dx = \int_X \xi \cdot \frac{d^2}{dx^2} \overline{\eta}\, dx .$$

Since $\frac{d}{dx}\overline{\eta} = \overline{\frac{d}{dx}\eta}$ and $|x| = \overline{|x|}$,

$$\int_X |x|^2 \frac{d^2}{dx^2} \xi \cdot \overline{\eta}\, dx = \int_X \xi \cdot |x|^2 \frac{d^2}{dx^2} \overline{\eta}\, dx = \int_X \xi \cdot \overline{|x|^2 \frac{d^2}{dx^2} \eta}\, dx .$$

We see that

$$\langle \Delta\xi, \eta \rangle = \langle \xi, \Delta\eta \rangle.$$

Namely, Δ is Hermitian. Thus its eigenvalues $\{\lambda\}$ are discrete. We shall think of the eigenvalue problems on the analogy of Sturm-Liouville problem:

$$\Delta\xi(x) - \lambda\xi(x) = 0; \quad \xi(x) = 0 \text{ on } \partial X.$$

We can show that $\lambda \leq 0$. Counting multiplicity, we will denote $\{\lambda\}$ by

$$-\infty \longleftarrow \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq 0.$$

Let the eigen-space be

$$E(\lambda) = \{\phi_i(x) \,|\, \Delta\phi_i(x) - \lambda\phi_i(x) = 0\}.$$

It must be a subtle problem what $L^2(X)$ is. We shall start with the following statement.

Thesis 1.1.

$$L^2(X) = \bigoplus_\lambda E(\lambda).$$

The action of $C_{\mathbb{Q}} = \mathrm{A}_{\mathbb{Q}}^* / \mathbb{Q}^*$ on the functions on X is

$$(U(g)\phi)(x) = \phi(g^{-1}x) \quad \forall g \in C_{\mathbb{Q}},\ x \in X.$$

Now, one computes

$$(U(g)|x|^2 \frac{d^2}{dx^2}\phi)(x) = (U(g)|\cdot|^2\phi'')(x) = \left|g^{-1}x\right|^2 \phi''(g^{-1}x).$$

It holds that $dgx = |g|dx$, so

$$\begin{aligned}|x|^2 \frac{d^2}{dx^2}(U(g)\phi)(x) = |x|^2 \frac{d^2}{dx^2}\phi(g^{-1}x) &= |x|^2 \frac{d^2 g^{-1}x}{dx^2} \frac{d^2}{d\left(g^{-1}x\right)^2}\phi(g^{-1}x) \\ &= \left|g^{-1}x\right|^2 \phi''(g^{-1}x).\end{aligned}$$

It turns out that $U(g)$ and Δ are commutative. Hence they shares the same eigen-space. We have set the eigen-space:

$$E(\lambda) = \left\{\phi_i(x) \,|\, \Delta\phi_i(x) - \lambda\phi_i(x) = 0\right\}.$$

Then we may think that $(U, E(\lambda))$ gives an irreducible representation. We will have

$$U = \bigoplus_{\lambda} U_\lambda;\ \ (U, V) = \bigoplus_{\lambda} (U, E(\lambda)).$$

From the above Thesis 1.1, we will obtain

$$U = \bigoplus_{\lambda} U_\lambda;\ \ (U, L^2(X)) = \bigoplus_{\lambda} (U, E(\lambda)).$$

Our main problem is whether

$$\mathrm{tr} U_\lambda(g) = \sum_{i=1}^{\infty} \langle (U(g)\phi_i)(x),\ \phi_i(x)\rangle$$

exists or not.

Think of a certain function h on X, which satisfies that

$$h(x) = \sum_{i=1}^{\infty} a_i \phi_i(x); \quad a_i = \int_X h(y)\phi_i(y)dy.$$

Put $\lambda_i(g) = \langle U(g)\phi_i(x), \phi_i(x)\rangle$ and $\mathrm{t}\in[0, \infty)$. Here $\lambda_i(g)\in\mathbb{C}$. Set

$$h(\mathrm{t}; x) = \sum_{i=1}^{\infty} e^{\mathrm{t}\lambda_i(g)} \cdot a_i\phi_i(x).$$

We can compute as follows;

$$h(\mathrm{t}; x) = \int_X \left\{ \sum_{i=1}^{\infty} e^{\mathrm{t}\lambda_i(g)} \cdot \phi_i(x)\phi_i(y) \right\} h(y)dy.$$

