PROOF OF THE COLLATZ CONJECTURE

KURMET SULTAN

ABSTRACT: The article provides proof of the Collatz conjecture. It is proved that the calculation of the Collatz function $C(n)$ based on numbers of the form $6m \pm 1,\ m \in \mathbb{N}$ is equivalent to the calculation based on any positive integers. It is further proved that if, on the basis of elements of the set $G = \{g | g = 6n \mp 1, n \in N\}$ and 1, we perform the inverse calculation by the formulas $((6n \pm 1) \cdot 2^q - 1)/3$ and $(2^q - 1)/3$ then to each number of the form $6n \pm 1$ and 1 there will correspond an infinite number of integers of the form 3t, $6m - 1$ and $6m + 1$. Then it is shown that if we construct a graph of numbers by combining equal numbers $6n \pm 1$ and $6m \pm 1$, which are elements of the set G and predecessor numbers, respectively, then a graph tree is formed. A graph tree, each vertex of which corresponds to numbers of the form $6m \pm 1$, is a proof of the Collatz conjecture, since any vertex of it is connected with a finite vertex associated with a unit.

Keywords: Collatz conjecture, 3n + 1 problem, Syracuse problem, proof.

MSC CLASSIFICATION CODE: 11D04

1. INTRODUCTION

The Collatz conjecture, also known as the *3 n + 1* problem and the Syracuse problem, is one of the unresolved problems in mathematics. The following papers devoted to the *3 n + 1* problem [1, 2, 3] can be noted.

The Collatz function $C(n)$ is defined on the natural numbers as follows:

$$C(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

For purposes of explaining the Collatz conjecture, take any natural number n. If the number is even, then divide it by 2, and if the number is odd, then multiply by 3 and add 1, we obtain even number *3 n + 1*. On the number obtained, we perform the same actions, and so on. The Collatz conjecture is that regardless of the initial number *n* is taken, we eventually arrive at unity.

2. THE STARTING NUMBER

Theorem 1. *The calculation of the Collatz function C (n), when only numbers of the form* $6m \mp 1$ *are used for calculation, is equivalent to the calculation of C (n) using any positive integers.*

Theorem 1 is based on the following regularity:

If the calculation of the Collatz function is carried out according to the formula

$$C = (3n + 1)/2^q, \text{ where } n, q \in N, \tag{2}$$

until an integer is obtained, depending on the type of the number n, the following results will be obtained for the Collatz function *C (n)*:

1) If n is even, then *C (n)* is odd or 1;
2) If n is an odd number multiple of 3, then *C (n)* is an odd number of the form $6m \mp 1$ or 1;
3) If n is an odd number of the form $6m \mp 1$, then *C (n)* is an odd number of the form $6n \mp 1$ or 1.

Proof of Theorem 1.

1.1. Obviously, if you start calculating the Collatz function from even numbers, you get an odd number or 1.

1.2. If we begin the calculation of the Collatz function with odd numbers that are multiples of 3, we get a number of the form $6m \mp 1$ or 1.

This is explained as follows. If odd numbers of the form $3t$ are multiplied by 3 and add 1, then it is obvious that we get numbers having the form 3s + 1. Moreover, if a number having the form $3s + 1$is odd, then, of course, it will be a number of the form $6m + 1$. And if a number of the form $3s + 1$ is an even number, then when dividing by 2 (one or several times) until an odd number is obtained, a number of the form $6m \mp 1$ or 1 is formed, since a number of the form $3s + 1$ is not divisible by 3.

1.3. It is clear that all numbers of the form $6m + 1$ and $6m - 1$ are odd numbers. If we multiply the odd number by 3 and add 1, then naturally we get an even number of the form $3s + 1$. And when dividing even numbers of the form $3s + 1$ by two (one or several times) until an odd number is obtained, the resulting odd numbers will be of the form $6n \mp 1$ or 1, since numbers of the form $3s + 1$ are not divided by 3.

It follows from the above that Theorem 1 is proved.

