Dalzell Integral

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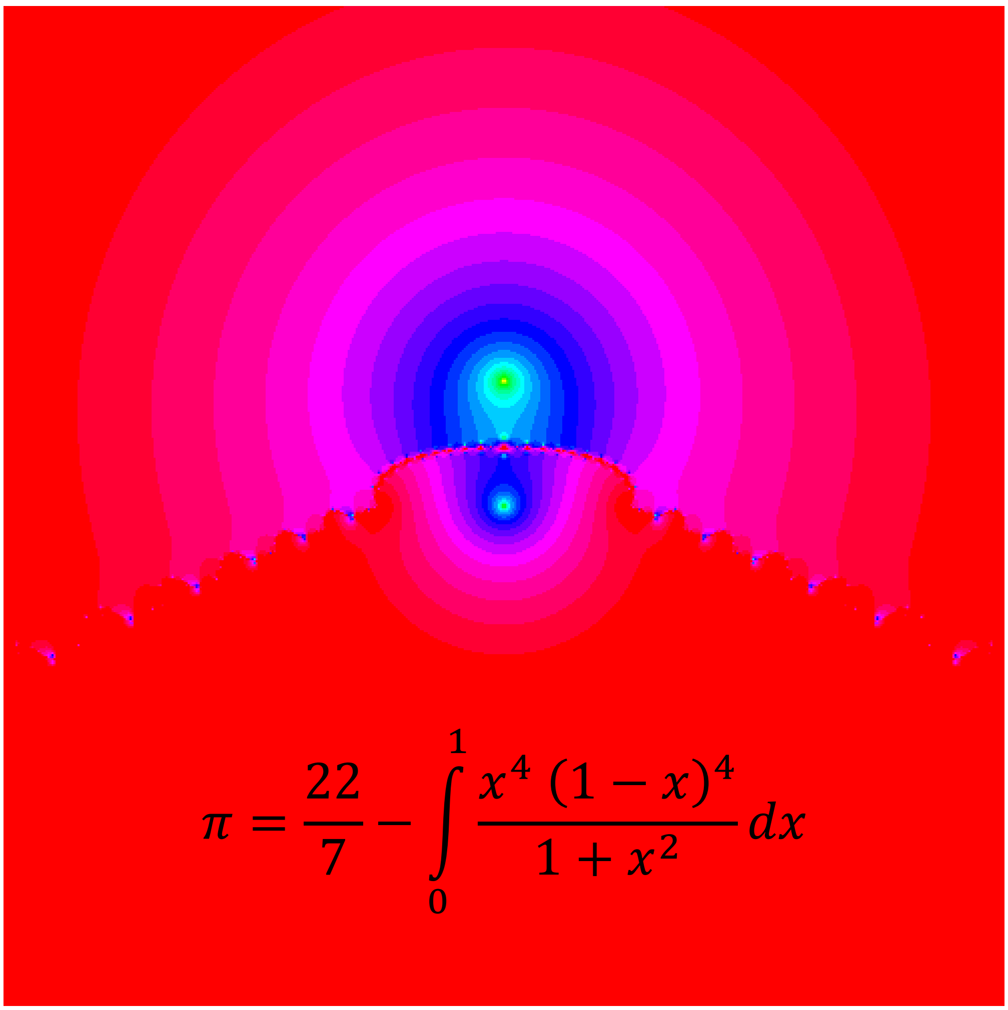

abstract

This note presents some formulas related with Dalzell integral.

1. Introduction

Dalzell integral:

$$\pi = \frac{22}{7} - \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx \tag{1}$$

This formula appears in Dalzell 1944 , 1971 , in this note we present some formulas related with Dalzell integral.