We may say that $U(g)\phi_i(x) = \phi_i(g^{-1}x) = \lambda_i(g)\phi_i(x)$. Thus, we can define the following operator

$$(e^{\mathrm{t}U_\lambda(g)}h)(x) = h(\mathrm{t}; x).$$

Let

$$p_\lambda(\mathrm{t}; x, y) = \sum_{i=1}^{\infty} e^{\mathrm{t}\lambda_i(g)} \cdot \phi_i(x)\phi_i(y).$$

So, $e^{\mathrm{t}U_\lambda(g)}$ has the integral expression:

$$(e^{\mathrm{t}U_\lambda(g)}h)(x) = \int_X p_\lambda(\mathrm{t}; x, y)h(y)dy.$$

When $\mathrm{t} \longrightarrow 0^+$ then

$$h(\mathrm{t}; x) = h(x) = \int_X \lim_{\mathrm{t}\to 0^+} p_\lambda(\mathrm{t}; x, y)h(y)dy.$$

Thus

(1) $\lim_{\mathrm{t}\to 0^+} p_\lambda(\mathrm{t}; x, y) = \delta(x-y) = \delta_x(y)$.

This implies that

$$\delta(x-y) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(y).$$

We will see that

(2) the symmetry

$$p_\lambda(\mathrm{t}; x, y) = p_\lambda(\mathrm{t}; y, x).$$

Put

$$p_\lambda(\mathrm{t}; x, y) = \sum_{i=1}^{\infty} a_i\phi_i(y); \quad a_i = \int_X p_\lambda(\mathrm{t}; x, z)\phi_i(z)dz.$$

We compute as follows;

$$
\begin{aligned}
p_\lambda(\mathrm{t+s};\, x,\, y) &= \sum_{i=1}^{\infty} e^{\mathrm{s}\lambda_i(g)} \cdot e^{\mathrm{t}\lambda_i(g)} \phi_i(x)\phi_i(y) = \sum_{i=1}^{\infty} e^{\mathrm{s}\lambda_i(g)} \cdot a_i \phi_i(y) \\
&= \sum_{i=1}^{\infty} e^{\mathrm{s}\lambda_i(g)} \cdot \phi_i(y) \int_X p_\lambda(\mathrm{t};\, x,\, z)\phi_i(z)dz \\
&= \int_X \left\{ \sum_{i=1}^{\infty} e^{\mathrm{s}\lambda_i(g)} \cdot \phi_i(y)\phi_i(z) \right\} p_\lambda(\mathrm{t};\, x,\, z)dz \\
&= \int_X p_\lambda(\mathrm{s};\, y,\, z) p_\lambda(\mathrm{t};\, x,\, z)dz .
\end{aligned}
$$

So,

(3) for all $\mathrm{t},\, \mathrm{s} \in [0, \infty)$

$$
p_\lambda(\mathrm{t+s};\, x,\, y) = \int_X p_\lambda(\mathrm{t};\, x,\, z) p_\lambda(\mathrm{s};\, z,\, y)dz .
$$

From the theory of semi-group, we see that

$$
e^{\mathrm{t}\sum_{i=1}^{\infty} \lambda_i(g)} = \int_X p_\lambda(\mathrm{t};\, x,\, x)dx .
$$

[Remark]

$$
\int_X p_\lambda(\mathrm{t};\, x,\, y)dy = \int_X \sum_{i=1}^{\infty} e^{\mathrm{t}\lambda_i(g)} \cdot \phi_i(x)\phi_i(y)dy .
$$

On the other hand,

$$
\int_X p_\lambda(\mathrm{t};\, x,\, x)dx = \int_X \sum_{i=1}^{\infty} e^{\mathrm{t}\lambda_i(g)} \cdot \phi_i(x)\phi_i(x)dx .
$$

Then

$$
\sum_{i=1}^{\infty} \lambda_i(g) = \frac{d}{d\mathrm{t}} e^{\mathrm{t}\sum_{i=1}^{\infty} \lambda_i(g)} \Big|_{\mathrm{t}=0}
$$

Therefore,

$\mathrm{tr}U_\lambda(g)$ exists if and only if $\frac{d}{d\mathrm{t}} \int_X p_\lambda(\mathrm{t};\, x,\, x)dx \Big|_{\mathrm{t}=0}$ exists.

