The Irrationality of Trigonometric and Hyperbolic Functions

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Introduction

Niven shows that for rational, non-zero r, $\cos r$ and $\cosh r$ are irrational [6]. His method is similar to that of his famous irrationality of π proof: functions are defined, integrals involving integration by parts are used, and a contradiction is arrived at [5]. Parks makes a similar argument, arguably simpler, for the cosine case [7]. Zhou recently proved the cosine and hyperbolic cosine results using recursive integrals [8]. In this article, our pattern does not involve integrals; just multiplication and derivatives of polynomials are needed.

The pattern is to start with an exponential equation where a sum of exponential values equals a rational number. So, for $\cosh$, using the identity $2\cosh r = e^r + e^{-r}$, this equation is $e^r + e^{-r} = a/b$. Then a polynomial, f, is defined. It has a zero root of multiplicity $p-1$, p a prime, and the exponents in the sum, r, $-r$, for $\cosh$, as additional roots of multiplicity p. The sum of the derivatives of $f(z)$ is given by $F(z)$. Using $e^z F(0) = F(z) + \epsilon$, proven below, and simple multiplication, this gives $0 = F(0)(e^r + e^{-r} - a/b) = -a/bF(0) + F(r) + F(-r) + \epsilon$. As the ϵ value grows power wise in the degree of f and multiplicity in f translates into factorial values in F, division by $(p-1)!$ gives a contradiction for large enough p. Details follow.

As the transform $e^z F(0) = F(z) + \epsilon$ is good for complex variables, as well as real, the identity $2\cos r = e^{ri} + e^{-ri}$ and the same procedure applies to this case as well. Using other identities, the corresponding irrationality of other trigonometric and hyperbolic functions are obtained. Corresponding results for inverses of these functions are easily proven as well.

The methods used in this article are also used in [3, 4] to show the transcendence of e and π and the irrationality of their natural number powers.

Lemmas

All polynomials are integer polynomials, z is a complex number, n and j are non-negative integers, and p is a prime number.

Definition 1. Given a polynomial $f(z)$, lowercase, the sum of all its derivatives is designated with $F(z)$, uppercase.

Definition 2. For non-negative integers n, let $\epsilon_n(z)$ denote the infinite series

$$\frac{z}{n+1} + \frac{z^2}{(n+1)(n+2)} + \cdots + \frac{z^j}{(n+1)(n+2)\dots(n+j)} + \dots.$$

Lemma 1. *If $f(z) = cz^n$, then*

$$F(0)e^z = F(z) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n.

Proof. As $F(z) = c(z^n + nz^{n-1} + \cdots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^z &= cn!(1 + z/1 + z^2/2! + \cdots + z^n/n! + \dots) \\ &= cz^n + cnz^{(n-1)} + \cdots + cn! + cz^{n+1}/(n+1)! + \dots \\ &= F(z) + cz^n(z/(n+1) + z^2/(n+1)(n+2) + \dots) \\ &= F(z) + f(z)\epsilon_n(z). \end{aligned}$$

Now $f(z)$ has polynomial growth in n and $\epsilon_n(z) \leq e^z$, so the product has polynomial growth in n. □

Lemma 2. *If F is the sum of the derivatives of the polynomial $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ of degree n, then*

$$e^z F(0) = F(z) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f.

Proof. Let $f_j(z) = c_j z^j$, for $0 \leq j \leq n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^{n} (f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n,$$

where F_j is the sum of the derivatives of f_j. Using Lemma 1,

$$e^z F_j(0) = F_j(z) + \epsilon \tag{3}$$

and summing (3) from $k = 0$ to n, gives

$$e^z F(0) = F(z) + n\epsilon.$$

As the finite sum of functions with polynomial growth in n also has polynomial growth in n, we arrive at (2). $\square$

Lemma 3. *If polynomial $f(z)$ has a root r of multiplicity p, then $f^{(k)}(r) = 0$ for $0 \leq k \leq p-1$ and each term of $f^{(k)}(r)$, $p \leq k \leq n$ is a multiple of $p!$.*

Proof. Suppose $r = 0$ then, for some n we have $f(z) = z^p(b_n z^n + \cdots + b_0)$. Now $f(z)$ has $b_0 z^p$ as its term with minimal exponent. Using the derivative operator, $D(z^n) = nz^{n-1}$, repeatedly, we see the 0 through $p-1$ derivatives of $f(z)$ will have a positive exponent of z in each term. This implies that $r = 0$ is a root for these derivatives. Using the product of p consecutive natural numbers is divisible by $p!$, terms of subsequent derivatives will be multiples of $p!$.