2. Integrals

$$\pi = \frac{22}{7} - \int_0^1 \frac{x^4(1-x)^4}{2-2x+x^2}\,dx \tag{2}$$

$$\pi = \frac{22}{7} - \int_0^{\pi/4} (\tan x)^4 (1-\tan x)^4\,dx \tag{3}$$

$$\pi = \frac{22}{7} - \int_0^{\ln\left(1+\sqrt{2}\right)} \sinh^4 x\left(1-\sinh x\right)^4 \operatorname{sech} x\,dx \tag{4}$$

$$\pi = \frac{22}{7} - \frac{1}{64}\int_0^1 x^3 \tan^{-1}\left(\frac{4\sqrt{1-x}}{4+x}\right)dx \tag{5}$$

$$\pi = \frac{22}{7} - \frac{1}{64}\int_0^1 (1-x)^3 \tan^{-1}\left(\frac{4\sqrt{x}}{5-x}\right)dx \tag{6}$$

$$\pi = \frac{22}{7} - \frac{1}{32}\int_0^1 x\left(1-x^2\right)^3 \tan^{-1}\left(\frac{4x}{5-x^2}\right)dx \tag{7}$$

$$\pi = \frac{22}{7} - 4\int_0^1 x^3(2x-1)(1-x)^3 \tan^{-1} x\,dx \tag{8}$$

$$\pi = \frac{22}{7} - \frac{1}{64}\int_0^1 \frac{\left(5+x^2\right)\left(1-x^2\right)^4}{25+6x^2+x^4}\,dx \tag{9}$$

$$\pi = \frac{22}{7} - \frac{1}{64}\int_0^1 \frac{x^4(2-x)^4(6-2x+x^2)}{(4+x^2)(8-4x+x^2)}dx \tag{10}$$

$$\pi = \frac{22}{7} - \frac{1}{256}\int_0^{\pi/2} \frac{(\sin(2x))^9}{1+(\sin x)^4}dx \tag{11}$$

$$\pi = \frac{22}{7} - \frac{1}{256}\int_0^{\pi/2} \frac{(\sin(2x))^9}{1+(\cos x)^4}dx \tag{12}$$

$$\pi = \frac{22}{7} - \int_1^{\infty} \frac{(x-1)^4}{x^8(1+x^2)}dx \tag{13}$$

3. Series

$$\pi = \frac{22}{7} - 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^{-k}}{(n+k+5)(n+k+6)(n+k+7)(n+k+8)(n+k+9)} \tag{14}$$

$$\pi = \frac{22}{7} - \frac{1}{1600}\sum_{n=0}^{\infty}\left(-\frac{6}{25}\right)^n\left(\frac{5}{2n+1}-\frac{19}{2n+3}+\frac{26}{2n+5}-\frac{14}{2n+7}+\frac{1}{2n+9}+\frac{1}{2n+11}\right)c_n \tag{15}$$

$$c_n = \sum_{k=0}^{[n/2]}(-1)^k\left(\frac{5}{6}\right)^{2k}\binom{n-k}{k} \tag{16}$$

$$\pi = \frac{22}{7} - \frac{2}{15}\sum_{n=0}^{\infty}3^{-n}\sum_{k=0}^{n}(-2)^k\binom{n}{k}\binom{2k+9}{5}^{-1} \tag{17}$$

$$\pi = \frac{22}{7} - \frac{2}{15}\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n\sum_{k=0}^{n}(-2)^{-k}\binom{n}{k}\binom{2n-2k+9}{5}^{-1} \tag{18}$$

$$\pi = \frac{22}{7} - \frac{1}{10}\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k+9}{5}^{-1} \tag{19}$$

$$\pi = \frac{22}{7} - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n \binom{2n+9}{5}^{-1} \qquad (20)$$

$$\pi = \frac{22}{7} - \frac{1}{320}\sum_{n=0}^{\infty}2^{-2n}\left(\frac{1}{2n+1} - \frac{4}{2n+3} + \frac{6}{2n+5} - \frac{4}{2n+7} + \frac{1}{2n+9}\right)c_n \qquad (21)$$

$$c_n = \sum_{k=0}^{n}(-1)^k \left(\frac{4}{5}\right)^{2n-k} \binom{2n-k}{k} \qquad (22)$$

4. Series – hypergeometric functions

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty}\frac{F\left(\{n, n+4\}, \{2n+8\}, -1\right)}{\binom{2n+6}{n+3}(2n+7)} \qquad (23)$$

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty}\frac{F\left(\{n, n+4\}, \{2n+8\}, 1/2\right)}{2^n \binom{2n+6}{n+3}(2n+7)} \qquad (24)$$