We will think of $\int_X p_\lambda(\mathrm{t};\ x,\ y)dy$ where $p_\lambda(\mathrm{t};x,y) = \sum_{i=1}^{\infty} e^{\mathrm{t}\lambda_i(g)} \cdot \phi_i(x)\phi_i(y)$. This integral is given formally. Especially, it is important whether this integral, as the function of t, converges or not. Now, we have seen that

$$\delta(x-y) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(y).$$

So, in the neighborhood $\mathrm{t} = 0$, we may hope that $p_\lambda(\mathrm{t};x,y)$ has *a nice property*. We must be allowed to think that $\int_X p_\lambda(\mathrm{t};\ x,\ y)dy$ is meaningful. By stating its usefulness and effectiveness, we shall think that whether the integral converges or not is solved.

We have seen that

$$p_\lambda(\mathrm{t+s};x,y) = \int_X p_\lambda(\mathrm{t};\ x,\ z)p_\lambda(\mathrm{s};\ z,\ y)dz.$$

It satisfies Chapman-Kolmogorov equation. It turns out that

$$p_\lambda(\mathrm{t};x,y) = \lim_{\mathrm{s}\to 0^+} p_\lambda(\mathrm{t+s};x,y) = \int_X p_\lambda(\mathrm{t};\ x,\ dz) \lim_{\mathrm{s}\to 0^+} p_\lambda(\mathrm{s};\ y,\ z).$$

Here, $p_\lambda(\mathrm{t};x,z)dz = p_\lambda(\mathrm{t};x,dz)$. So we can say that

$$p_\lambda(\mathrm{t};x,y) = \int_X \delta(y-z)p_\lambda(\mathrm{t};\ x,\ dz).$$

We will rewrite the above formula from a stochastic view. If $p_\lambda(\mathrm{t};x,dz)$ gives a stochastic measure then

$$p_\lambda(\mathrm{t};x,y) = E[\lim_{\mathrm{s}\to 0^+} p_\lambda(\mathrm{s};y,z)].$$

We hope to be given a stochastic model of $p_\lambda(\mathrm{t};x,y)$. To be brief, there exists a stochastic measure $\nu_\lambda(\mathrm{t};x,dy)$, which satisfies that

$$p_\lambda(\mathrm{t};x,dy) = \alpha(\mathrm{t};x,y)\nu_\lambda(\mathrm{t};x,dy)$$

for a certain function $\alpha(\mathrm{t};x,y)$. The stochastic measure $\nu_\lambda(\mathrm{t};x,dy)$ determines a stochastic process on X:

$$B^\lambda{}_\mathrm{t} = \{x^\lambda{}_\mathrm{t} \mid \mathrm{t}\in[0,\infty)\}.$$

Set $x = x^{\lambda}{}_0$. The probability of the event where $x^{\lambda}{}_t$ is contained in $Y \subseteq X$ is given as

$$\mathrm{P}_x(x^{\lambda}{}_t \in Y) = \int_Y v_{\lambda}(t;\, x,\, x^{\lambda}{}_t)\, dx^{\lambda}{}_t .$$

[Remark] Suppose that $\{s_t\}$ satisfies the stochastic differential equation:

$$ds_t = a(t,\, s_t)dt + b(t,\, s_t)dx^{\lambda}{}_t.$$

It turns out that

$$E(s_{t+\Delta} - s_t) = a(t,\, s_t)\Delta + o(\Delta),$$
$$V(s_{t+\Delta} - s_t) = b^2(t,\, s_t)\Delta + o(\Delta).$$

Then,

$$a(t,\, s_t) = \frac{dE(s_t)}{dt} \text{ and } b^2(t,\, s_t) = \frac{dV(s_t)}{dt}.$$

We may be allowed to consider that a stochastic process $\{s_t\}$ is given. Thus, we may think that the stochastic process $B^{\lambda}{}_t$ is given as the solution of

$$ds_t = 0dt + 1dx^{\lambda}{}_t.$$

We will rewrite $\alpha(t;\, x,\, y)$ as $\alpha(x^{\lambda}{}_t)$. Then

$$\begin{aligned} p_{\lambda}(t;\, x,\, y) &= E[\alpha(x^{\lambda}{}_t) \lim_{s\to 0^+} p_{\lambda}(s;\, y,\, x^{\lambda}{}_t)] \\ &= \int_X \left\{\alpha(x^{\lambda}{}_t) \lim_{s\to 0^+} p_{\lambda}(s;\, y,\, x^{\lambda}{}_t)\right\} v_{\lambda}(t;\, x,\, dx^{\lambda}{}_t). \end{aligned}$$

We shall call it "a stochastic interpretation".