If $r \neq 0$, then $f(z) = (z-r)^p Q(z)$, for some polynomial $Q(z)$. Let $g(z) = f(z+r) = z^p Q(z+r)$. As $g^{(k)} = f^{(k)}$ for all k, $g^{(k)}(0) = f^{(k)}(r)$, and the $r = 0$ case applies. $\square$

A Leibniz table can be used to give an example of the result of Lemma 3. Suppose $f(z) = (z-r)^3 Q(z)$, where $Q(z)$ is a polynomial of degree 2. Table 1 indicates $f^{(k)}(r) = 0$ for $0 \leq k \leq 2$ and each term of $f^{(k)}(r)$ for $3 \leq k \leq 5$ is a multiple of $3!$; the right column indicates this. For more on Leibniz tables see [2].

	$(z-r)^3$	$3(z-r)^2$	$6(z-r)$	$3!$
$Q(z)$	0 *0*	0 *1*	0 *2*	$3!bQ(r)$ *3*
$Q'(z)$	0 *1*	0 *2*	0 *3*	$3!bQ'(r)$ *4*
$Q''(z)$	0 *2*	0 *3*	0 *4*	$3!bQ''(r)$ *5*

Table 1: First interior cell values give evaluations at $z = r$ and the second the *order* of the derivative. The 'b' factor is a binomial coefficient.

Lemma 4. *If a and b are Gaussian integers and $p > |a|$, then $|a(p-1)! + bp!|$ is a non-zero integer divisible by $(p-1)!$, but not by p.*

Proof. As $a(p-1)! + bp!$ is of the form $A - B + (C - D)i$ with $A - B \neq 0$ or $C - D \neq 0$, A, B, C, and D integers, both results follow. □

Lemma 5. *Let $b_j z^j + \cdots + b_0$ be a polynomial of degree greater than 1. Then there exists a p such that if $f(z) = z^{p-1}(b_j z^j + \cdots + b_0)$, then $p \nmid |F(0)|$.*

Proof. We can write

$$f(z) = z^p(b_j z^{j-1} + \dots b_1) + z^{p-1}b_0. \tag{4}$$

Using Lemma 3, as $r = 0$ is a root of multiplicity p of $z^p(b_j z^{j-1} + \dots b_1)$, its first $p-1$ derivatives, evaluated at $r = 0$ are 0 and then its terms are multiples of $p!$. Similarly the first $p-2$ derivatives of $z^{p-1}b_0$ are 0 at 0 and the $p-1$ derivative is $(p-1)!b_0$ and subsequent derivatives are multiples of $p!$. If $p > |b_0|$, as all terms but one in $F(0)$ are a multiple of p, the result follows. □

Example

We can illustrate a simple example of these lemmas. Given $e^{r_1} + e^{r_2} = a$, where r_1, r_2, and a are whole numbers, define polynomial $f(z) = z^{p-1}[(z-r_1)(z-r_2)]^p$. Then $0 = F(0)(e^{r_1} + e^{r_2} - a)$ and, using Lemma 2, $0 = F(r_1) + F(r_2) - aF(0) + \epsilon$.

Using a Leibniz table and letting $p = 5$, the derivatives of z^{p-1} and $[(z - r_1)(z - r_2)]^p$ are along the top row and left most column in Table 2. The interior cells added give $F(z)$. Consider $F(0) + F(r_1) + F(r_2)$. When $z = 0$, the first row, second through fifth column values are all 0. The only non-zero interior values occur in the last column. The first term in this column is $4!(r_1 r_2)^5$; if $r_1 r_2 < 5$, then this term is not divisible by 5. The remaining terms all have $5!$ factors. When $z = r_j$, $j = 1, 2$, using Lemma 3, all non-zero derivatives have all terms multiples of $5!$. This means $F(0) + F(1) + F(2)$ has all terms divisible by $(p-1)!$ and all but one term divisible by p. Using Lemma 4, we can conclude it is not zero, when $p > r_1 r_2$, and an integer when divided by $(p-1)!$. For large enough p this gives a contradiction of $0 = aF(0) + F(r_1) + F(r_2) + \epsilon$.

