Spatial Distance and the Theory of Relativity

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Abstract. We derive the element of spatial distance in terms of the time coordinate and a new tensor related to the Kronecker delta.

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1. Introduction

Landau and Lifshitz, in *The Classical Theory of Fields* (First english edition 1951) [1, p. 234, paragraph 84], deduced the following formula[1]

$$\mathrm{d}l^2 = \left(-g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}}\right)\mathrm{d}x^i\mathrm{d}x^j,$$

where

$$h_{ij} = -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \tag{1}$$

is the three-dimensional metric tensor, determining the metric, i. e., the geometric properties of the space[2].

In this paper, we derive mathematically the formula

$$\mathrm{d}l^2 = \left(-g_{00} + \frac{g_{0i}g_{0j}}{g_{ij}}\right)(\mathrm{d}x^0)^2,$$

where

$$t_{ij} = -g_{00} + \frac{g_{0i}g_{0j}}{g_{ij}}. \tag{2}$$

1. Henceforward, we consider the Greek indices can be 0, 1, 2, 3 and Latin indices only $1, 2, 3$.

2. In *The Classical Theory of Fields*, Landau and Lifshitz used the symbol $\gamma_{\alpha\beta}$ instead of h_{ij}, such as Zelmanov (*Chronometric Invariants*) and we did here.

By analogy of the previous paragraph, we can say that t_{ij} is the time metric tensor, determining the metric, i. e., the geometric properties of the time.

This will be the subject of the second Section.

In third Section, we derive a new tensor related to the Kronecker delta, from the idea of Landau and Lifshitz [1, p. 235, (84.8)].

2. The element dl of spatial distance in terms of the time coordinate

In special relativity, the interval ds is defined by

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta. \tag{3}$$

If we consider s to be a function of x^α and x^β, i. e., $s = f(x^\alpha, x^\beta)$, then we can rewrite the interval ds as

$$\mathrm{d}s = \frac{\partial s}{\partial x^\alpha}\mathrm{d}x^\alpha + \frac{\partial s}{\partial x^\beta}\mathrm{d}x^\beta, \tag{4}$$

by a definition of differential calculus, see [2, p. 946, (7)]. The squaring of (4), give us

$$\mathrm{d}s^2 = \left(\frac{\partial s}{\partial x^\alpha}\mathrm{d}x^\alpha\right)^2 + 2\frac{\partial s}{\partial x^\alpha}\frac{\partial s}{\partial x^\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \left(\frac{\partial s}{\partial x^\beta}\mathrm{d}x^\beta\right)^2. \tag{5}$$

On the other hand, in [1, p. 233, (84.4)], separating the space and time coordinates, we have for the interval

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i \mathrm{d}x^j + 2g_{0i}\mathrm{d}x^0 \mathrm{d}x^i + g_{00}(\mathrm{d}x^0)^2. \tag{6}$$

Comparing term by term between (5) and (6), we can deduce that

$$\frac{\partial s}{\partial x^\beta}\mathrm{d}x^\beta = \sqrt{g_{00}}\mathrm{d}x^0, \tag{7}$$

$$\frac{\partial s}{\partial x^\alpha}\frac{\partial s}{\partial x^\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{0i}\mathrm{d}x^0 \mathrm{d}x^i \tag{8}$$

and

$$\left(\frac{\partial s}{\partial x^\alpha}\mathrm{d}x^\alpha\right)^2 = g_{ij}\mathrm{d}x^i \mathrm{d}x^j. \tag{9}$$

From (7), we obtain

$$\frac{\partial s}{\partial x^{\beta}}=\sqrt{g_{00}}\frac{\mathrm{d}x^{0}}{\mathrm{d}x^{\beta}}. \tag{10}$$

Substituting the right hand side of (10) into the left hand side of (8), we encounter

$$\begin{aligned}\frac{\partial s}{\partial x^{\alpha}}\left(\sqrt{g_{00}}\frac{\mathrm{d}x^{0}}{\mathrm{d}x^{\beta}}\right)\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}&=g_{0i}\mathrm{d}x^{0}\mathrm{d}x^{i}\\ \Rightarrow\frac{\partial s}{\partial x^{\alpha}}&=\frac{g_{0i}}{\sqrt{g_{00}}}\frac{\mathrm{d}x^{i}}{\mathrm{d}x^{\alpha}}.\end{aligned} \tag{11}$$

We set the right hand side of (11) into the left hand side of (6), and find

$$\begin{aligned}\left(\frac{g_{0i}}{\sqrt{g_{00}}}\frac{\mathrm{d}x^{i}}{\mathrm{d}x^{\alpha}}\mathrm{d}x^{\alpha}\right)^{2}&=g_{ij}\mathrm{d}x^{i}\mathrm{d}x^{j}\\ \Rightarrow\mathrm{d}x^{j}&=\frac{(g_{0i})^{2}}{g_{ij}\,g_{00}}\mathrm{d}x^{i}.\end{aligned} \tag{12}$$

We put the right hand side of (12) in the right hand side of (6) and encounter

$$\mathrm{d}s^{2}=\frac{(g_{0i})^{2}}{g_{00}}(\mathrm{d}x^{i})^{2}+2g_{0i}\mathrm{d}x^{0}\mathrm{d}x^{i}+g_{00}(\mathrm{d}x^{0})^{2}. \tag{13}$$