2. REVERSE CALCULATION

Definition 1. *The numbers from which the calculation of the Collatz function yields the considered number is called the predecessor numbers.*

The predecessor numbers for any element of the set $G = \{g | g = 6n \mp 1, n \in N\}$ and the number 1 can be established by performing the inverse calculation. For these purposes, you can use the following formulas

$$r_g = (g \cdot 2^q - 1)/3\,, r_1 = (1 \cdot 2^q - 1)/3\,,\ \text{where } g = 6n \mp 1, n\ \in N. \tag{3}$$

It follows from Theorem 1 that, in the inverse calculation of the Collatz function, each element of the set $G = \{g | g = 6n \mp 1, n \in N\}$ and the number 1 are associated with elements of the sets

$$R_g = \{v, r | v = 3t;\ r = 6m \mp 1;\ t, m \in N\}, \tag{4}$$

$$R_1 = \{v_1, r_1 | v_1 = 3t;\ r_1 = 6m \mp 1, t, m \in N\}, \tag{5}$$

Moreover, each set R_g associated with a certain element g of the set G and the number 1 are disjoint. In other words, the elements of the set R_g associated with one element g of the set G will not be repeated in other sets. This is proved as follows: Let two elements belonging to two sets R_i and R_j be equal, then we obtain the following equality

$$(g_1 \cdot 2^{q_1} - 1)/3 = (g_2 \cdot 2^{q_2} - 1)/3 \quad \text{or} \quad g_1/g_2 = 2^{q_2}/2^{q_1}.$$

The last equality does not have a solution in integers, since the right-hand side of the equality is an even number or 1 (if $q_1 \leq q_2$), and the left-hand side of the equality is odd or non-integer, since g_1 and g_2 are odd numbers, and $g_1 \neq g_2$.

It is easy to prove that the elements of the set R_1 are not contained in the sets R_g, since if $q_1 \leq q_2$, then

$$(1 \cdot 2^{q_1} - 1)/3 \neq (g \cdot 2^{q_2} - 1)/3 \text{ or } 1/g \neq 2^{q_2}/2^{q_1}.$$

Note that if we separate elements corresponding to numbers of the form $6m \mp 1$ from the sets R_g and R_1 and combine them into one set, we obtain the following set $K = \{k | k = 6m \mp 1, m \in N\}$. This means that the sets G and K consist of the same numbers, i.e. consist of the same elements. This raises the following question:

Are all the predecessor numbers really contained in the set K?

Note that if one or more elements of the set K are not predecessor numbers, this means that the graphs whose root vertices correspond to these elements cannot be connected to a common graph.

Theorem 2. *All elements of the set $K = \{k | k = 6m \mp 1, m \in N\}$ are mapped to elements of the set $G = \{g | g = 6n \mp 1, n \in N\}$ and the number 1, that is, are precursor numbers.*

Theorem 2 is proved as follows. Suppose that one or more elements of the set K are not predecessor numbers, then when calculating the Collatz function based on these elements, the resulting numbers should not be of the form $6n \mp 1$ or should not be the number 1, which contradicts clause 1.3 of the proof of Theorem 1, which means Theorem 2, proved.

If the above is presented graphically, then we get infinitely many graphs corresponding to each element of the set G and the number 1, each of which looks like a bunch of balls tied to one ball. A diagram of the graphs of the number 1 and a number of the form $6n \mp 1$ with six numbers is shown in Fig. 1.

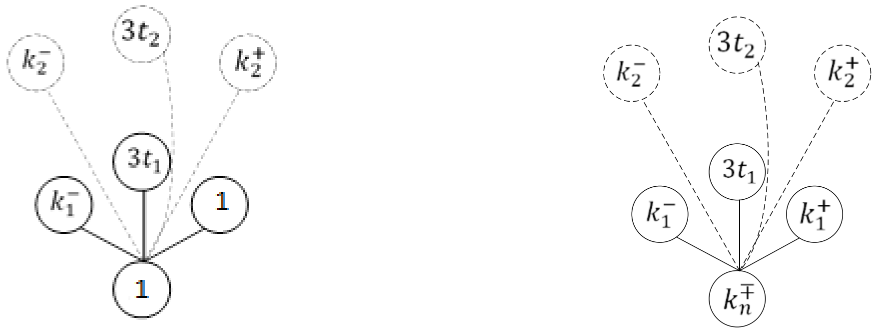

FIGURE 1. The graphs corresponding to the number 1 and one element of the set G

Note: $k^- = 6m - 1$, $k^+ = 6m + 1$, $k^\mp = 6n \mp 1$.