Applications

Theorem 1. *For non-zero rational r, $\cosh r$ is irrational.*

All 0 at $z = 0$.

		z^4	$4z^3$	$12z^2$	$4!z$	$4!$
These are 0 at r_1, r_2.	$Q(z)^5$					$4!P^{(0)}(0)$
	$5Q(z)^4\dots$					$5!\dots$
	$P^{(2)}(z)$					$\vdots$
	$P^{(3)}(z)$					
	$P^{(4)}(z)$					
These are $5!A$ at r_1, r_2.	$P^{(5)}(z)$					
	$P^{(6)}(z)$					
	$P^{(7)}(z)$					
	$P^{(8)}(z)$					
	$P^{(9)}(z)$					
	$P^{(10)}(z)$					

Table 2: Leibniz table for $z^4P(z)$, where $P(z) = [(z-r_1)(z-r_2)]^5$. The first two derivatives are expressed with $Q(z) = (z-r_1)(z-r_2)$ to show how a $4!$ factor is generated for one cell only.

Proof. Suppose not. Suppose $2\cosh r = a/b$ where a/b is a rational number. As 0 is not in the range of $\cosh$, we can assume $a/b \neq 0$. Using the exponents of $2\cosh = e^r + e^{-r}$, define

$$f(z) = d^{3p-1}z^{p-1}[(z+r)(z-r)]^p = (dz)^{p-1}(dz-c^2)^p$$

where $r = c/d$. Then $f(z)$ is an integer polynomial. Next

$$0 = F(0)\left(e^r + e^{-r} - \frac{a}{b}\right).$$

Using Lemma 2,

$$|b\epsilon| = |bF(r) + bF(-r) - aF(0)| \tag{5}$$

and, using Lemmas 3, 4, and 5, this gives a contradiction: for large enough p, dividing (5) by $(p-1)!$ gives a left side less than 1 equals a right side of at least 1. We've used if p doesn't divide a whole number, that whole number is greater than 0. $\square$

As Lemma 3 applies to complex polynomials as well as real, the proof for $\cos r$ is similar.

Theorem 2. *For non-zero rational* r*,* $\cos r$ *is irrational.*

Proof. Suppose not. Suppose $2\cos r = a/b$ where a/b is a rational number. We exclude $a/b = 0$ as $\cos k\pi/2 = 0$ has $\cos$ with an irrational argument. Using the exponents of $2\cos = e^{ri} + e^{-ri}$, define

$$f(z) = d^{3p-1}z^{p-1}[(z+ri)(z-ri)]^p = (dz)^{p-1}((dz)^2 + c^2)^p$$

where $r = c/d$. Then $f(z)$ is an integer polynomial. Next

$$0 = F(0)\left(e^{ri} + e^{-ri} - \frac{a}{b}\right).$$

Exactly as in the $\cosh$ case, we have, using Lemma 2,

$$|b\epsilon| = |bF(ri) + bF(-ri) - aF(0)|$$

and, using Lemmas 3, 4, and 5, this gives a contradiction. □

Note: As $\cos\pi = -1$, Theorem 2 does imply that π is irrational. This does imply that $k\pi/2$ is also irrational – the case missing from Theorem 2.

Other functions

Once $\cos r$ and $\cosh r$ are proven irrational, sec r and sech r are easy consequences, being reciprocals of these functions [6]. As $\cos 2r = \cos^2 r - \sin^2 r = 1 - 2\sin^2 r$, the rationality of $\sin r$ would imply that of $\cos$, a contradiction. Assume $\tan r$ is rational. Then using

$$\cos 2r = \frac{1-\tan^2 r}{1+\tan^2},$$

$\cos r$ would be rational too, a contradiction. As csc r and cot r are the reciprocals of $\sin r$ and $\tan r$, the former two are proven irrational. Similarly, the hyperbolic functions all follow the same program. Inverse functions have an easy proof: if $f^{-1}(r) = a/b$ then $f(f^{-1}(r)) = r = f(a/b)$, a contradiction of $f(r)$ is irrational.

References

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