Let $\mathrm{d}s\rightarrow 0$ in (13), solve and obtain

$$\mathrm{d}x^{i}=-\frac{g_{00}}{g_{0i}}\mathrm{d}x^{0}. \tag{14}$$

Obviously, substituting (14) into (12), we get

$$\mathrm{d}x^{j}=-\frac{g_{0i}}{g_{ij}}\mathrm{d}x^{0}. \tag{15}$$

The element $\mathrm{d}l$ of spatial distance is given by [1, p. 234]

$$\mathrm{d}l^{2}=\left(-g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}}\right)\mathrm{d}x^{i}\mathrm{d}x^{j}, \tag{16}$$

where

$$h_{ij}=-g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}}.$$

From (14), (15) and (16), it follows that

$$\mathrm{d}l^2 = \left(-g_{00} + \frac{g_{0i}g_{0j}}{g_{ij}}\right)(\mathrm{d}x^0)^2, \tag{17}$$

where

$$t_{ij} = -g_{00} + \frac{g_{0i}g_{0j}}{g_{ij}}.$$

this is the sought expression, which defines the element $\mathrm{d}l$ of spatial distance in terms of the time coordinate.

Now, we will derive a relation between the two tensors, h_{ij} and t_{ij}. From (1), we find

$$\begin{aligned} h_{ij} &= -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \\ \Leftrightarrow \frac{g_{00}}{g_{ij}} h_{ij} &= \frac{g_{00}}{g_{ij}}\left(-g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}}\right) \\ &= -g_{00} + \frac{g_{0i}g_{0j}}{g_{ij}}. \end{aligned} \tag{18}$$

Replace the left hand side of (2) into the right hand side of (18)

$$\frac{g_{00}}{g_{ij}} h_{ij} = t_{ij} \Rightarrow h_{ij} = \frac{g_{ij}}{g_{00}} t_{ij}. \tag{19}$$

3. On a new tensor related to the Kronecker delta

Notice that, surely, from $g^{ij}g_{jl} = \delta^i_l$, we obtain

$$g^{ij}\,g_{jl} + g^{i0}\,g_{0l} = \delta^i_l,\; g^{ij}\,g_{j0} + g^{i0}\,g_{00} = 0,\; g^{0j}\,g_{j0} + g^{00}\,g_{00} = 1, \tag{20}$$

see [1, p. 235, formulas (84.8)]. If we choice the two first equations from (20), it becomes a system of equations

$$\begin{cases} g^{ij}\,g_{jl} + g^{i0}\,g_{0l} = \delta^i_l \\ g^{ij}\,g_{j0} + g^{i0}\,g_{00} = 0. \end{cases} \tag{21}$$

Assuming g^{ij} and g^{i0} as variables and solving (21), we encounter the solutions

$$g^{ij} = \frac{g_{00}\,\delta^i_l}{g_{jl}\,g_{00} - g_{0l}\,g_{j0}} \tag{22}$$

and

$$g^{i0} = \frac{g_{j0}\,\delta^i_l}{g_{0l}\,g_{j0} - g_{jl}\,g_{00}}. \tag{23}$$

Rearranging the terms of (22) and (23), and seeing that $g_{j0} = g_{0j}$, we find

$$-g^{ij}\left(-g_{jl} + \frac{g_{0l}\,g_{0j}}{g_{00}}\right) = \delta^i_l \tag{24}$$

and

$$-g^{i0}\left(-g_{0l} + \frac{g_{jl}\,g_{00}}{g_{0j}}\right) = \delta^i_l. \tag{25}$$

Hereinafter, we define the following tensors

$$h_{jl} = -g_{jl} + \frac{g_{0l}\,g_{0j}}{g_{00}} \tag{26}$$

and

$$z_{jl} = -g_{0l} + \frac{g_{jl}\,g_{00}}{g_{0j}}. \tag{27}$$

The tensor h_{jl} was studied by Landau and Lifshitz, in *The Classical Theory of Fields* [1, p. 234ss], and it is associated with the spatial distance. The eminent physicist, Abraham Zelmanov, used the tensor h_{jl} in his theory of *Chronometric Invariants* [2, p. 14ss]. It is worth noting that in the physical or mathematical literature there is no mention of the tensor z_{jl}.

Now, we will derive a relation between the two tensors, h_{jl} and z_{jl}. From (27), we find

$$\begin{aligned} z_{jl} &= -g_{0l} + \frac{g_{jl}\,g_{00}}{g_{0j}} \\ &= \frac{g_{00}}{g_{0j}}\left(g_{jl} - \frac{g_{0l}\,g_{0j}}{g_{00}}\right) \\ &= -\frac{g_{00}}{g_{0j}}\left(-g_{jl} + \frac{g_{0l}\,g_{0j}}{g_{00}}\right). \end{aligned} \tag{28}$$

Replace the left hand side of (26) into the right hand side of (8)

$$z_{jl} = -\frac{g_{00}}{g_{0j}}h_{jl} \Rightarrow h_{jl} = -\frac{g_{0j}}{g_{00}}z_{jl}. \tag{29}$$

4. Conclusion

Now, it all boils down to looking for applications of these formulas, presented in this paper.

REFERENCES

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