The actual to each element of the set G and the number 1 is associated with infinitely many numbers of the form $3t$ and $6m \mp 1$.

3. PROOF OF CONJECTURE

Since each number of the form $6m \mp 1$, which is an element of the set K and corresponding to the top vertices of the graph of elements of the set G and the number 1, is equal to a certain element of the set G, all graphs can be combined into one common graph by connecting equal numbers. The combination of graphs is as follows:

First, numbers of the form $6m \mp 1$ corresponding to the upper vertices of the graph of the number 1 are determined, then graphs of elements of the set G whose root vertices are equal to the top vertices of the graph 1 are selected. After that, the selected graphs of the elements of the set G are installed on the upper vertices of the graph by connecting the equal numbers, corresponding to the top vertices of graph 1 and the root vertices of the graphs of the selected elements of G. Next, other graphs of other elements of the set G, n the selection of equal numbers. Further, repeating such a procedure, we obtain one volumetric infinitely growing fractal graph tree, each vertex of which will be associated with the number 1.

Fig. 2 shows an example of combining two planar graphs corresponding to elements 5 and 13 of G.

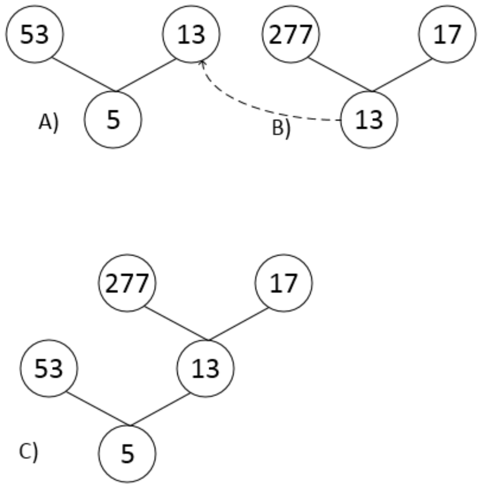

FIGURE 2. An example of combining two graphs

Note that multiples of 3 do not participate in the formation of a common graph, although each graph, including the graph of number 1, contains such numbers. Nevertheless, for completeness, they can also be shown on a common graph. The general graph-tree including numbers of multiples of 3 will have the form, as shown in Figure 3.

From the tree graph shown in Figure 3, it follows that each vertex has a multiple of 3. At the same time, numbers multiple of 3, as shown in Figure 3, do not affect the formation of the structure of the graph. If you start the calculation with numbers that are multiples of 3, then the path is joined with the vertex corresponding to a number of the form $6n \mp 1$, and then the path will continue along the graph structure. This structure corresponds to the statement that when calculating the Collatz function on the basis of multiples of 3, we obtain numbers of the form $6n \mp 1$.

In Fig.3 the two lower shaded vertices with the signs $K -/+$ and $3t$, which are connected with the vertex 1, symbolize that there are infinitely many numbers of the form $6n \mp 1$and numbers that are multiples of 3, which, when calculating the Collatz function, are infinite. Moreover, such numbers of the form $6n \mp 1$ form their branches in the tree graph.

The arrow starting from vertex 1 and directed there (loop) shows that if the calculation of the Collatz function starts from the number 1, then we get 1.