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty}\frac{F\left(\{n+4, n+8\}, \{2n+8\}, 1/2\right)}{2^{n+4} \binom{2n+6}{n+3}(2n+7)} \qquad (25)$$

$$\pi = \frac{22}{7} - 16\sum_{n=1}^{\infty}\frac{F\left(\{n+4, n+8\}, \{2n+8\}, -1\right)}{\binom{2n+6}{n+3}(2n+7)} \qquad (26)$$

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n \frac{F\left(\{n/2, (n+1)/2\}, \{(2n+9)/2\}, 1/9\right)}{\binom{2n+6}{n+3}(2n+7)} \qquad (27)$$

$$\pi = \frac{22}{7} - 16\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n+8} \frac{F\left(\{(n+8)/2, (n+9)/2\}, \{(2n+9)/2\}, 1/9\right)}{\binom{2n+6}{n+3}(2n+7)} \qquad (28)$$

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty} \frac{F\left(\left\{n/2, (n+8)/2\right\}, \left\{(2n+9)/2\right\}, -1/8\right)}{\left(\sqrt{2}\right)^{n} \binom{2n+6}{n+3} (2n+7)} \tag{29}$$

$$\pi = \frac{22}{7} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{F\left(\left\{(n+1)/2, (n+9)/2\right\}, \left\{(2n+9)/2\right\}, -1/8\right)}{\left(\sqrt{2}\right)^{n+1} \binom{2n+6}{n+3} (2n+7)} \tag{30}$$

$$\pi = \frac{22}{7} - \frac{1}{5} \sum_{n=1}^{\infty} \frac{F\left(\left\{5, 2n+2\right\}, \left\{n+10\right\}, 1/2\right)}{2^{n+2} \binom{n+9}{5}} \tag{31}$$

$$\pi = \frac{22}{7} - \frac{1}{160} \sum_{n=1}^{\infty} \frac{F\left(\left\{-n+8, n+5\right\}, \left\{n+10\right\}, 1/2\right)}{\binom{n+9}{5}} \tag{32}$$

$$\pi = \frac{22}{7} - \frac{8}{5} \sum_{n=1}^{\infty} \frac{F\left(\left\{5, -n+8\right\}, \left\{n+10\right\}, -1\right)}{2^{n} \binom{n+9}{5}} \tag{33}$$

Remark : $F\left(\left\{a,b\right\}, \left\{c\right\}, x\right) \equiv {}_2F_1\left(\left\{a,b\right\}, \left\{c\right\}, x\right)$ is the hypergeometric function.

5. Hypergeometric function ${}_3F_2$.

$$\pi = \frac{22}{7} - \frac{1}{630} {}_3F_2\left(\left\{1, \frac{5}{2}, 3\right\}, \left\{5, \frac{11}{2}\right\}, -1\right) \tag{34}$$

6. Series of logarithms

$$\pi = \frac{22}{7} - \left(16\ln 2 - \frac{621}{56}\right) - \left(240\ln 2 - \frac{83843}{504}\right) - \left(2352\ln 2 - \frac{456479}{280}\right) - \dots \tag{35}$$

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty} f_n \tag{36}$$

$$f_n = 8 \sum_{k=0}^{n+3} \sum_{m=0}^{k} \binom{n+3}{k} \binom{k}{m} (-1)^{n-k+m+1} 2^{n-k} 3^{k-m} g(n,k,m) \tag{37}$$

$$g(n,k,m) = \begin{cases} \ln 2 & ,k+m-n+1=0 \\ \dfrac{2^{k+m-n+1}-1}{k+m-n+1} & ,k+m-n+1 \neq 0 \end{cases} \tag{38}$$

7. Series of inverse tangents

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty} \left(\frac{q(n)}{21n^7 (n+1)^7} - 4\tan^{-1}\left(\frac{1}{n^2+n+1} \right) \right) \tag{39}$$

$$\begin{aligned} q(n) = 84n^{12} + 504n^{11} + 1176n^{10} + 1260n^9 + 497n^8 - 112n^7 - 42n^6 \\ + 112n^5 + 28n^4 - 42n^3 - 14n^2 + 7n + 3 \end{aligned} \tag{40}$$

$$\pi = \frac{22}{7} - \left(\frac{3461}{2688} - 4\tan^{-1}\frac{1}{3} \right) - \left(\frac{1113155}{1959552} - 4\tan^{-1}\frac{1}{7} \right) - \ldots \tag{41}$$

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