Especially, we can say that

$$\begin{aligned} p_{\lambda}(t;\, x,\, x) &= E[\alpha(x^{\lambda}{}_t) \lim_{s\to 0^+} p_{\lambda}(s;\, x,\, x^{\lambda}{}_t)] \\ &= \int_X \left\{\alpha(x^{\lambda}{}_t) \lim_{s\to 0^+} p_{\lambda}(s;\, x,\, x^{\lambda}{}_t)\right\} v_{\lambda}(t;\, x,\, dx^{\lambda}{}_t). \end{aligned}$$

Then

$$\lim_{t\to 0^+} p_{\lambda}(t;\, x,\, x) = \lim_{t\to 0^+} \int_X \left\{\alpha(x^{\lambda}{}_t) \lim_{s\to 0^+} p_{\lambda}(s;\, x,\, x^{\lambda}{}_t)\right\} v_{\lambda}(t;\, x,\, dx^{\lambda}{}_t).$$

We have shown that $\lim_{t\to 0^+} p_{\lambda}(t;\, x,\, y) = \delta(x-y)$. So

$$\lim_{s\to 0^+} p_{\lambda}(s;\, x,\, x^{\lambda}{}_t) = \begin{cases} \infty & \cdots\ x^{\lambda}{}_t = x \\ 0 & \cdots\ x^{\lambda}{}_t \neq x. \end{cases}$$

Thus

$$\alpha(x^{\lambda}{}_{\mathrm{t}})\lim_{\mathrm{s}\to 0^+} p_{\lambda}(\mathrm{s};\, x,\, x^{\lambda}{}_{\mathrm{t}}) = \begin{cases} \infty & \cdots \ x^{\lambda}{}_{\mathrm{t}} = x \\ 0 & \cdots \ x^{\lambda}{}_{\mathrm{t}} \neq x\,. \end{cases}$$

It must be allowed to think that $\int_X \left\{\alpha(x^{\lambda}{}_{\mathrm{t}})\lim_{\mathrm{s}\to 0^+} p_{\lambda}(\mathrm{s};\, x,\, x^{\lambda}{}_{\mathrm{t}})\right\} \nu_{\lambda}(\mathrm{t};\, x,\, dx^{\lambda}{}_{\mathrm{t}})$ is almost identified with $\int_X \delta(x-y)dy$. Thus,

$$E[\alpha(x^{\lambda}{}_{\mathrm{t}})\lim_{\mathrm{s}\to 0^+} p_{\lambda}(\mathrm{s};\, x,\, x^{\lambda}{}_{\mathrm{t}})] = \int_X \left\{\alpha(x^{\lambda}{}_{\mathrm{t}})\lim_{\mathrm{s}\to 0^+} p_{\lambda}(\mathrm{s};\, x,\, x^{\lambda}{}_{\mathrm{t}})\right\} \nu_{\lambda}(\mathrm{t};\, x,\, dx^{\lambda}{}_{\mathrm{t}}) < \infty$$

and it must be satisfied independently of t. So we may say that $\lim_{\mathrm{t}\to 0^+} p_{\lambda}(\mathrm{t};\, x,\, x) < \infty$.

[Remark] We have $\lim_{\mathrm{t}\to 0^+} p_{\lambda}(\mathrm{t};\, x,\, y) = \delta(x—y)$. Therefore,

$$\lim_{\mathrm{t}\to 0^+} p_{\lambda}(\mathrm{t};\, x,\, x) = \delta(0) = \infty.$$

The above stochastic view is to consider $p_{\lambda}(\mathrm{t};\, x,\, x)$ as an expected value i.e. an average. We shall say that to think of the average avoids being $\lim_{\mathrm{t}\to 0^+} p_{\lambda}(\mathrm{t};\, x,\, x) = \delta(0) = \infty$.

Therefore,

$$\lim_{\mathrm{t}\to 0^+} \int_X p_{\lambda}(\mathrm{t};\, x,\, x)dx = c\int_X dx = c\cdot \textit{the volume of } X.$$

Suppose that the volume of X is finite. Then $\lim_{\mathrm{t}\to 0^+} \int_X p_{\lambda}(\mathrm{t};\, x,\, x)dx$ is finite. Thus,

if the volume of X is finite then $\int_X p_{\lambda}(\mathrm{t};\, x,\, x)dx$ is defined at $\mathrm{t} = 0$.