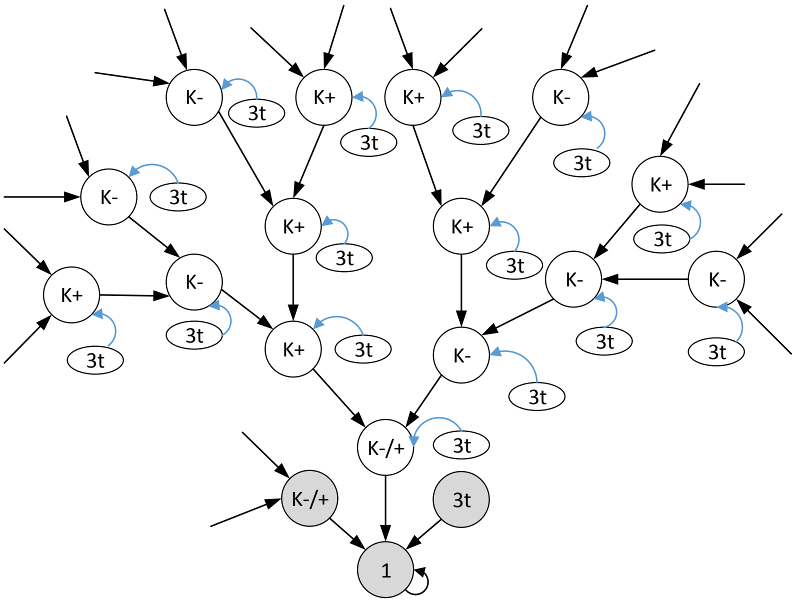

FIGURE 3. Oriented graph tree, including multiples of 3

If even common numbers (2t) are shown on the general tree graph, then we get the graph shown in Fig. 4.

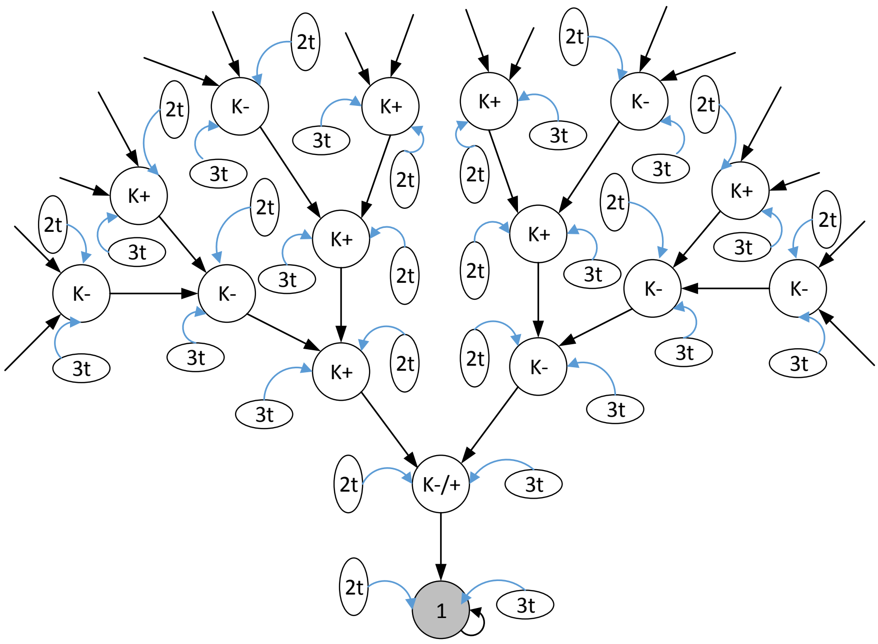

FIGURE 4. Oriented graph tree with even and odd numbers.

Note that the graphs shown in Figures 3 and 4 are based on the first three precursor numbers of elements of the set G and the number 1, therefore they are flat. In fact, a common graph tree is three-dimensional, since to each element of the set G and the number 1 there correspond infinitely many predecessor numbers.

Thus, it is proved that if the number 1 and all elements of the set $G = \{g | g = 6n \mp 1, n \in N\}$, as well as the numbers corresponding to the number 1 and the elements of the set G, are combined in the form of a graph, then a common three-dimensional graph tree, each vertex of which is associated with a finite vertex associated with a unit. It follows that the Collatz conjecture is true, and it is proved.

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