Question 405: pi and G

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abstract
This note presents some formulas involving pi and G (Catalan constant).

I. Introduction

Pi constant:

1. $$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Catalan constant:

2. $$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.91596559\ldots$$

In Ref.1, page 375 (formulas 3521.1,3521.2) appears:

3. $$\int_0^{\infty}\frac{x}{\sinh x}\,dx = \frac{\pi^2}{4}$$

4. $$\int_0^{\infty}\frac{x}{\cosh x}\,dx = 2G$$

Using (3) and (4) we obtain some formulas involving pi and G.

II. Formulas

5. $$\int_0^{\infty}\int_0^{\infty}\frac{xy}{\sinh(x+y)+\sinh(x-y)}\,dx\,dy = \frac{\pi^2 G}{4}$$

6. $$\int_0^{\infty}\int_{-y}^{y}\frac{y^2-x^2}{\sinh y+\sinh x}\,dx\,dy = 2\pi^2 G$$

7. $$\int_0^\infty \int_0^y \frac{(y^2 - x^2)\sinh y}{(\sinh y)^2 - (\sinh x)^2}\, dx\, dy = \pi^2 G$$

8. $$\int_0^\infty \int_0^1 \frac{(1 - x^2)\, y^3 \sinh y}{(\sinh y)^2 - (\sinh(x\,y))^2}\, dx\, dy = \pi^2 G$$

9. $$\int_0^\infty \frac{\sqrt{x + \sqrt{1 + x^2}}\, \ln\left(x + \sqrt{1 + x^2}\right)}{x\sqrt{1 + x^2}}\, dx = \frac{\pi^2}{2} + 4\,G$$

10. $$\int_0^\infty \frac{\ln\left(x + \sqrt{1 + x^2}\right)}{x\sqrt{1 + x^2}\,\sqrt{x + \sqrt{1 + x^2}}}\, dx = \frac{\pi^2}{2} - 4\,G$$

11. $$\int_0^1 \frac{-\ln(x)}{1 - x^4}\, dx = \int_1^\infty \frac{x^2 \ln(x)}{x^4 - 1}\, dx = \frac{\pi^2}{16} + \frac{G}{2}$$

12. $$\int_0^1 \frac{-x^2 \ln(x)}{1 - x^4}\, dx = \int_1^\infty \frac{\ln(x)}{x^4 - 1}\, dx = \frac{\pi^2}{16} - \frac{G}{2}$$

13. $$\frac{\pi^2}{8} + G = \int_0^\infty \frac{x\, e^x}{\sinh(2\,x)}\, dx$$

14. $$\frac{\pi^2}{8} - G = \int_0^\infty \frac{x\, e^{-x}}{\sinh(2\,x)}\, dx$$

15. $$\frac{\pi^2}{8} + G = \int_0^a \frac{x\, e^x}{\sinh(2\,x)}\, dx + 2 \sum_{n=0}^\infty \frac{(1 + (4\,n + 1)a)e^{-(4\,n + 1)a}}{(4\,n + 1)^2}$$

$$a \geq 0$$

16. $$\frac{\pi^2}{8} - G = \int_0^a \frac{x\, e^{-x}}{\sinh(2\,x)}\, dx + 2 \sum_{n=0}^\infty \frac{(1 + (4\,n + 3)a)e^{-(4\,n + 3)a}}{(4\,n + 3)^2}$$

$$a \geq 0$$

17. $$\frac{\pi^4}{16} - 4\,G^2 = \int_0^{\infty}\int_{-y}^{y} \frac{(y^2 - x^2)\, e^x}{\cosh(2\,y) - \cosh(2\,x)}\, dx\, dy$$

18. $$\frac{\pi^2}{4} = \ln\left(\prod_{n=1}^{\infty}\left(\tanh\left(\frac{n+1}{2}\right)\coth\left(\frac{n}{2}\right)\right)^n\right) + \int_0^1 \left(\sum_{n=0}^{\infty}\frac{x}{\sinh(x+n)}\right) dx$$

19. $$2\,G = 2\sum_{n=1}^{\infty} n \tan^{-1}\left(\frac{e^{-n}(1 - e^{-1})}{1 + e^{-2n-1}}\right) + \int_0^1 \left(\sum_{n=0}^{\infty}\frac{x}{\cosh(x+n)}\right) dx$$

References.

[1] GRADSHTEYN, I.S., and RYZHIK, I.M.: TABLE of INTEGRALS,SERIES, and PRODUCTS. seventh edition. Edited by Alan Jeffrey and Daniel Zwillinger. Academic Press , 2007.