Next, we will think of

$$\lim_{\Delta\to 0^+} \frac{\int_X p_{\lambda}(\mathrm{t}+\Delta;\, x,\, x)dx - \int_X p_{\lambda}(\mathrm{t};\, x,\, x)dx}{\Delta}.$$

We can compute

$$\begin{aligned}\lim_{\Delta\to 0^+} \frac{\int_X p_{\lambda}(\mathrm{t}+\Delta;\, x,\, x)dx - \int_X p_{\lambda}(\mathrm{t};\, x,\, x)dx}{\Delta} &= \lim_{\Delta\to 0^+} \frac{\int_X p_{\lambda}(\mathrm{t}+\Delta;\, x,\, x) - p_{\lambda}(\mathrm{t};\, x,\, x)dx}{\Delta} \\ &= \lim_{\Delta\to 0^+} \frac{\int_X \left\{\int_X p_{\lambda}(\mathrm{t};\, x,\, z)p_{\lambda}(\Delta;\, x,\, z)dz\right\} - p_{\lambda}(\mathrm{t};\, x,\, x)dx}{\Delta}.\end{aligned}$$

Here $\lim_{\Delta \to 0^+} p_\lambda(\Delta;\ x,\ z) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(z) = \delta(x - z)$. Taking

$$\lim_{t \to 0} \frac{e^{t\lambda_i(g)} - 1}{t} = \frac{d}{dt} e^{t\lambda_i(g)} \Big|_{t=0} = \lambda_i(g)$$

into account, it turns out that when t approaches 0 then $e^{t\lambda_i(g)}$ rapidly approaches 1. So we may say that

(a) when $\Delta \longrightarrow 0^+$ then $p_\lambda(\Delta;\ x,\ z) = \sum_{i=1}^{\infty} e^{\Delta\lambda_i(g)} \phi_i(x)\phi_i(z)$ rapidly approaches $\sum_{i=1}^{\infty} \phi_i(x)\phi_i(z) = \delta(x - z)$.

Thus,

(b) when $\Delta \longrightarrow 0^+$ then $\int_X \left\{ \int_X p_\lambda(t;\ x,\ z) p_\lambda(\Delta;\ x,\ z)\, dz \right\} - p_\lambda(t;\ x,\ x)\, dx$ rapidly approaches zero.

Therefore,

$\int_X p_\lambda(t;\ x,\ x)\, dx$ is a differential function at $t = 0$.

Thesis 3.1.

If the volume of X is finite then $\frac{d}{dt} \int_X p_\lambda(t;\ x,\ x)\, dx \Big|_{t=0}$ exists.

We know that

$$\mathrm{A}_{\mathbb{Q}}/\mathbb{Q} \cong \prod_{p < \infty} \mathbb{Z}_p \times [0,\ 1] \text{ and } \mathrm{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \cong \prod_{p < \infty} \mathbb{Z}_p^* \times \mathbb{R}^*_{>0}.$$

For $r \in \mathbb{Q}_p$, $r = p^\rho \frac{m}{n}$ ($p \nmid m,\ p \nmid n$). Just imagine as follows;

(i) $\mathbb{Z}_p^*$ corresponds to a unite circle,

(ii) $\mathbb{Z}_p$ corresponds to a unite disk.

From this imagination, we can say that

$\prod_{p<\infty} \mathbb{Z}_p$ is a set of countable unite disks $\mathbb{Z}_p$, namely a cylinder.

Then we can also say that

$\prod_{p<\infty} \mathbb{Z}_p^*$ is the surface of the cylinder except two inner disks.

Now, the thickness of a unite disk $\mathbb{Z}_p$ is zero. Thus the thickness of the cylinder is $0 \cdot \infty = c$, namely the thickness of the cylinder must be finite. We shall consider that

$$\mathrm{A}_{\mathbb{Q}}/\mathbb{Q} = \text{the cylinder} \times [0, 1]$$

and

$$\mathrm{A}_{\mathbb{Q}}^*/\mathbb{Q}^* = \text{the surface of the cylinder except two inner disks} \times \mathbb{R}^*_{>0} .$$

Intuitively thinking, we may say that *X exists in the middle of $\mathrm{A}_{\mathbb{Q}}/\mathbb{Q}$ and $\mathrm{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$*. Since we may say that both $\mathrm{A}_{\mathbb{Q}}/\mathbb{Q}$ and $\mathrm{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ have a finite volume, the volume of X must be finite.

Thesis 3.2.

The volume of X is finite.

With Thesis 3.1, we can say that $\left.\frac{d}{dt}\int_X p_\lambda(\mathrm{t};\, x,\, x)dx\right|_{\mathrm{t}=0}$ exists. Therefore, we can confirm that U_λ is trace-class. This fact ensures considering U as a trace-class operator.

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