Question 411: Integrals

Edgar Valdebenito

abstract
This note presents a collection of integrals for pi

Collection of Integrals for Pi

$$\frac{11}{32}\pi=\int_0^1\sqrt{\frac{1}{\sqrt{1-\sqrt{x}}}-1}\,dx \tag{1}$$

$$\frac{973}{4096}\pi=\int_0^1\sqrt{\frac{1}{\sqrt{1-\sqrt{1-\sqrt{x}}}}-1}\,dx \tag{2}$$

$$\frac{39766083}{134217728}\pi=\int_0^1\sqrt{\frac{1}{\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{x}}}}}-1}\,dx \tag{3}$$

$$2\sqrt{\pi}\left(\sqrt{2}-1\right)=\int_0^1\left(\sqrt{-\frac{1}{\ln\left(\frac{1+x}{2}\right)}}-\sqrt{-\frac{1}{\ln\left(\frac{1-x}{2}\right)}}\right)x\,dx \tag{4}$$

$$\frac{\sqrt{\pi}\,e^{-2}}{4}=\int_1^\infty\left(\sqrt{x+\sqrt{x^2-1}}-\sqrt{x-\sqrt{x^2-1}}\right)e^{-2x}\,dx \tag{5}$$

$$\frac{\pi}{4}=\int_0^1\sin^{-1}\left(\frac{x^2}{\sqrt{x^4+(1-x)^4}}\right)dx \tag{6}$$

$$\frac{\pi}{4} = \int_0^1 \cos^{-1}\left(\frac{x^2}{\sqrt{x^4+(1-x)^4}}\right)dx \tag{7}$$

$$\frac{\pi}{\sqrt{2}} - \frac{\Gamma(1/4)^2}{4\sqrt{\pi}} = \int_1^{\sqrt{2}} \sin^{-1}\sqrt{2-2x^{-2}}\,dx \tag{8}$$

$$\frac{\pi}{2} - \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}} = \int_0^1 \frac{x\sin^{-1}x}{\left(2-x^2\right)^{3/2}}\,dx \tag{9}$$

$$\frac{\pi}{2} = \int_0^1 \left(\cos^{-1}\left(\frac{x-\sqrt{4-5x+x^2}}{2}\right) - \cos^{-1}\left(\frac{x+\sqrt{4-5x+x^2}}{2}\right)\right)dx \tag{10}$$

$$\begin{aligned}\int_0^1 \sin^{-1}\left(\sqrt{\frac{2+x^2-x\sqrt{8+x^2}}{2}}\right)dx &= \frac{\pi}{\sqrt{2}}\left(F\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\{1\},\frac{1}{2}\right) - F\left(\left\{-\frac{1}{2},\frac{1}{2}\right\},\{1\},\frac{1}{2}\right)\right)\\ &= \frac{1}{4\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2 - \pi\sqrt{2\pi}\,\Gamma\left(\frac{1}{4}\right)^{-2}\end{aligned} \tag{11}$$

$$\begin{aligned}\int_0^1 \sin^{-1}\left(\frac{1}{2}\sqrt{x^2+x\sqrt{8+x^2}}\right)dx &= \frac{\pi}{2}\left(1-2F\left(\left\{-\frac{1}{2},\frac{1}{2}\right\},\{1\},\frac{1}{2}\right) + F\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\{1\},\frac{1}{2}\right)\right)\\ &= \frac{\pi}{2} - 2\pi\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)^{-2}\end{aligned} \tag{12}$$

$$\frac{\pi}{4} = \int_0^\infty \left(1-\sqrt{\frac{\sqrt{1+4x^2}-1}{2x^2}}\right)dx \tag{13}$$

$$\frac{\sqrt{3}}{\pi\sqrt[3]{4}}\Gamma\left(\frac{2}{3}\right)^3 = \int_0^\infty \left\{1-\left(\sqrt[3]{1+x^{-2}\sqrt[3]{1+x^{-2}\sqrt[3]{1+x^{-2}\ldots}}}\right)^{-1}\right\}dx \tag{14}$$

$$\frac{3\pi^2}{32} - \ln 2 = \int_0^{\pi/4} \left\{x\sin\left(x+x\sin\left(x+x\sin\left(x+\ldots\right)\right)\right)\right\}dx \tag{15}$$

$$\pi = \int_0^\infty \left(\sqrt[3]{\sqrt{\frac{1}{27}+\frac{1}{4x^2}}+\frac{1}{2x}} - \sqrt[3]{\sqrt{\frac{1}{27}+\frac{1}{4x^2}}-\frac{1}{2x}} \right)^2 dx \qquad (16)$$

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{4}\Gamma\left(\frac{1}{4}\right)^2 - \Gamma\left(\frac{3}{4}\right)^2\right) = \int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\,dx \qquad (17)$$

$$\frac{1}{2\sqrt{2\pi}}\left(\frac{1}{4}\Gamma\left(\frac{1}{4}\right)^2 + \Gamma\left(\frac{3}{4}\right)^2\right) - \frac{1}{2} = \int_1^\infty \frac{1}{1+x^2+\sqrt{x^4-1}}\,dx \qquad (18)$$

$$\frac{\pi}{10}-\frac{3\ln 2}{10}+\frac{\ln 3}{5}+\frac{1}{18} = \int_{1/6}^{1/2}\left(\sqrt[3]{\frac{1}{2x}-\frac{26}{27}+\sqrt{\frac{25}{27}-\frac{26}{27x}+\frac{1}{4x^2}}}+\sqrt[3]{\frac{1}{2x}-\frac{26}{27}-\sqrt{\frac{25}{27}-\frac{26}{27x}+\frac{1}{4x^2}}}\right)dx \qquad (19)$$

$$\frac{\pi}{10}-\frac{3\ln 2}{10}+\frac{\ln 3}{5}+\frac{1}{18} = \int_{1/6}^{1/2}\left(\sqrt[3]{\frac{1}{x}-\frac{52}{27}+\frac{1}{3}\sqrt[3]{\frac{1}{x}-\frac{52}{27}+\frac{1}{3}\sqrt[3]{\frac{1}{x}-\frac{52}{27}+\ldots}}}\right)dx \qquad (20)$$

$$\pi = \int_0^{1/2}\sqrt{\frac{\sqrt[3]{2}\left(4x^2+2\sqrt[3]{2}\left(243x-2x^3+9x\sqrt{729-12x^2}\right)^{2/3}\right)}{x\left(243x-2x^3+9x\sqrt{729-12x^2}\right)^{1/3}}-40}\;dx \qquad (21)$$

Remarks :

- ❖ $\Gamma(x)$ is the gamma function
- ❖ $F(\{a,b\},\{c\},x) \equiv {}_2F_1(\{a,b\},\{c\},x)$ is the hypergeometric function.

References

1. Boros, G. and Moll, V.: Irresistible Integrals, Cambridge University Press, 2004.
2. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products. seventh edition. Edited by Alan Jeffrey and Daniel Zwillinger. Academic Press, 2007.